Improper affine spheres and the Hessian one equation

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Abstract

The Hessian one equation and its complex resolution provides an important tool in the study of improper affine spheres. Conversely, the properties of these surfaces play an important role in the development of geometric methods for the study of their PDEs. We review some results of this good interplay and present our extension of the classical Ribaucour transformations to this subject. In particular, we construct new solutions and families of improper affine spheres, periodic in one variable, with any even number of complete embedded ends and singular set contained in a compact set. Also, we compare the Cauchy problem for the elliptic and non-elliptic Hessian equation, with some results about their admissible singularities, mainly, isolated singularities and singular curves with cuspidal edges and swallowtails.

1 Introduction

- Affine spheres are the umbilical surfaces of the equiaffine theory in $\mathbb{R}^3$, ($SL(3,\mathbb{R})$-invariants).
- Locally, they are the graphs of the solutions of some Monge-Ampère equations.
- The study of their PDEs, with geometric methods, was initiated by Calabi, Pogorelov and Cheng-Yau.

1.1 Preliminaries

If $f:\Omega\subset\mathbb{R}^2\longrightarrow\mathbb{R}^3$ is a solution of the Hessian one equation

$$f_{xx}f_{yy}-f_{xy}^2=1,$$

then its graph $\psi=\{(x,y,f(x,y)):(x,y)\in\Omega\}$ is an improper affine sphere in $\mathbb{R}^3$.

That is, ψ has constant affine normal

$$\xi=\frac{1}{2}\Delta_h\psi=(0,0,1),$$

where

$$h=\kappa^{\frac{-1}{4}}\sigma$$

is the affine metric, (the $SL(3,\mathbb{R})$-invariant metric obtained with the Gauss curvature κ and the second fundamental form σ of ψ).

In fact, from the Hessian one equation, the affine metric

$$h=f_{xx}dx^2+f_{yy}dy^2+2f_{xy}dxdy$$

(second fundamental form of a flat surface in $\mathbb{H}^3$) and the affine conormal

$$N=\kappa^{\frac{-1}{4}}N_e=(-f_x,-f_y,1)$$

satisfy

$$1=\sqrt{det(h)}=det(\psi_x,\psi_y,\xi)=det(N_x,N_y,N).$$

Also, $h=-\langle dN,d\psi\rangle$, $\langle N,\xi\rangle=1$ and $\langle N,d\psi\rangle=0$.

Thus, for a conformal parameter z, we have $h=2\rho|dz|^2$ with

$$\rho=\langle N,\psi_{z\bar z}\rangle=-\imath[\psi_z,\psi_{\bar z},\xi]=-\imath[N_z,N_{\bar z},N]$$

and $\xi=(0,0,1)$. Hence,

$$\psi_z=\imath N\times N_z,\qquad N_{\bar z}=-\imath\xi\times\psi_{\bar z}$$

and

$$\Phi=\frac{1}{2}\Big(N+\imath\xi\times\psi\Big)$$

is a holomorphic curve, such that $N=\Phi+\overline{\Phi}$.

In particular, ψ is an affine maximal surface $\equiv N_{z\bar z}=0$ and

$$\psi_{z\bar z}=\imath N_{\bar z}\times N_z=\rho\xi.$$

1.2 Weierstrass-type Representation Formulas

Theorem 1.1 *Calabi (1988). If ψ is an affine maximal surface (improper affine sphere), then*

$$\psi=2Re\int\imath(\Phi+\overline{\Phi})\times\Phi_z dz,$$

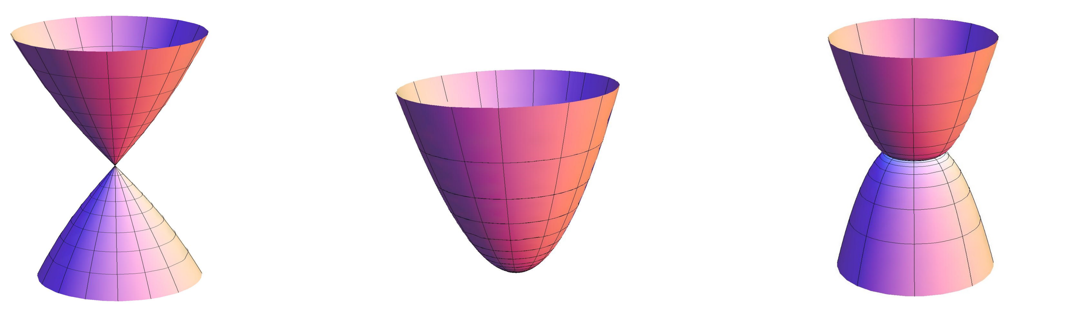
with ϕ a holomorphic (planar) curve and $-\imath[\Phi+\overline{\Phi},\Phi_z,\overline{\Phi_z}]>0$.

Theorem 1.2 *Ferrer, Martínez, M (1996). If ψ is an improper affine sphere in $\mathbb{R}^3\equiv\mathbb{C}\times\mathbb{R}$, then*

$$\psi=\Big(G+\overline{F},\frac{1}{2}|G|^2-\frac{1}{2}|F|^2+Re(GF)-2Re\int FdG\Big)$$

with F and G holomorphic functions, such that $N=(\overline{F}-G,1)$ and $h=|dG|^2-|dF|^2>0$.

Examples 1.3 *Rotational IAS: $G=z$, $F=\frac{a}{z}$, $|z|^2>|a|$.*



Isolated singularity ($a<0$), complete ($a=0$), cuspidal edge $a>0$.

1.3 Applications

- An extension of a theorem by Jörgens and a maximum principle at infinity for IAS, (Ferrer, Martínez, M 99).

$$f(x,y)=\mathcal{E}(x,y)+a\log|z|^2+0(1).$$

- The space of IAS with fixed compact boundary, (FMM 00).
- Flat surfaces in $\mathbb{H}^3$, (Gálvez, Martínez, M 00).
- Flat fronts in $\mathbb{H}^3$, with admissible singularities, (isolated singularities, cuspidal edges and swallowtails), (Kokubu, Umehara, Yamada 04).
- Improper affine maps, with admissible singularities, where $|dG|=|dF|\neq 0$, (Martínez 05). That is, $h=|dG|^2-|dF|^2\geq 0$, but

$$|d\Phi|^2=2(|dG|^2+|dF|^2)>0.$$

- The space of solutions to the Hessian one equation in the finitely punctured plane, (Gálvez, Martínez, Mira 05). Explicit construction for two singularities, with the annular Jacobi theta functions.
- The Cauchy problem for IAS and the Hessian one equation, (Aledo, Chaves, Gálvez 07). Isolated singularities are in 1-1 correspondence with planar convex analytic Jordan curves.
- Complete flat surfaces in $\mathbb{H}^3$ with two isolated singularities, (Corro, Martínez, M 10).
- Generalized Weyl problem, (Gálvez, Martínez, Teruel 14).

2 Ribaucour transformations

Definition 2.1 *Two improper affine maps $\psi,\widetilde{\psi}:\Sigma\longrightarrow\mathbb{R}^3$ are R-associated if there is a differentiable function $g:\Sigma\longrightarrow\mathbb{R}$ such that*

1. $(\psi+gN)\times\xi=(\widetilde{\psi}+g\widetilde{N})\times\xi$.
2. $dGdF=d\widetilde{G}d\widetilde{F}$.

Theorem 2.1 *(Martínez, M, Tenenblat 15). Equivalently*

$$(\widetilde{F},\widetilde{G})=\Big(F+\frac{1}{cR},G+R\Big),$$

where $c\in\mathbb{R}-\{0\}$ and R is a holomorphic solution of the Riccati equation

$$dR+dG=cR^2dF\qquad\Big(\Longleftrightarrow d\Big(\frac{1}{cR}\Big)+dF=\frac{1}{cR^2}dG\Big).$$

Consequences 2.2 *:*

- *$\widetilde{\psi}$ has a new end at p_o if and only if p_o is either a zero or a pole of R.*

  $(Q(p_o)\neq 0\Longrightarrow$ complete, embedded and of revolution type).
- *The singular set of $\widetilde{\psi}$ is the nodal set of the harmonic function $\log|dG|-\log(c^2|R|^4|dF|)$.*
- *If ψ is helicoidal, then $FG=-a^2$ and*

$$R=\frac{\exp(z)}{2ac}\frac{1+b+(1-b)k\exp(bz)}{1+k\exp(bz)}$$

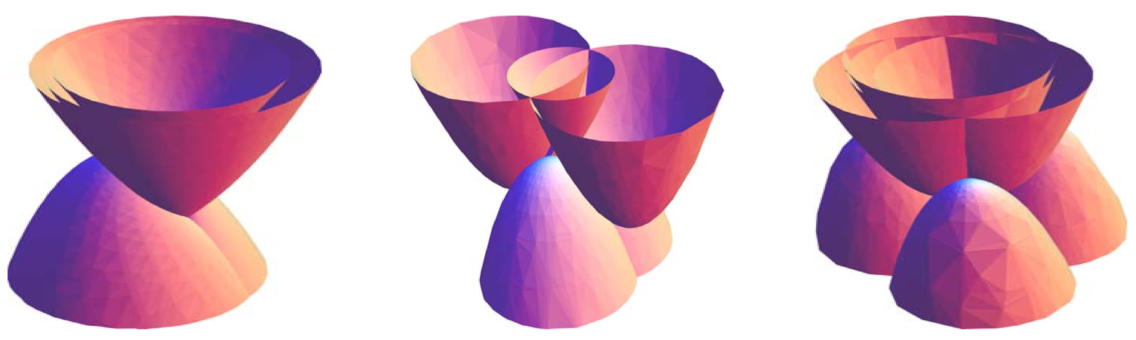
with $a,k\in\mathbb{C}$, $c\in\mathbb{R}-\{0\}$ and $b=\sqrt{1+4a^2c}\neq 0$.

In particular, if

$$b=\frac{n}{m}\in\mathbb{Q}-\{0,1\}$$

is irreducible, then $\widetilde{\psi}$ is $2m\pi$-periodic in one variable and has $2n$ complete embedded ends of revolution type.

Examples 2.3 *R-helicoidal*



$(n,m)=(1,3),\qquad(2,1)\qquad$ *and* $\qquad(3,2).$

The singular set is contained in a compact set.

3 Cauchy problem

Björling-type problem 3.1 *Find all (definite and indefinite) IAS containing a curve α in $\mathbb{R}^3$ with a prescribed affine conormal U along it.*

1. Note that h definite implies

$$0<h(\alpha'(s),\alpha'(s))=-\langle\alpha'(s),U'(s)\rangle,$$

with $\{\alpha,U\}$ analytic curves, (Aledo, Chaves, Gálvez 07).

2. In the indefinite case, $\langle\alpha',U'\rangle$ vanishes when α' is an asymptotic (also known as characteristic) direction.

3.1 Non-characteristic Cauchy problem

- First, we exclude asymptotic (characteristic) data.
- We consider

$$f_{xx}f_{yy}-f_{xy}^2=\varepsilon=\pm 1$$

and the ε-complex numbers (Inoguchi, Toda 04)

$$\mathbb{C}_\varepsilon=\{z=s+jt\ :\ s,t\in\mathbb{R},\ j^2=-\varepsilon,\ j1=1j\}.$$

Thus

$$\Phi=\frac{1}{2}\Big(N+j\xi\times\psi\Big)=\frac{1}{2}\Big(-f_x-jy,-f_y+jx,1\Big)$$

is a holomorphic curve and

$$\psi=2Re\int j(\Phi+\overline{\Phi})\times\Phi_z dz.$$

- If $\psi:\Sigma\longrightarrow\mathbb{R}^3$ is an IAS with $\xi=(0,0,1)$ and $\beta:I\longrightarrow\Sigma$ is a curve, then $\alpha=\psi\circ\beta$, $U=N\circ\beta$ and $\lambda=-\langle\alpha',U'\rangle$ satisfy

$$\begin{cases}1=\langle\xi,U\rangle,\\ 0=\langle\alpha',U\rangle,\\ \lambda=\langle\alpha'',U\rangle.\end{cases}$$

Definition 3.1 *A pair of (analytic) curves $\alpha,U:I\longrightarrow\mathbb{R}^3$ is a non-characteristic admissible pair if verify the above conditions with $\lambda:I\longrightarrow\mathbb{R}^+$.*

Theorem 3.1 *(M 14).*

1. *If $\{\alpha,U\}$ is a non-characteristic admissible pair, then there exits a unique IAS ψ containing $\alpha(I)$ with affine conormal U along α.*
2. *There exits a unique solution to the Cauchy problem*

$$\begin{cases}f_{xx}f_{yy}-f_{xy}^2=\varepsilon,\\ f(x,0)=a(x),\qquad a''(x)>0,\\ f_y(x,0)=b(x).\end{cases}$$

Consequences 3.2 *:*

1. *If $[\alpha',\alpha'',\xi]\neq 0$, then α and λ determine*

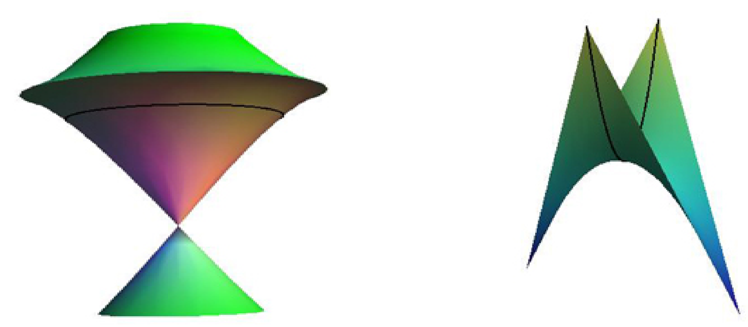
$$U=\frac{\alpha'\times(\alpha''-\lambda\xi)}{[\alpha',\alpha'',\xi]}\qquad and\qquad\psi.$$

2. *In particular, any revolution IAS can be recovered with one their circles α and the affine metric along it. Moreover, α is geodesic when $\lambda=r^2$ and $\varepsilon=-1$.*
3. *In general, α is geodesic of some IAS if and only if*

$$[\alpha',\alpha'',\xi]=-\varepsilon[U',U'',\xi],$$

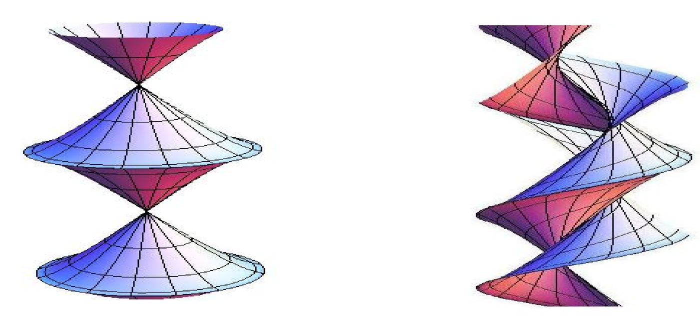
with $\lambda=m\in R^+$.

Examples 3.3 *IAS admitting a geodesic planar curve*



Any symmetry of a non-characteristic admissible pair induces a symmetry of the IAS generated by it.

Examples 3.4 *IAS which are invariant under a one-parametric group of equiaffine transformations*



Isolated singularities and cuspidal edges.

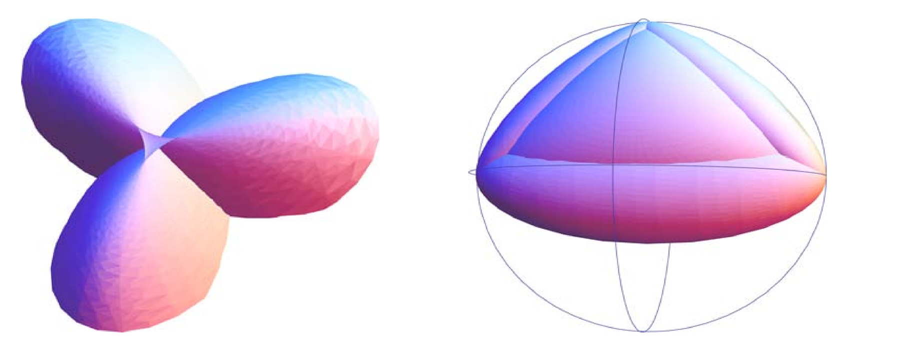
Where are the swallowtails?

3.2 Prescribed singular curves

Theorem 3.5 *If $[\alpha',\alpha'',\alpha''']^2\neq-\varepsilon[\alpha',\alpha'',\xi]^4\neq 0$, then there exists a unique improper affine map ψ with α as (cuspidal edge) singular curve.*

Theorem 3.6 *If $[\alpha',\alpha'',\alpha''']^2\neq-\varepsilon[\alpha',\alpha'',\xi]^4\neq 0$ on $I-\{0\}$ and 0 is a zero of α', $\alpha'\times\alpha''$, $[\alpha',\alpha'',\xi]$ and $[\alpha',\alpha'',\alpha''']$ of order 1, 2, 2 and 3 respectively, then $\alpha(0)$ is a swallowtail of ψ.*

Examples 3.7 *Three swallowtails*



Improper affine map, flat front (Martínez, M 14).

3.3 Characteristic Cauchy problem

- The uniqueness fails when $\alpha(s)=\psi(u(s),v(s))$ is tangent to an asymptotic curve.
- Two solutions agree on a domain which contains $\alpha(I)$ except its characteristic points without sign (Martínez, M 15).