# Improper affine spheres and the Hessian one equation [1]

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**Abstract**

Improper affine spheres have played an important role in the development of geometric methods for the study of the Hessian one equation. Here, we review most of the advances we have made in this direction during the last twenty years.

## 1 Introduction

Differential Geometry and Partial Differential Equations (PDEs) are related by a productive tie by means of which both theories well out. On the one hand, many results from the theory of hypersurfaces and submanifolds use, in a fundamental way, tools from PDEs theory. Among them, we can emphasize topics like maximum principles, regularity theorems, height estimates, asymptotic behaviour at infinity, representation theorems, or results of existence and uniqueness with fixed boundary conditions.

On the other hand, many classic PDEs are linked to interesting geometric problems. Even more, apart from being source of inspiration in the search for interesting PDEs, the geometry allows, in many cases, to integrate these equations, to establish non trivial properties of the solutions and to give superposition principles which determine new solutions in terms of already known solutions.

One of the biggest contributions from geometry to PDEs theory are Monge-Ampère equations. Such equations are totally non linear PDEs which model interesting geometric aspects related to the curvature, and its study has become a topic of great mathematical importance, see [15, 24].

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Among the most outstanding Monge-Ampère equations we can quote the Hessian one equation

$$u_{xx}u_{yy} - u_{xy}^2 = \varepsilon, \qquad \varepsilon \in \{-1, 1\}, \qquad (x, y) \in \Omega \subseteq \mathbb{R}^2. \tag{$H_\epsilon$}$$

The equation $(H_\epsilon)$ has been studied from many perspectives by several authors and the situation changes completely if we take $\varepsilon = 1$ (elliptic case) or $\varepsilon = -1$ (hyperbolic case). When $\varepsilon = 1$, we know there exists a representation for the solutions in terms of holomorphic data and their most relevant global properties has been studied, such as the behaviour of the ends, the flexibility or the existence of symmetric solutions (see [9, 10]). In [3, 11] the space of solutions of the exterior Dirichlet problem is endowed with a suitable structure and in [12], the space of solutions of the equation defined on $\mathbb{R}^2$ with a finite number of points removed was described, and endowed with the structure of a differential manifold. In [1] the Cauchy problem was solved, and this solution was used, among other things, to classify the solutions which have an isolated singularity. Thanks to classical subjects of the PDEs theory, such as the continuity method, it seems possible to extend some of the above results to more general classes of Monge-Ampère equations. However, the hyperbolic case is much more complicated and we can not expect classification results as in the definite case. For example, $u(x, y) = xy + g(x)$ is an entire solution for any real function $g$.

From a geometric point of view, the equation $(H_\epsilon)$ arises as the equation of improper affine spheres, that is, surfaces with parallel affine normals. Although Affine Differential Geometry has a long history whose origins date back to 1841 in a work by Transon on the normal affine of a curve, it has been during the last thirty years when this theory has experienced a remarkable development. In its research geometric, analytic and complex techniques mix and in order to understand the geometry of this important family of surfaces deeply, the global rigidity has been weakened in different ways. On the one hand, in [9, 10], using methods of complex analysis, their behaviour at the infinity has been described noting that there exists a closely relationship between their ends and the ones of a minimal surface with finite total curvature. On the other hand and concerning with $(H_\epsilon)$, a geometric theory of smooth maps with singularities (improper affine maps) has been developed opening an interesting range of global examples. In most of the cases the singular set determines the surface and, generically, the singularities are cuspidal edges and swallowtails, see [8, 13, 17].

In very recent works, [18, 19, 20, 21] we have solved the problem of finding all indefinite improper affine spheres passing through a given regular curve of $\mathbb{R}^3$ with a prescribed affine co-normal vector field along this curve; the problem is well-posed when the initial data are non-characteristic and the uniqueness of the solution can fail at characteristic directions. We also have learnt how to obtain easily improper affine maps with a prescribed singular set, how to construct global examples with the desired singularities and how to use the classical theory of Ribaucour transformations to obtain new solutions of the elliptic hessian one equation.

Here, we want to make a short survey with some of the above mentioned advances in the geometric study of $(H_\epsilon)$.

## 2 A complex (split-complex) representation formula

If $u : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a solution of $(H_\epsilon)$, then its graph

$$\psi = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$$

is an improper affine sphere in $\mathbb{R}^3$ with constant affine normal $\xi = (0, 0, 1)$, affine metric $h$,

$$h := u_{xx}\ dx^2 + u_{yy}\ dy^2 + 2u_{xy}\ dxdy, \tag{2}$$

and affine conormal $N$,

$$N := (-u_x, -u_y, 1). \tag{3}$$

From (2) and (3) it is easy to check that the following relations hold,

$$h = - < dN, d\psi >, \quad < N, \xi >= 1, \quad < N, d\psi >= 0, \tag{4}$$

$$\sqrt{|\det(h)|} = \det[\psi_x, \psi_y, \xi] = -\det[N_x, N_y, N], \tag{5}$$

see [16, 22] for more details. Conversely, up to unimodular transformations, any improper affine sphere in $\mathbb{R}^3$ is, locally, the graph over a domain in the $x, y$-plane of a solution to $(H_\epsilon)$.

In the elliptic (hyperbolic) case the affine metric $h$ induces a Riemann (Lorentz) surface structure on $\Omega$ known as the *underlying conformal structure of* $u(x, y)$.

It follows from $(H_\epsilon)$ that,

$$(du_x)^2 + \varepsilon\ dy^2 = u_{xx}h, \qquad (du_y)^2 + \varepsilon\ dx^2 = u_{yy}h, \tag{6}$$

and the two first coordinates of $\psi$ and $N$ provide conformal parameters for $h$. Actually, consider $\mathbb{C}_\varepsilon$ the complex (split-complex) numbers, that is

$$\mathbb{C}_\varepsilon = \{z = s + j\ t\ :\ s, t \in \mathbb{R},\ j^2 = -\varepsilon,\ j1 = 1j\}, \tag{7}$$

then it is not difficult to prove, see [5, 10, 20], that $\Phi : \Omega \longrightarrow \mathbb{C}^3_\varepsilon$,

$$\Phi := N + j\ \xi \times u, \tag{8}$$

is a planar holomorphic (split-holomorphic) curve. In fact, $\Phi = (-B, A, 1)$ where

$$A := -u_y + j\ x, \qquad B := u_x + j\ y, \tag{9}$$

are holomorphic (split-holomorphic) functions on $\Omega$. Moreover, from $(H_\epsilon)$ and (2),

$$|d\Phi|^2 = |dA|^2 + |dB|^2 = (u_{xx} + u_{yy})h, \tag{10}$$

and $|d\Phi|^2$ and $h$ are in the same conformal class always that $u_{xx} + u_{yy}$ has a sign.

From (2) and (9), the metric $h$ is given by

$$h := \mathrm{Im}(dA\overline{dB}) = |dG|^2 - |dF|^2 \tag{11}$$

where $2F = -B - \varepsilon j\ A$ and $2G = B - \varepsilon j\ A$, and the immersion $\psi$ may be recovered as

$$\psi := -\frac{1}{2}\operatorname{Im}\int(\Phi + \overline{\Phi}) \times d\Phi = (G + \overline{F}, \frac{|G|^2}{2} - \frac{|F|^2}{2} + 2\operatorname{Re}\int Gd\overline{F}), \tag{12}$$

with the two first coordinates of $\psi$ identified as numbers of $\mathbb{C}_\varepsilon$ in the standard way.

Conversely, if we consider the class of improper affine maps, (IAM), as the improper affine spheres with admissible singularities, where the affine metric is degenerated, but the affine conormal is well defined, then we have

**Theorem 2.1.** **[9, 17]** *Let $\Sigma$ be a Riemann surface and $\psi : \Sigma \longrightarrow \mathbb{C}_\varepsilon \times \mathbb{R}$ be an improper affine map, then $\psi$ can be represented as*

$$\psi = \Big(G + \overline{F}, \frac{1}{2}|G|^2 - \frac{1}{2}|F|^2 + Re(GF) - 2Re\int FdG\Big) \tag{13}$$

*with $F, G : \Sigma \longrightarrow \mathbb{C}_\varepsilon$ holomorphic (split-holomorphic) functions. In this case, the affine conormal and the affine metric of $\psi$ are given, respectively, by $N = (\overline{F} - G, 1)$ and*

$$h = |dG|^2 - |dF|^2.$$

The pair $(G, F)$ is called the *Weierstrass data of $\psi$*.

## 3 The elliptic case

The above complex representation has been an important tool in the description of global examples and in the methods used to understand the geometry of this important family of surfaces deeply. Classical examples can be described as follows:

1. *The elliptic paraboloid:* It can be obtained by taking, $\Sigma = \mathbb{C}$ and Weierstrass data $(z, kz)$, where $k$ is constant. (see Figure 1).

2. *Rotational improper affine maps:* They are obtained by taking $\Sigma = \mathbb{C}\backslash\{0\}$ and Weierstrass data $(z, \pm r^2/z)$, $r \in \mathbb{R}\backslash\{0\}$, (see Figure 1).

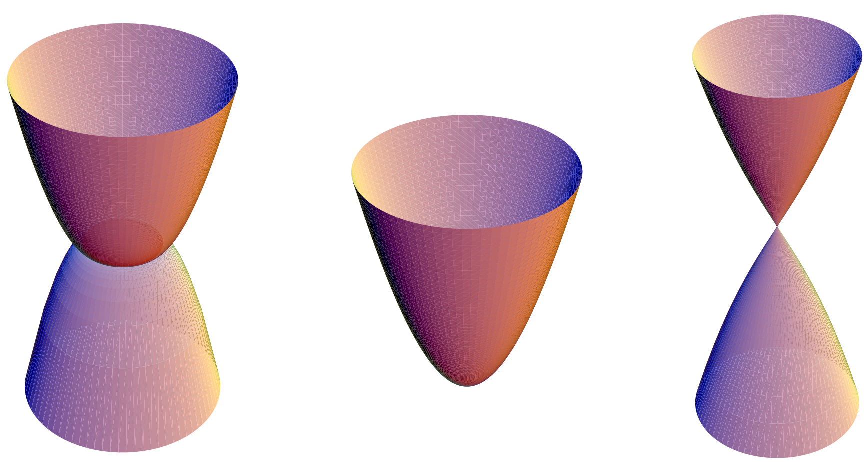

Figure 1: Elliptic paraboloid and rotational improper affine maps

3. *Non-rotational improper affine maps with two-ends.* Others complete examples with two embedded ends can be obtained by taking $\Sigma = \mathbb{C}\backslash\{0\}$ and Weierstrass data: $(z, az + b/z + c)$, where $b \in \mathbb{R}$, $a, c \in \mathbb{C}$, $|a| \neq 1$ (see Figure 2).

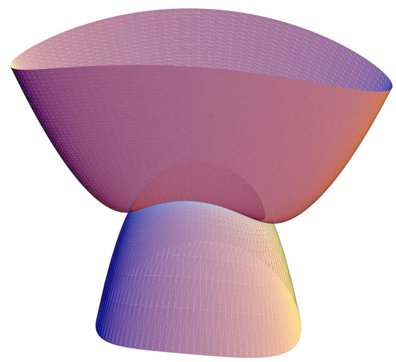

Figure 2: Non-rotational complete improper affine map with two embedded ends

4. *Some multivalued improper affine map:* By considering $\Sigma = \mathbb{C}\backslash\{0\}$ and Weierstrass data $(z, \pm\sqrt{-1}r^2/z)$, $r \in \mathbb{R}\backslash\{0\}$ we obtain multivalued complete improper affine maps with a vertical period and two embedded ends. (see Figure 3)

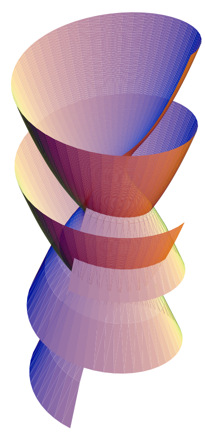

Figure 3: Multivalued complete improper affine map with two embedded ends

5. *Improper affine map with one non-embedded end:* By taking $\Sigma = \mathbb{C}$ and the Weierstrass data $(z, z+z^2)$ we obtain a complete improper affine map with only one end which is non-embedded and non affinely equivalent to the elliptic paraboloid (see Figure 4)

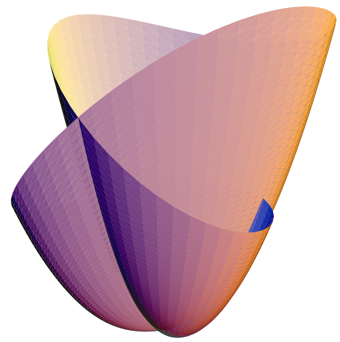

Figure 4: Complete improper affine map with only one non-embedded end

### 3.1 Global behaviour

The most elemental elliptic model is the elliptic paraboloid, it is the only elliptic improper affine sphere which is complete from the affine point of view, [4, 6] and also from the Euclidean one [14].

Using methods of complex analysis, [17], one can weakened the above global rigidity and prove that any complete IAM is conformally equivalent to a compact Riemann surface $\overline{\Sigma}$ with a finite number of points (ends) removed. In this case the Weierstrass data extend meromorphically to the ends and a complete end $p$ is embedded if and only if the Weirestrass data have at most a single pole at $p$. In this context, the classical result of Jörgens is extended to prove that the elliptic paraboloid is the unique embedded complete IAM with only one end and any complete IAM with exactly two embedded ends is affine equivalent to either a rotational IAM or to one of the examples described above in 3.

These global results and (13), let us to prove:

- Any solution of $(H_\epsilon)$ has the following behaviour at infinity:

$$u(x,y) \approx \mathcal{E}(x,y) + a\log|z|^2,$$

  where $\mathcal{E}(x,y)$ is a quadratic polynomial, [10].

- The moduli space of the exterior Dirichlet problem associated to $(H_\epsilon)$ is either empty or a 5-dimensional differentiable manifold, [11].

The conformal representation (13) and the existence of two classical holomorphic functions with the fundamental property of mapping bijectively a once punctured bounded domain in $\mathbb{C}$ with $n$ boundary components onto $\mathbb{C}$ with $n$ vertical (resp. horizontal) slits is used in [12] to construct solutions of the elliptic Hessian one equation in the punctured plane $\mathbb{R}^2\backslash\{p_1,\cdots,p_n\}$ so that $p_j$'s are non removable isolated singularities of such solutions. These solutions are, up to equiaffine transformations, the only elliptic solutions to $(H_\epsilon)$ that are globally defined over a finitely punctured plane and so, there is an explicit one-to-one correspondence between the conformal equivalence classes of $\mathbb{C}$ with n disks removed and the space of solutions to the elliptic Hessian one equation in a finitely punctured plane with exactly $n$ non-removable isolated singularities.

### 3.2 Ribaucour's type transformations

It is well known that geometric transformations are also useful to construct new surfaces from a given one. Recently, the classical study about Ribaucour transformations developed by Bianchi, [2], has been applied to provided global description of new families of complete minimal surfaces obtained from minimal surfaces which are invariant under a one-parametric group of transformations, see [7]. For the particular case of IAM we can introduce a Ribaucour's type transformation as follows:

Let $\psi : \Sigma \longrightarrow \mathbb{C} \times \mathbb{R}$ be an IAM, we say that an IAM $\widetilde{\psi}$ is *R-associated* to $\psi$ if there is a differentiable function $g : \Sigma \longrightarrow \mathbb{R}$ such that

1. $(\psi + gN) \times \xi = (\widetilde{\psi} + g\widetilde{N}) \times \xi$.
2. $dGdF = d\widetilde{G}d\widetilde{F}$,

where $(G, F)$ and $(\widetilde{G}, \widetilde{F})$ are Weierstrass data for $\psi$ and $\widetilde{\psi}$, respectively.

In [19] we prove that $\psi$ and $\widetilde{\psi}$ are R-associated if and only if:

$$(\widetilde{F}, \widetilde{G}) = \Big(F + \frac{1}{cR}, G + R\Big),$$

where $c \in \mathbb{R} - \{0\}$ and $R$ is a holomorphic solution of the following Riccati equation:

$$dR + dG = cR^2 dF \quad \Big(\text{equivalently,} \;\; d\Big(\frac{1}{cR}\Big) + dF = \frac{1}{cR^2} dG\Big), \tag{14}$$

Using (13) and standar properties of the solutions of (14) we have that if $\widetilde{\psi}$ is *R-associated* to $\psi$, then

- $\widetilde{\psi}$ has a new end at $p_0$ if and only if $p_0$ is either a zero or a pole of $R$.
- The singular set of $\widetilde{\psi}$ is the nodal set of the harmonic function
$$\log |dG| - \log(c^2 |R|^4 |dF|).$$
- If $\psi$ is helicoidal (which means that $FG = -a^2$ for some constant $a \in \mathbb{C}$) we have that the general solution of (14) can be written as
$$R = \frac{\exp(z)}{2ac} \frac{1 + b + (1 - b)k \exp(bz)}{1 + k \exp(bz)}$$
with $k \in \mathbb{C}$, $c \in \mathbb{R} - \{0\}$ and $b = \sqrt{1 + 4a^2 c} \neq 0$.
In particular, if
$$b = \frac{n}{m} \in \mathbb{Q} - \{0, 1\}$$
is irreducible, then $\widetilde{\psi}$ is $2m\pi$-periodic in one variable, its singular set is contained in a compact set and has $2n$ complete embedded ends of revolution type, see Figure 5:

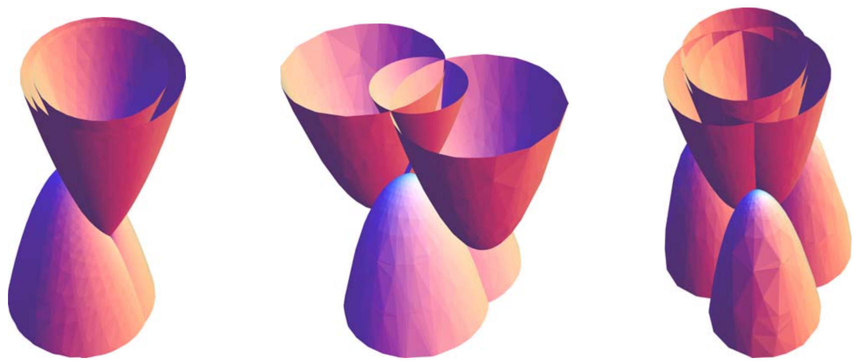

Figure 5: $a = 1,\ n = 1,\ m = 3; \quad a = 1,\ n = 2,\ m = 1; \quad a = 1\ n = 3,\ m = 2$

# 4 The hyperbolic case

The central problems of hyperbolic IAM are the same as for the elliptic case. Basically this is the problem of understanding, in a global an local way, the connection between intrinsic and extrinsic affine geometry. However in contrast to the elliptic case, the results here are in many respects far from complete. For example, we can not expect a classification result as Jörgens' Theorem. In fact,

$$u(x, y) = xy + g(x), \quad (x, y) \in \mathbb{R}^2$$

is an entire solution of the hyperbolic Hessian one equation for any real function $g$.

In the same way and concerning with continuous solutions of $(H_\epsilon)$ defined on a punctured plane we have, see [21]:

- Let $\mathcal{D}$ be a planar disk and $A : \mathcal{D} \longrightarrow \mathbb{C}_{-1}$ be a split-holomorphic function satisfying $A_z = H^2$ for some split-holomorphic function $H : \mathcal{D} \longrightarrow \mathbb{C}_{-1}$. If $z_0 \in \mathcal{D}$ is an isolated zero of $H$, then the indefinite improper affine map $\psi : \mathcal{D} \longrightarrow \mathbb{R}^3$ given, as in (12), by the split-holomorphic curve $\Phi(z) = (jz, A(z), 1)$, is well defined on $\mathcal{D}^* = \mathcal{D} - \{z_0\}$ and it has a non removable isolated singularity at $z_0$.

The above result let us to construct indefinite improper affine map $\psi : \mathbb{C}_{-1} \longrightarrow \mathbb{R}^3$ with a finite number of prescribed isolated singularities at the points $\{z_1, \cdots, z_n\}$. For this is enough to consider a split-holomorphic function $H : \mathbb{C}_{-1} \longrightarrow \mathbb{C}_{-1}$ with zeros at the points $\{z_1, \cdots, z_n\}$.

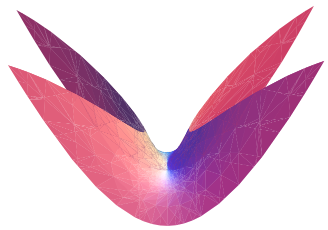

Figure 6: Entire solution on the puncture plane with $H(z) = z$

## 5 The Cauchy problem

In many situations, the classical theory of surfaces and the methods of complex analysis have allowed to study the classical Cauchy problem of the underlying associated PDE. From a geometric point of view the problem can be posed as follows:

> GEOMETRIC CAUCHY PROBLEM: Find the surfaces on a specific class of surfaces that pass through a given curve and whose tangent plane distribution along the curve is also a given distribution of oriented planes.

The problem is *well-posed* if there is a unique solution which depends continuously of the data. Most of the advances in this direction have been obtained in the elliptic case and for surfaces which admit a complex representation. In this case the surfaces are always real analytic and it is necessary to prescribed analytic initial data (one has well-posedness problems under the analytic regularity assumption but the problem is $\mathcal{C}^\infty$ ill-posed).

In contrast to the elliptic case, the problem when the underlying PDE associated is hyperbolic, is far from complete. If one considers lesser regular data $\mathcal{C}^\infty$ or even worst, many problems arise. Actually, these classes of surfaces can not be analytic and gluing procedures may create unexpected situations.

In the particular case of Improper Affine Spheres (IAS), that is improper affine maps without singularities, we want to study the following problem:

(**P**) Find all IAS containing a curve $\alpha(s)$ with a prescribed affine conormal $U(s)$ along it.

1. In the elliptic case

$$0 < h(\alpha'(s), \alpha'(s)) = -\langle \alpha'(s), U'(s) \rangle,$$

and we will consider $\{\alpha, U\}$ analytic data, [1].

2. In the hyperbolic case, $\langle \alpha', U' \rangle$ vanishes when $\alpha'$ is an asymptotic (also known as characteristic) direction. In this case we only consider regular data.

### 5.1 Non-characteristic Cauchy problem

If $\psi : \Sigma \longrightarrow \mathbb{C} \times \mathbb{R}$ is an IAS with affine normal $\xi = (0, 0, 1)$ and $\beta : I \longrightarrow \Sigma$ is a curve, then $\alpha = \psi \circ \beta$, $U = N \circ \beta$ and $\lambda = -\langle \alpha', U' \rangle$ satisfy

$$\left.\begin{array}{l} 1 = \langle \xi, U \rangle, \\ 0 = \langle \alpha', U \rangle, \\ \lambda = \langle \alpha'', U \rangle. \end{array}\right\} \tag{15}$$

A pair of (analytic ) curves $\alpha, U : I \longrightarrow \mathbb{R}^3$ is a *non-characteritic admissible pair* if it verifies the above conditions with $\lambda : I \longrightarrow \mathbb{R}^+$.

In [20] we have proved that any non-characteristic admissible pair determines a unique IAS $\psi$ containing $\alpha(I)$ with affine conormal $U$ along $\alpha$. As consequence,

- There exits a unique solution to the Cauchy problem associated to $(H_\epsilon)$.

If $[\alpha', \alpha'', \xi] \neq 0$, then, from (15), $U$ is determined by $\alpha$ and $\lambda$. In particular, any revolution IAS can be recovered with one their circles $\alpha$ and the affine metric along it. Moreover, $\alpha$ is geodesic when $\lambda = r^2$ and $\varepsilon = -1$. In general, $\alpha$ is geodesic of some IAS if and only if $[\alpha', \alpha'', \xi] = -\varepsilon[U', U'', \xi]$, with $\lambda = m \in R^+$ (see Figure 7).

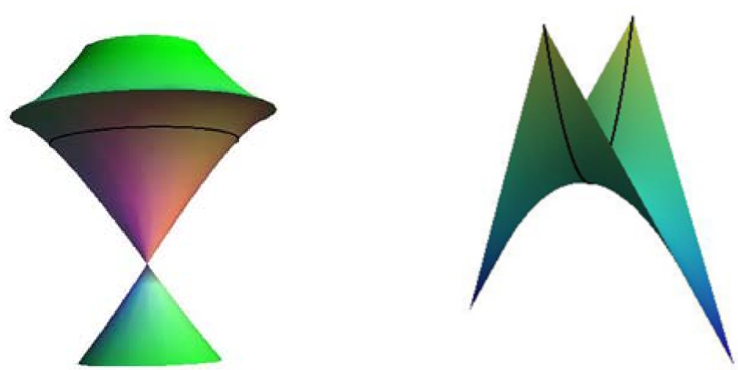

Figure 7: IAS admitting a geodesic planar curve

Any symmetry of a non-characteristic admissible pair induces a symmetry of the IAS generated by it.

### 5.2 Characteristic Cauchy problem

In the hyperbolic case we can consider a characteristic admissible pair $\{\alpha, U\}$, that is, $\langle \alpha', U' \rangle$ vanishes when $\alpha'$ is an asymptotic direction. In this situation and by using asymptotic parameters $(u, v)$, we can take,

$$N = (a + b, 1) \qquad \textit{and} \qquad \xi \times \psi = (b - a, 0)$$

for two harmonic planar curves $a(u)$ and $b(v)$ It is clear that an asymptotic curve $\psi(u, v_o)$ determines $a(u)$ and $N(u, v_o)$, but not $b(v)$. Thus, an admissible pair $\{\alpha, U\}$ generates many (hyperbolic) IAS, when $\langle \alpha', U' \rangle$ vanishes identically, in fact, the uniqueness fails when $\alpha(s) = \psi(u(s), v(s))$ is tangent to an asymptotic curve (see Figure 8).

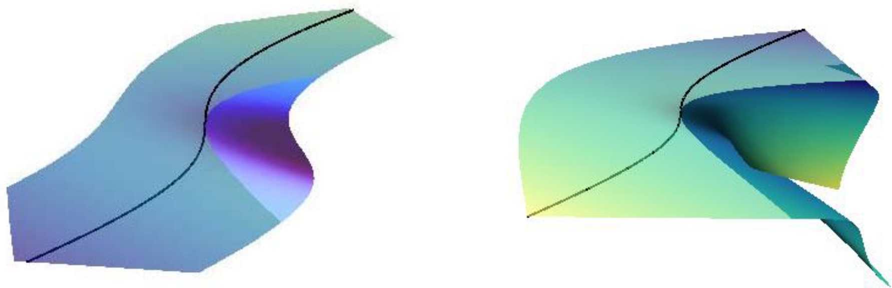

Figure 8: Asymptotic (characteristic) data

### 5.3 Prescribed singular curves

Following the same approach as in the non-characteristic case, see [1, 21], we can prove the existence of IAM with a prescribed curve of singularities (cuspidal edges or swallowtail) which determine the surface (see Figure 9).

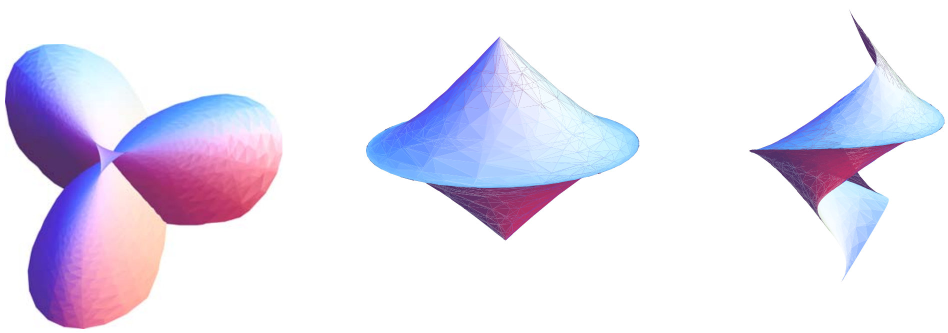

Figure 9: Three swallowtails and cuspidal edges

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