# A class of surfaces with flat equiaffine metric and their characterisation

A. Martínez, F. Milán, and L. Vrancken

**Abstract**

We study locally strongly convex surfaces with complete flat affine metric. We show how we can characterize all known examples by a tensorial condition involving the covariant derivative of the shape operator and the gradient of the Pick invariant.

**Key words**: *flat equiaffine metric, equiaffine differential geometry.*
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## 1 Introduction

In this paper we study nondegenerate affine surfaces $M^2$ into $\mathbb{R}^3$, equipped with its standard affine connection $D$. It is well known that on such a surface there exists a canonical transversal vector field $\xi$, which is called the affine normal. Using this tranversal it is possible to introduce an affine connection $\nabla$ and an nondegenerate bilinear form $h$ which are respectively called the induced affine connection and the affine metric.

Note that in general $\nabla$ is not the Levi Civita connection for the affine metric $h$. The well known theorem of Jörgens states that the only positive definite immersion of $\mathbb{R}^2, D$, where $D$ is the usual flat connection in $\mathbb{R}^3$ is the elliptic paraboloid. Here we will study a similar question. We study affine immersions for which the affine metric is a positive definite complete flat metric. In Section 3, we will show how to we can characterize and construct such immersions starting from solutions $g$ of one complex valued differential equation. We also show that for the known complete examples, provided the frame is set up correctly, the solutions are conformal to each other (with real conformality factor).

In Section 4, we then introduce a tensorial condition on a flat affine surface. We note that all previously constructed examples, including the non-complete ones satisfy that condition. We also show that in the complete case, it characterises completely the know examples, i. e. we show

**Theorem 1** *Let $M$ be an affine complete locally strongly convex surface with flat equiaffine metric satisfying*

$$\mathrm{trace}_h(\nabla S) + \mu\,\mathrm{grad}_h(J) = 0, \tag{1}$$

*for some constant $\mu$. Then $M$ is affine congruent to either the elliptic paraboloid, the surface $xyz = 1$ or the surface*

$$x(u,v) = \tfrac{-1}{\sqrt{3}}((\cosh(3u))^{\frac{1}{3}}\cosh(\sqrt{3}v), (\cosh(3u))^{\frac{1}{3}}\sinh(\sqrt{3}v), \int_0^u (\cosh(3t))^{-\frac{2}{3}}dt). \tag{2}$$

# 2 Preliminaries

In this paper, we follow the structural approach as introduced by K. Nomizu [5]. For a nondegenerate surface in the affine space $\mathbb{R}^3$ equipped with its usual flat connection $D$ and volume form given by the determinant function, it is well known how to introduce a canonical transversal vector field $\xi$ called the Blaschke normal. Using $\xi$, by the formulas of Gauss and Weingarten, we obtain the induced connection $\nabla$, the affine metric $h$ and the shape operator S by:

$$\begin{aligned} D_X Y &= \nabla_X Y + h(X,Y)\xi \\ D_X \xi &= -SX. \end{aligned}$$

The *affine mean curvature* $H$ is defined by $H = \frac{1}{2}\operatorname{trace} S$. The surface is called affine maximal if and only if $H$ vanishes identical. It is said that $M$ is an affine sphere if and only $S = HI$. If $H \neq 0$, $M$ is called a proper affine sphere, otherwise $M$ is called an improper affine sphere.

Note that, in general, the induced connection $\nabla$ is not the Levi Civita connection for the affine metric $h$. Indeed, the classical theorem of Berwald states that this happens only for quadrics. Therefore, on a generic affine surface, the difference tensor $K$ defined by

$$K(X,Y) = K_X Y = \nabla_X Y - \widehat{\nabla}_X Y,$$

where $\widehat{\nabla}$ is the Levi Civita connection of $h$ is a non vanishing symmetric tensor. The apolarity condition states that for every $X$, $\operatorname{trace} K_X$ vanishes. The difference tensor is related to the cubic form by

$$(\nabla h)(X,Y,Z) = -2h(K(X,Y),Z).$$

The Codazzi equation for $h$ states that the cubic form is symmetric in $X$, $Y$ and $Z$. From [5], we also recall the following equations:

$$\begin{aligned}
&R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY && \text{(Gauss } \nabla) \\
&(\nabla_X S)Y = (\nabla_Y S)X && \text{(Codazzi } \nabla) \\
&h(SX,Y) = h(X,SY) && \text{(Ricci)} \\
&\widehat{R}(X,Y)Z = \tfrac{1}{2}(h(Y,Z)SX - h(X,Z)SY + h(SY,Z)X \\
&\qquad\qquad - h(SX,Z)Y) - [K_X,K_Y]Z && \text{(Gauss } \widehat{\nabla}) \\
&(\widehat{\nabla}K)(X,Y,Z) - (\widehat{\nabla}K)(Y,X,Z) = \tfrac{1}{2}(h(Y,Z)SX - h(X,Z)SY \\
&\qquad\qquad\qquad\qquad - h(SY,Z)X + h(SX,Z)Y) && \text{(Codazzi } K)
\end{aligned}$$

Given an equiaffine immersion, the conormal map $N$, which is a map into the dual space, is defined by $N(\xi) = 1$ and $N(X) = 0$, where $X$ is a tangent vector field. It is well known, see [4] that this map can be considered as a centroaffine immersion. Identifying the dual space of $\mathbb{R}^3$ with $\mathbb{R}^3$, we have that the basic invariants are given by:

$$D_X N_\star(Y) = N_\star(\nabla^\star_X Y) + h(X, SY)N.$$

where $\nabla^\star$ is the conjugate connection. The Lelieuvre formula then states the following:

**Theorem 2** *Given a triple of functions $N = (N^1, N^2, N^3)$ defined on a simply connected domain $\Omega$ such that $|NN_uN_v| \neq 0$ and such that $\triangle U \parallel U$, where*

$$\triangle = |NN_uN_v|^{-1}(\tfrac{\partial^2}{\partial u^2} + \tfrac{\partial^2}{\partial v^2}).$$

*Then there exist an affine immersion $F : \Omega \to \mathbb{R}^3$, with affine conormal map $U$ and as Blaschke metric $h = |NN_uN_v|(du^2 + dv^2)$. Moreover, $F$ can be explicitly recovered by*

$$F = \int (N \times N_v)du - (N \times N_u)dv.$$

Note that also the converse is true on a locally strongly convex surface (see [3]).

Now we will assume that $M$ is a locally strongly convex surface with affine metric $h$. Let $\{u, v\}$ be locally defined isothermal coordinates on $M$, i.e. there exists a function $E$ such that

$$\begin{aligned} h(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial u}) &= h(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial v}) = E \\ h(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial v}) &= 0. \end{aligned}$$

We then introduce a complex coordinate $z = u + iv$. It immediately follows that

$$\begin{aligned} h(\partial, \bar\partial) &= \tfrac{1}{2}E, \\ h(\bar\partial, \bar\partial) &= h(\partial, \partial) = 0. \end{aligned}$$

where $\partial = \frac{\partial}{\partial z}$ and $\bar\partial = \frac{\partial}{\partial \bar z}$. From the apolarity condition it then follows that we can introduce complex valued functions $U$ and $B$ such that

$$\begin{aligned} K(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) &= U\tfrac{\partial}{\partial \bar z}, \\ K(\tfrac{\partial}{\partial \bar z}, \tfrac{\partial}{\partial \bar z}) &= \bar U\tfrac{\partial}{\partial z}, \\ K(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial \bar z}) &= 0, \\ S\tfrac{\partial}{\partial z} &= H\tfrac{\partial}{\partial z} + B\tfrac{\partial}{\partial \bar z} \\ S\tfrac{\partial}{\partial \bar z} &= H\tfrac{\partial}{\partial \bar z} + \bar B\tfrac{\partial}{\partial z}. \end{aligned}$$

In the special case that $M$ has flat affine metric, i.e. in the special case that we can choose isothermal coordinates $u$ and $v$ such that $E = 1$. It follows straightforwardly from the Gauss, Codazzi and Ricci equations that the previously defined functions satisfy:

$$
\begin{aligned}
H &= -2U\bar{U} \\
B &= -2\bar{\partial}U \\
B_{\bar{z}} &= H_z + U\bar{B}.
\end{aligned}
$$

Note that the above equations imply that the function $U$ satisfies:

$$U_{\bar{z}\bar{z}} = U_z\bar{U} + 2U(\bar{U})_z. \tag{3}$$

Conversely, given a solution of the above differential equation on a simply connected domain $\Omega$, taking for $h$ the standard metric and defining $K$ and $S$ as above it follows from the fundamental existence and uniqueness theorem, see [1] that there exists an affine immersion with $h$ as (flat) induced metric, $S$ as affine shape operator and $K$ as difference tensor.

It is also clear that in order for a surface to have a complete flat affine metric, we must have that the function $U$ solving the above differential equation, must be defined on the whole of $\mathbb{C}$.

# 3 A class of examples and their geometric properties

Throughout this section, we will assume that $M$ is a locally strongly convex surface in $\mathbb{R}^3$. In case that $M$ is a complete non compact affine surface with affine conormal $N = (N_1, N_2, N_3)$, it is well known that $M$ must be also Euclidean complete ([6]). In that case, we may assume that $N_3 > 0$, i.e. we may assume that $M$ is a vertical graph. We take a conformal parameter as introduced in the previous section. As we have chosen isothermal coordinates, the fact that $\xi$ is the affine normal implies that $|\xi x_u x_v| = E$, where $x$ denotes the immersion of $M$ into $\mathbb{R}^3$. Using the natural identification between the dual space of $\mathbb{R}^3$ and $\mathbb{R}^3$ given by $a \wedge b(c) = |cab|$, it follows immediately that

$$
\begin{aligned}
N_z &= i\xi \wedge x_z, && (4)\\
N_{\bar{z}} &= -i\xi \wedge x_{\bar{z}} && (5)\\
N &= \tfrac{1}{E}x_u \wedge x_v = -\tfrac{2}{E}ix_z \wedge x_{\bar{z}}. && (6)
\end{aligned}
$$

Deriving (4) with respect to $\bar{z}$ it then follows that

$$N_{z\bar{z}} = i\xi_{\bar{z}} \wedge x_z = iHx_z \wedge x_{\bar{z}} = -\frac{HE}{2}N. \tag{7}$$

As by our assumption, $N_3$ is a positive function, we can introduce a complex valued function $g$ by $N_1 + iN_2 = gN_3$. Using (7) it then follows that

$$\begin{aligned}
-\tfrac{HE}{2} gN_3 &= -\tfrac{HE}{2}(N_1 + iN_2) \\
&= (N_1 + iN_2)_{z\bar{z}} \\
&= (gN_3)_{z\bar{z}} \\
&= g_{z\bar{z}}N_3 + g_{\bar{z}}N_{3z} + g_z N_{3\bar{z}} + gN_{3z\bar{z}} \\
&= g_{z\bar{z}}N_3 + g_{\bar{z}}N_{3z} + g_z N_{3\bar{z}} - \tfrac{HE}{2} gN_3.
\end{aligned}$$

Hence we have

$$g_{z\bar{z}}N_3 + g_{\bar{z}}N_{3z} + g_z N_{3\bar{z}} = 0 \tag{8}$$
$$\bar{g}_{z\bar{z}}N_3 + \bar{g}_{\bar{z}}N_{3z} + \bar{g}_z N_{3\bar{z}} = 0. \tag{9}$$

Using the same technique, we also find that

$$\frac{iE}{2} = |NN_zN_{\bar{z}}| = \tfrac{1}{2} iN_3^3 D, \tag{10}$$

where $D = |g_z|^2 - |g_{\bar{z}}|^2$. Hence $D$ is a nonvanishing function and we have that $DN_3^3 = E$. If we now solve the equations (8) and (9) for $N_3$, we find that

$$\frac{N_{3z}}{N_3} = \frac{-g_z\bar{g}_{z\bar{z}} + \bar{g}_z + g_{z\bar{z}}}{D}. \tag{11}$$

As $DN_3^3 = E$ it follows that $3 \log N_3 = \log E - logD$. Hence we have

$$3\frac{N_{3z}}{N_3} = \frac{E_z}{E} - \frac{D_z}{D} = \frac{-3g_z\bar{g}_{z\bar{z}} + 3\bar{g}_z g_{z\bar{z}}}{D}.$$

From this it follows that we can express the conformal factor of the metric also in terms of the function $g$ as:

$$(\log E)_z = \frac{-2g_z\bar{g}_{z\bar{z}} + 2\bar{g}_z g_{z\bar{z}} + g_{zz}\bar{g}_{\bar{z}} - g_{\bar{z}}\bar{g}_{zz}}{|g_z| - |^2 g_{\bar{z}}|^2} \tag{12}$$

As $(\log E)_{z\bar{z}}$ is a real function, we must have that

$$Im\left(\tfrac{\partial}{\partial\bar{z}}\left(\frac{-2g_z\bar{g}_{z\bar{z}} + 2\bar{g}_z g_{z\bar{z}} + g_{zz}\bar{g}_{\bar{z}} - g_{\bar{z}}\bar{g}_{zz}}{|g_z| - |^2 g_{\bar{z}}|^2}\right)\right) = 0. \tag{13}$$

From this we can formulate the following theorem:

**Theorem 3** *Let $S$ be a simply connected Riemann surface and $g : S \to \mathbb{C}$ a local diffeomorphism. Then, there exists an affine conformal vertical graph $x : S \to \mathbb{R}^3$ with conormal map $N = (gN_3, N_3)$ if and only if $g$ satisfies (13). In this case, we have that $N_3 = (E/D)^{\frac{1}{3}}$ and the immersion can be recovered from the Lelieuvre's formula by using only $g$.*

**Proof:** We have already seen that the condition on $g$ is a necessary condition. To show that it is sufficient, we proceed as follows. Assume that $g$ satisfies (13). Then, we can define a positive function $E$ by

$$(\log E)_z = \frac{-2g_z\bar{g}_{z\bar{z}} + 2\bar{g}_z g_{z\bar{z}} + g_{zz}\bar{g}_{\bar{z}} - g_{\bar{z}}\bar{g}_{zz}}{|g_z| - |^2 g_{\bar{z}}|^2}$$

We define $N_3 = (E/D)^{\frac{1}{3}}$ and put $N_1 + iN_2 = gN_3$. A straightforward computation then shows that $N_{z\bar{z}}$ is parallel with $N$. Applying the Lelieuvre's formula then completes the proof. ∎

In case that $M$ has flat affine metric, there exists a global complex coordinate such that $E = 1$ and therefore by the previous theorem and (12), we have the following corollary:

**Corollary 1** *There exists a complete flat affine immersion which is a vertical graph with affine conormal $N$ if and only if there exists a local diffeomorphism $g : \mathbb{C} \to \mathbb{C}$ such that*

$$-2g_z(\bar{g})_{z\bar{z}} + 2\bar{g}_z g_{z\bar{z}} + g_{zz}\bar{g}_{\bar{z}} - g_{\bar{z}}\bar{g}_{zz} = 0, \tag{14}$$

*and $N$ is then recovered as $(N_3 g, N_3)$ with $N_3^3(|g_z|^2 - |g_{\bar{z}}|^2) = 1$.*

In the flat case, also the invariant $U$ can be easily recovered from the local diffeomorphism $g$ in the following way. We take $E = 1$ and then we have that

$$\begin{aligned} N_{zz} &= i((-Hx_z - Bx_{\bar{z}}) \wedge x_z + i(\xi \wedge Ux_{\bar{z}}) \\ &= -Bix_{\bar{z}} \wedge x_z + Ui\xi \wedge x_{\bar{z}} \\ &= -UN_{\bar{z}} - \tfrac{B}{2}N. \end{aligned}$$

As on one hand we have that

$$|N_{zz}NN_z| = -U|N_{\bar{z}}NN_z| = -U|NN_zN_{\bar{z}}| = -\tfrac{i}{2}U,$$

and on the other hand, we have that

$$|N_{zz}NN_z| = \tfrac{1}{2}i \begin{pmatrix} (N_1 + iN_2)_{zz} & (N_1 + iN_2) & (N_1 + iN_2)_z \\ (N_1 - iN_2)_{zz} & (N_1 - iN_2) & (N_1 - iN_2)_z \\ N_{3zz} & N_3 & N_{3z} \end{pmatrix} = -\tfrac{1}{2}iN_3^3(g_{zz}\bar{g}_z - \bar{g}_{zz}g_z).$$

Hence we deduce that

$$U = \frac{(g_{zz}\bar{g}_z - \bar{g}_{zz}g_z)}{(g_z\bar{g}_{\bar{z}} - g_{\bar{z}}\bar{g}_z)} \tag{15}$$

We now proceed with some examples. The surface $x_1x_2x_3 = 1$ which is a flat homogeneous affine surface is besides the paraboloid the easiest affine surface.

**Example 1** We consider the following parametrization:

$$x(u,v) = \frac{1}{\sqrt{3}2^{\frac{1}{3}}}(e^{-u}e^{-\sqrt{3}v}, e^{-u}e^{\sqrt{3}v}, e^{2u}),$$

of the surface $x_1x_2x_3 = \frac{1}{6\sqrt{3}}$. As it follows by a straightforward computation that

$$|x_u x_v x_{uu}| = |x_u x_v x_{vv}| = 1$$

it follows that $z = u + iv$ is an isothermal coordinate with $E = 1$. In particular the surface has flat affine metric. The conormal immersion is given by

$$N(u,v) = -\frac{1}{\sqrt{3}2^{\frac{1}{3}}}(e^{u}e^{\sqrt{3}v}, e^{u}e^{-\sqrt{3}v}, -e^{-2u})$$

As the third component is never vanishing, we also see that we can write the surface as a vertical graph. Introducing the local diffeomorphism, which for this example we will denote by $g_0$, as before, we find that

$$g_0 = e^{3u}(e^{\sqrt{3}v} + ie^{-\sqrt{3}v}),$$

from which it follows that $U = 1$.

**Example 2** From [2], we recall the following example:

$$x(u,v) = \frac{-1}{\sqrt{3}}((\cosh(3u))^{\frac{1}{3}}\cosh(\sqrt{3}v), (\cosh(3u))^{\frac{1}{3}}\sinh(\sqrt{3}v), \int_0^u (\cosh(3t))^{-\frac{2}{3}}dt). \quad (16)$$

As it follows again by a straightforward computation that

$$|x_u x_v x_{uu}| = |x_u x_v x_{vv}| = 1$$

we see that $z = u + iv$, $E = 1$ and the surface has flat affine metric. As it is defined for all values of $u$ and $v$, the metric is complete. The conormal immersion is given by

$$N(u,v) = \frac{1}{\sqrt{3}}(\cosh(3u))^{-\frac{1}{3}}(-\cosh(\sqrt{3}v), \sinh(\sqrt{3}v), \sinh(3u))$$

From the above equation, we see that we can write the surface as a global graph in the $x_1$-direction (and not in the vertical direction). However, locally we can still write it as a graph in the vertical direction. Doing so, and introducing the complex diffeomorphism in the same way as before, we find that

$$g = -\frac{1}{\sinh(3u)}(\cosh(\sqrt{3}v) - i\sinh(\sqrt{3}v)).$$

Comparing $g$ with $g_0$ in the previous example, we remark that

$$\frac{g}{g_0} = \frac{i-1}{e^{6u}-1}. \quad (17)$$

**Example 3** The other positive definite example from [2], can be parametrized by:

$$x(u,v) = \tfrac{1}{\sqrt{3}}((\cos(3u))^{\frac{1}{3}}\cos(\sqrt{3}v), (\cos(3u))^{\frac{1}{3}}\sin(\sqrt{3}v), \int_0^u (\cos(3t))^{-\frac{2}{3}}dt). \tag{18}$$

As it follows again by a straightforward computation that

$$|x_u x_v x_{uu}| = |x_u x_v x_{vv}| = 1$$

we see that $z = u + iv$, $E = 1$ and the surface has flat affine metric. As $x_u$ and $x_v$ are not well defined everywhere, it follows that the surface is not complete. The conormal immersion is given by

$$N(u,v) = -\tfrac{1}{\sqrt{3}}(\cos(3u))^{-\frac{1}{3}}(\cos(\sqrt{3}v), \sin(\sqrt{3}v), \sin(3u))$$

Locally we can still write it as a graph in the vertical direction. Doing so, and introducing the complex diffeomorphism in the same way as before, we find that

$$g = \frac{1}{\sin(3u)}e^{i\sqrt{3}v}.$$

Similarly, to Example 2, it is also possible to consider the following example:

**Example 4** We define

$$x(u,v) = \tfrac{-1}{\sqrt{3}}((\sinh(3u))^{\frac{1}{3}}\cosh(\sqrt{3}v), (\sinh(3u))^{\frac{1}{3}}\sinh(\sqrt{3}v), \int_0^u (\sinh(3t))^{-\frac{2}{3}}dt). \tag{19}$$

As it follows again by a straightforward computation that

$$|x_u x_v x_{uu}| = |x_u x_v x_{vv}| = 1$$

we see that $z = u+iv$, $E = 1$ and the surface has flat affine metric. As the tangent space is not defined for all values of $u$ and $v$, the metric is not complete. The conormal immersion is given by

$$N(u,v) = \tfrac{1}{\sqrt{3}}(\sinh(3u))^{-\frac{1}{3}}(-\cosh(\sqrt{3}v), \sinh(\sqrt{3}v), \cosh(3u))$$

We can still write it locally as a graph in the vertical direction. Doing so, and introducing the complex diffeomorphism in the same way as before, we find that

$$g = -\frac{1}{\cosh(3u)}(\cosh(\sqrt{3}v) - i\sinh(\sqrt{3}v)).$$

Comparing $g$ with $g_0$ in the previous example, we remark that

$$\frac{g}{g_0} = \frac{i-1}{e^{6u}+1}. \tag{20}$$

In view of the Example 2 and Example 4, it makes sense to try to determine the surfaces with flat affine metric, which can be locally written as a vertical graph, and for which the corresponding local diffeomorphism $g$ satisfies:

$$g = \frac{1}{\phi} g_0 e^{i\varphi}, \tag{21}$$

where $\varphi$ is a constant and $\phi$ is a real valued positive function. Note that the value of $\varphi$ has no geometric meaning and simply corresponds to a rotation in the horizontal plane.

Substituting (21) into (14) we find that the function $\phi$ has to satisfy the following system of differential equations:

$$3\phi_{\bar{z}} + \phi_{zz} - 2\phi_{z\bar{z}} = 0. \tag{22}$$

Rewriting (22) as two real differential equations, we find that

$$\begin{aligned} &6\phi_u - \phi_{uu} - 3\phi_{vv} = 0 \\ &3\phi_v = \phi_{uv}. \end{aligned}$$

Solving the second equation implies that there exists a function $\tilde{f}$, which only depends on $u$ such that

$$\phi_u = 3\phi + \tilde{f}(u).$$

Substituting this expression into the first equation, we find that

$$\begin{aligned} 3\phi_{vv} &= 18\phi + 6\tilde{f}(u) - 3\phi_u - \tilde{f}'(u) \\ &= 9\phi + 3\tilde{f}(u) - \tilde{f}'(u). \end{aligned}$$

Therefore, deriving once more with respect to $v$ yields:

$$\phi_{vvv} = 3\phi_v.$$

Hence there exists functions $A_1$, $A_2$ and $A_3$ depending only on the variable u, such that

$$\phi = A_1(u) + A_2(u) e^{\sqrt{3}v} + A_3(u) e^{\sqrt{3}v}.$$

Substituting this expression again in our system of differential equations it follows that

$$\begin{aligned} A_2' &= 3A_2 \\ A_3' &= 3A_3 \\ A_1'' &= 6A_1'. \end{aligned}$$

Hence there exists constants $p$, $q$, $r$ and $s$ such that

$$\phi = p e^{6u} - q + r e^{3u + \sqrt{3}v} - s e^{3u - \sqrt{3}v}.$$

Taking $\phi$ as above, it follows that

$$D = \frac{6\sqrt{3}\, e^{6u-\sqrt{3}v}\,(e^{6u}\, p + q)}{\left(e^{6u-\sqrt{3}v}\, p - e^{-\sqrt{3}v}\, q + e^{3u}\,\left(r - e^{-2\sqrt{3}v}\, s\right)\right)^3}$$

$$U = -1 + \frac{2\, q}{e^{6u}\, p + q}.$$

Note that as $U$ does not depend on the value of $r$ and $s$, it follows from the existence and uniqueness theorem, see [1] that also the immersions for different values of $r$ and $s$ coincide. We therefore without any loss of generality may assume that $r = s = 0$. In the same way, as $U$ depends only of the quotient of $p$ and $q$, we can introduce a new constant $\lambda$ by $\lambda = \frac{p}{q}$. Hence, denoting by $U_\lambda$ the invariant associated to the immersion with the parameter $\lambda$ we get

$$U_\lambda = -1 + \frac{2}{e^{6u}\,\lambda + 1}$$

Note that by a translation of the $u$-coordinate it is sufficient to consider the cases that $\lambda = 0, 1, -1$. It is now clear that the first case corresponds to Example 1, the second case to Example 2, whereas the third case corresponds to the non complete example given by Example 4.

# 4 Characterisation results

Throughout this section we shall assume that $M$ is a locally strongly convex surface with flat equiaffine metric. If $M$ is an affine sphere, it is well known, see [7], [8] it is well known that $M$ is congruent with an open part of a paraboloid or an open part of the surface $xyz = 0$. Both of those examples satisfy trivially the condition that

$$\mathrm{trace}_h(\nabla S) + \mu\, \mathrm{grad}_h(J) = 0, \tag{23}$$

for any constant $\mu \in \mathbb{R}$. This means that for any local orthonormal basis $\{e_1, e_2\}$, we have that

$$(\nabla_{e_1} S e_1) + (\nabla_{e_2} S e_2) + \mu e_1(J) e_1 + e_2(J) e_2 = 0, \tag{24}$$

In the previous section we have seen that our new class of examples also satisfies the above condition for the special value of $\mu = 3$.

We will show that, assuming that $M$ is not a paraboloid or the previously mentioned surface $xyz = 1$, that this property characterises completely our new example. First, we show:

**Theorem 4** *Let $M$ be a surface with flat induced metric and without umbilical points. Assume that there exists a constant $\mu$ such that*

$$(\nabla_{e_1} S e_1) + (\nabla_{e_2} S e_2) + \mu e_1(J) e_1 + e_2(J) e_2 = 0,$$

*then $\mu = 3$.*

**Proof:** As $M$ is a flat surface without umbilical points, we have that there exists a globally defined orthonormal basis $\{e_1, e_2\}$ such that

$$\begin{aligned} Se_1 &= \lambda_1 e_1 \\ Se_2 &= \lambda_2 e_2. \end{aligned}$$

We now introduce local functions $a, b$ and $\alpha$, $\beta$ by

$$\widehat{\nabla}_{e_1} e_1 = a e_2 \qquad \widehat{\nabla}_{e_2} e_2 = b e_1$$

and

$$K(e_1, e_1) = \alpha e_1 + \beta e_2,$$

The symmetry properties of $K$ together with the apolarity condition then imply that

$$K(e_1, e_2) = \beta e_1 - \alpha e_2, \qquad K(e_2, e_2) = -\alpha e_1 - \beta e_2.$$

As $M$ has flat affine metric, the following relation between the Pick invariant and the mean curvature follows immediately:

$$\lambda_1 + \lambda_2 = 2H = -2J = -4(\alpha^2 + \beta^2). \tag{25}$$

It follows from the above equation, that we can introduce functions $r$, $s$ and $\theta$ by

$$\begin{aligned} \lambda_1 &= -2r^2 - \tfrac{1}{2}s, \\ \lambda_2 &= -2r^2 + \tfrac{1}{2}s, \\ \alpha &= r\cos\theta, \\ \beta &= r\sin\theta. \end{aligned}$$

Note that as we assumed that $M$ has no umbilical points, and therefore the cubic form cannot vanish on an open dense subset, we must have that $r$ is a non vanishing function on an open dense subset of $M$. In the remainder of the proof, we will restrict ourselves to this open and dense subset. As $M$ has flat affine metric, it follows that

$$e_2(a) + e_1(b) - a^2 - b^2 = 0. \tag{26}$$

Rewriting (24), using the introduced functions, we find that

$$4r(\mu - 1)e_1(r) - \tfrac{1}{2}e_1(s) + s(b - r\cos\theta) = 0 \tag{27}$$

$$4r(\mu - 1)e_2(r) + \tfrac{1}{2}e_2(s) - s(a + r\sin\theta) = 0. \tag{28}$$

We now compute the Codazzi equations

$$(\widehat{\nabla}K)(e_2, e_1, e_1) - (\widehat{\nabla}K)(e_1, e_2, e_1) = \tfrac{1}{2}s$$

and $(\nabla_{e_1}S)e_2 = (\nabla_{e_2})Se_1$. This gives

$$4re_2(r) + \tfrac{1}{2}e_2(s) + rs\sin\theta - as$$
$$e_1(s) - 8re_1(r) - 2s(b + r\cos\theta)$$
$$-\tfrac{1}{2}s - 3br\cos\theta + \cos\theta e_1(r) + r\cos\theta e_2(\theta) - 3ar\sin\theta + e_2(r)\sin\theta - r\sin\theta e_1(\theta)$$
$$3ar\cos\theta - \cos\theta e_2(r) + r\cos\theta e_1(\theta) - 3br\sin\theta + \sin\theta e_1(r) + r\sin\theta e_2(theta).$$

If $\mu = 2$, it follows immediately from comparing the above equations that $rs\cos\theta = rs\sin\theta = 0$. Hence $s = 0$ and $M$ is umbilical which is a contradiction. Therefore, we may assume that $\mu \neq 2$. In that case, solving the obtained equations, together with (27) and (28) for the derivatives of the functions $r$, $s$ and $\theta$, we obtain after a lengthy but straightforward computation that

$$e_1(\theta) = -3a + \frac{s(\mu-3)\sin\theta}{2r(\mu-2)} \tag{29}$$
$$e_2(\theta) = 3b + \frac{s(\mu-3)\cos\theta}{2r(\mu-2)} \tag{30}$$
$$e_1(r) = \frac{s\cos\theta}{2(\mu-2)} \tag{31}$$
$$e_2(r) = \frac{s\sin\theta}{2(\mu-2)} \tag{32}$$
$$e_1(s) = 2sb + \frac{2s\mu r}{\mu-2}\cos\theta \tag{33}$$
$$e_2(s) = 2sa - \frac{2s\mu r}{\mu-2}\sin\theta. \tag{34}$$

Computing now $[e_1, e_2](r)$ and $[e_1, e_2]\theta$ in two different ways, making also use of (26), we find that

$$-2a(\mu-2)\cos\theta + 2(b(\mu-2) + r\mu\cos\theta)sin\theta = 0 \tag{35}$$
$$(\mu-3)(-4+\mu)s + 8br(\mu-2)\cos\theta + 4r^2\mu\cos 2\theta - 16ar\sin\theta + 8ar\mu\sin\theta. \tag{36}$$

Using now that $\mu \neq 3$, we find by solving (35) and (36) for $a$ and $b$ that

$$a = \frac{(-s(\mu-4) + 4r^2\mu)\sin\theta}{8r(\mu-2)}$$
$$b = -\frac{(s(\mu-4) + 4r^2\mu)\cos\theta}{8r(\mu-2)}$$

Substituting these expressions in (26) it follows that $0 = \frac{r^2\mu^2}{(\mu-2)^2}$ which is a contradiction. ∎

In the remainder of this section, we will now assume that $\mu = 3$. Note that up to (35) all equations in the proof of the previous theorem remain valid. From (35), it follows that we can introduce a function $\kappa$ such that

$$a := \kappa \sin\theta + 3r\cos^2\theta\sin\theta$$
$$b := \kappa \cos\theta - 3r\sin^2\theta\cos\theta.$$

It then follows from (26) and by computing $[e_1, e_2]s$ in two different ways that

$$e_1(\kappa) = \tfrac{1}{4}\cos\theta(-8\kappa^2 - 48\kappa r + 3s + 3(4\kappa r - 6r^2 - s)\cos(2\theta) + 18r^2\cos(4\theta))$$
$$e_2(\kappa) = \tfrac{1}{4}\sin\theta(-8\kappa^2 + 48\kappa r - 3s + 3(4\kappa r + 6r^2 - s)\cos(2\theta) + 18r^2\cos(4\theta)).$$

It is now elementary to verify that computing $[e_1, e_2]\kappa$ in two different ways does not produce any new equations.

We will now proceed with inducing flat coordinates in the following way. We put:

$$f_1 := \cos(\tfrac{1}{3}\theta)e_1 + \sin(\tfrac{1}{3}\theta)e_2$$
$$f_2 := -\sin(\tfrac{1}{3}\theta)e_1 + \cos(\tfrac{1}{3}\theta)e_2.$$

It follows immediately from the previous differential equations that $\widehat{\nabla}_{f_i} f_j = 0$, for $i, j \in \{1, 2\}$. This means that there exist flat coordinates $u$ and $v$ such that $f_1 = \frac{\partial}{\partial u}$ and $f_2 = \frac{\partial}{\partial v}$. Using the coordinates $u$ and $v$ the differential equations for the functions $r$, $s$, $\kappa$ and $\theta$ can be rewritten in terms of $z = u + iv$ as:

$$\partial r = \frac{s}{4}\, e^{-\frac{2i}{3}\theta}$$
$$\partial s = \frac{s}{2}\left(2\,\kappa + 6\,r\,\cos 2\,\theta + 3\,\mathrm{i}\,r\,\sin 2\,\theta\right) e^{-\frac{2\,\mathrm{i}}{3}\,\theta}$$
$$\partial\theta = \frac{1}{2}\left(-3\,\mathrm{i}\,\kappa - 9\,r\,\cos\theta\,\sin\theta\right)\, e^{-\frac{2\,\mathrm{i}}{3}\,\theta}$$
$$\partial\kappa = \frac{1}{16}\left(18r^2 - 2\,e^{4\,\mathrm{i}\,\theta}\,\left(8\,g^2 + 9\,r^2\right) + 3\,e^{2\,\mathrm{i}\,\theta}\,\left(4\,g\,r - s\right) + e^{6\,\mathrm{i}\,\theta}\,\left(-84\,g\,r + 3\,s\right)\right)\, e^{-\frac{14\,\mathrm{i}}{3}\,\theta},$$

where $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. Note that $h(\partial, \bar{\partial}) = \frac{1}{2}$. Therefore, if we introduce the complex invariants $U$ and $B$, such that

$$K(\partial, \partial) = U\bar{\partial}$$
$$S\partial = H\partial + B\bar{\partial},$$

we get that

$$U = r \tag{37}$$
$$B = -\tfrac{1}{2} e^{\frac{2\,\mathrm{i}}{3}\,\theta} s. \tag{38}$$

Note that the function $U$ is real. It is also easy to verify that the above functions $U$ and $B$ satisfy indeed the system of differential equations for a surface with flat affine metric derived in the previous section. In particular, we have that

$$\begin{aligned} H &= -2U\bar{U} = -2UU \\ B &= -2\bar{\partial}U \end{aligned}$$

and the function $U$ satisfies the differential equation

$$\bar{\partial}\bar{\partial}U = (\partial U)\bar{U} + 2U\partial(\bar{U}) = 3U\partial U. \tag{39}$$

Note that given $U$, we can easily recover $B$ from the previous equations. Rewriting this differential equation for $U$ in real form, using the $u$ and $v$ derivatives, we obtain from (39) that

$$\tfrac{\partial^2 U}{\partial u^2} - \tfrac{\partial^2 U}{\partial v^2} = 6U\tfrac{\partial U}{\partial u} \tag{40}$$

$$\tfrac{\partial^2 U}{\partial u \partial v} = -3U\tfrac{\partial U}{\partial v}. \tag{41}$$

First we will consider a special case, namely we will assume that the function $U$ depends only on the variable $u$. In that case (41) is trivially satisfied, whereas (40) implies that there exists a constant $A_1$ such that

$$U' = 3(U^2 + A_1)$$

In order to solve the above system, we consider 3 subcases. First, we assume that $A_1 = 0$. In that case, we get that there exist a constant $A_2$ such that

$$U = -\tfrac{1}{3}\frac{1}{u + A_2}.$$

It is clear that by a translation of the $u$-coordinate, we may assume that $A_2 = 0$. As $U$ is not defined on the whole of $\mathbb{R}^2$, it is clear that the corresponding surface is not complete. Next, we assume that $A_1 = -\nu_1^2$ for some positive number $\nu_1$. Clearly, $U = \pm\nu_1$ are solution which correspond to Example 1. Assume therefore that $U^2 - \nu^2 \neq 0$. In that case, we get that

$$\frac{U'}{U - \nu_1} - \frac{U'}{U + \nu_1} = 6\nu_1$$

Integrating the above expression shows that there exists a constant $\nu_2$ such that

$$|\frac{U - \nu_1}{U + \nu_1}| = e^{6\nu_1 u + \nu_2}.$$

Again, by applying a translation, we may assume that $\nu_2 = 0$. If $\frac{U-\nu_1}{U+\nu_1}$ is positive, we get that

$$U = -\lambda\tfrac{\cosh(3\lambda u)}{\sinh 3\lambda u},$$

whereas if $\frac{U-\nu_1}{U+\nu_1}$ is negative, we get that

$$U = \lambda \tfrac{\sinh(3\lambda u)}{\cosh 3\lambda u},$$

Note that in the first case, we have that the surface is locally congruent (after applying an homothety, i.e. after assuming that the mean curvature at a well chosen initial point equals a well chosen value) to Example 4, whereas in the second case, we get that our surface is affine congruent with Example 2. Finally, in case that $A_1 = \nu_2$. We proceed in the same way. In that case, again after applying a translation, we find that

$$U = \lambda \tfrac{\sin(3\lambda u)}{\cos 3\lambda u},$$

This yields, up to affine congruence Example 3.

Let us now consider the general case, i.e. the function $U$ which solves (40) and (41) depends on both the variables $u$ and $v$. In this case, we can proceed as follows. Introducing an auxiliary function $f$, we can write the system of differential equations as

$$\begin{aligned} \tfrac{\partial^2 U}{\partial u^2} &= -3U\tfrac{\partial U}{\partial u} + f \\ \tfrac{\partial^2 U}{\partial v^2} &= -9U\tfrac{\partial U}{\partial u} + f \\ \tfrac{\partial^2 U}{\partial u \partial v} &= -3U\tfrac{\partial U}{\partial v}. \end{aligned}$$

Note that last equation shows that the function $f = \frac{\partial U}{\partial u} + \frac{3}{2}U^2$ depends only on the variable $u$. Moreover from the first equation it then follows that $g' = f$ (and therefore also $f$ depends only on the variable $u$). From the integrability condition from the above differential equation it then follows that

$$g' = g - 3(\tfrac{\partial U}{\partial v})^2 + 9(\tfrac{\partial U}{\partial u})^2.$$

Substituting $\frac{\partial U}{\partial u} = -\frac{3}{2}U^2 + f$ into this equation, we find that

$$(\tfrac{\partial U}{\partial v})^2 = 2Uf' + \tfrac{27}{4}U^4 - 9U^2 f + 3f^2 - \tfrac{1}{3}f''.$$

Deriving oncemore with respect to $u$, we find that the function $f$ satisfies the following differential equation:

$$f''' = 24ff'.$$

Integrating this two times we find that there exists constants $c$ and $d$ such that

$$(f')^2 = 8f^3 + 2cf + d. \tag{42}$$

Conversely, it is easy to verify that given a solution for (42), we look at the following system of ordinary differential equations:

$$\begin{aligned} \tfrac{\partial U}{\partial u} &= -\tfrac{3}{2}U^2 + f, \\ \tfrac{\partial U}{\partial v} &= \sqrt{2Uf' + \tfrac{27}{4}U^4 - 9U^2 f + 3f^2 - \tfrac{1}{3}f''}, \end{aligned}$$

It is then easy to verify that the integrability conditions of this system are verified and that a solution has to satisfy (40) and (41).

We conclude by calculating to what correspond in the previously considered special case, the integration constants $c$ and $d$. Note that in the special case

$$U' = 3(U^2 + A_1),$$

and therefore, we have that

$$\begin{aligned} f &= \tfrac{9}{2}U^2 + 3A_2 \\ f' &= 27U^3 + 27UA_2 \\ f'' &= 243U^4 + 324U^2A_2 + 81A_2^2. \end{aligned}$$

From this it follows that $c = -27A_2^2$ and $d = -54A_2^3$.

**Theorem 5** *Let $M$ be an affine complete locally strongly convex surface with flat equiaffine metric satisfying*

$$\operatorname{trace}_h(\nabla S) + \mu \operatorname{grad}_h(J) = 0, \tag{43}$$

*for some constant $\mu$. Then $M$ is affine congruent to either the elliptic paraboloid, the surface $xyz = 1$ or the surface*

$$x(u,v) = \tfrac{-1}{\sqrt{3}}((\cosh(3u))^{\frac{1}{3}}\cosh(\sqrt{3}v), (\cosh(3u))^{\frac{1}{3}}\sinh(\sqrt{3}v), \int_0^u (\cosh(3t))^{-\frac{2}{3}}dt). \tag{44}$$

We will divide the proof in several propositions. We assume that $M$ is not affine congruent with either $xyz = 1$ or the elliptic paraboloid. Note that it is easy to see that the previously introduced function $U$ is globally defined and that the surfaces $xyz = 1$ and the elliptic paraboloid correspond to taking a constant solution for the $U$. We also have seen that the only complete solution which does not depend on the variable $v$ was given by

$$U = \lambda \tfrac{\sinh(3\lambda u)}{\cosh 3\lambda u},$$

This yields, up to affine congruence Example 2.

We now look at solutions for

$$(f')^2 = p(f) = 8f^3 + 2cf + d. \tag{45}$$

Note that, as $U$ is globally defined, $f$ is a globally defined function of the variable $u$. We now look at the equation of degree 3 given by $p(x) = 8x^3 + 2cx + d$ we denote by $r_m$ (resp. $r_M$) the minimum (resp. the maximum) zero of $p(x)$, then we have

**Lemma 1** *$f$ is bounded with*

1. *$r_m \leq f \leq 0$, if $c > 0$, or*
2. *$r_m \leq f \leq \sqrt{\frac{-c}{12}}$, if $c \leq 0$.*

*Moreover* $r_m \le f \le r_M$.

**Proof:** From (**??**), $p(f) \ge 0$ and $f$ is bounded from below by $r_m$, that is, $r_m \le f$.

Using (**??**), $f$ is bounded from above (see [CY], Theorem 8). In fact, if $c > 0$, then $f'' \ge 12f^2$ and $f \le 0$; if $c \le 0$, then, $f \le \sqrt{-c/12}$.

Now we'll see that $f \le r_M$. Otherwise, there exists $u_0 \in \mathbb{R}$ s.t. $f(u_0) > r_M$. Since $p(f)$ has no zeros in $]r_M, \infty[$, we conclude either $f' > 0$ in $[u_0, \infty[$ or $f' < 0$ in $]-\infty, u_0]$ which contradicts that $f$ is bounded (see Figure 1) ∎

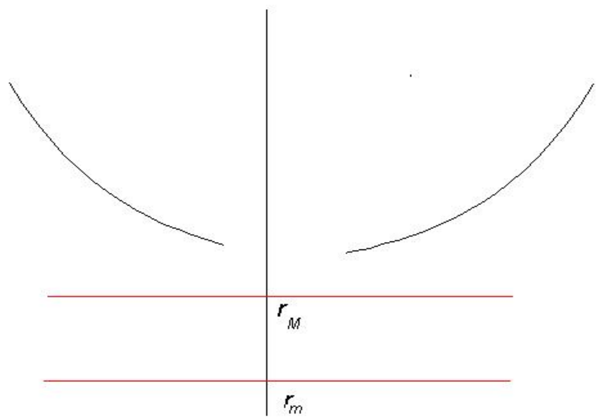


*Figura 1*

**Corollary 2** *If* $r_m = r_M$, *then* $f$ *is constant.*

Now and on we'll assume $p(x)$ has three zeros $r_m \le r \le r_M$. In this case, as the sum of the roots equals zero, it is clear that

$$\begin{aligned} c &= 4(r_m r + r_m r_M + r r_M) \\ &= -4(r_m^2 + r r_m + r^2) \\ &= -4(r_m^2 + r_M r_m + r_M^2) \qquad\qquad = -4(r_M^2 + r r_M + r_M^2) \end{aligned}$$

Hence $c \le 0$. Moreover from Lemma 1 and because $p(f) \ge 0$ we have (see Figure 2)

**Corollary 3** *If* $r_m = r < r_M$, *then* $f$ *is constant.*

**Corollary 4** *Either* $r_m \le f \le r$ *or* $f = r_M$.

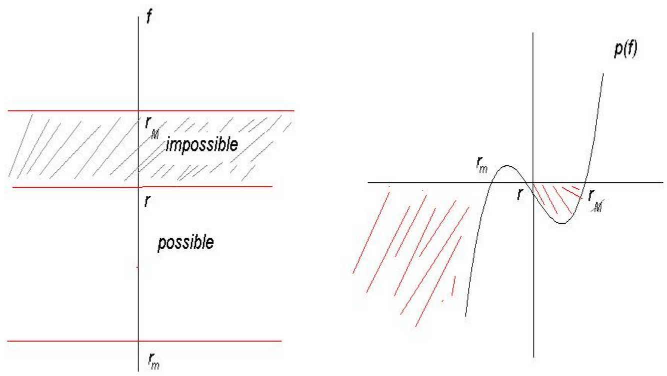


Figura 2

Therefore, we can consider in the following that $r_m \leq f \leq r$.

**Proposition 1** *We have* $-\sqrt{\frac{-c}{12}} > r_m$.

**Proof:** Note that as the sum of the roots equals zero, $r_m \leq 0$. First, we consider the case that $r_m \leq r \leq 0$. In that case, we have that

$$r_m^2 \geq \tfrac{1}{3}(r_m^2 + r r_m + r^2) = -\tfrac{c}{12},$$

where equality holds if and only if $r = r_m$. In the case that $r_m \leq 0 < r \leq r_M$, we have that

$$r_m^2 > \tfrac{1}{3}(r^2 + r r_M + r_M^2) = -\tfrac{c}{12}.$$

∎

**Proposition 2** *If $f$ is nonconstant, Then either $12r^2 + c \neq 0$ and there exist $u_0, u_1 \in \mathbb{R}$ with $f(u_0) = r_m$ and $f(u_1) = r$ or there is $b \in \mathbb{R}$ s.t. up to translation*

$$f(u) = \frac{b^2}{2}\tanh^2(bu) - b^2/3.$$

**Proof:** First, we will show, by reasoning as in Proposition 1, that $-\sqrt{\frac{-c}{12}} < r,$. Indeed, if $r > 0$, there is nothing to prove, whereas, if $r < 0$, we have

$$r^2 \leq \tfrac{1}{3}(r_m^2 + r r_m + r^2) = -\tfrac{c}{12},$$

where equality holds if and only if $r = r_m$ (in which case $f$ would be constant by a previous corollary).

Therefore, if $f$ is non constant we must have that

$$r_m < -\sqrt{\frac{-c}{12}} < r.$$

This implies that $f$ can not be asymptotic to $f = r_m$ otherwise $f''(r_m) = 12r_m^2 + c = 0$ which is impossible (see Figure 4). Consequently, if $f$ is nonconstant, $f$ must admit a local minimum and therefore there exists a point $u_0$ such $f(u_0) = r_m$. Next either there exists also $u_1 \in \mathbb{R}$ with $f(u_1) = r$ or $f$ has a minimum at $u_0$ and $f$ decrease in $[u_0 - \varepsilon, u_0[$ and increase in $]u_0, u_0 + \varepsilon]$, that is to the right of $u_0$, $f$ increase as a convex function until $f = -\sqrt{\frac{-c}{12}}$ where change to concave and follow increasing as concave function until it is asymptotic to $f = r$ at $+\infty$. Similarly, to the left of $u_0$ we go in a convex way until $f = -\sqrt{\frac{-c}{12}}$ change to concave and it is asymptotic to $f = r$ at $-\infty$ (see Figure 5).

From the above discussion, $f''(r) = 12r^2 + c = 0$ and $f'(r) = 8r^3 + 2c\,r + d = 0$. Hence we must have that $r = \sqrt{\frac{-c}{12}}$ and as

$$r^2 \geq \tfrac{1}{3}(r^2 + rr_M + r_M^2) = -\tfrac{c}{12}$$

with equality if and only if $r = r_M$, we know that there exists a constant $b$ such that $r_m = -b^2/3$ and $r = r_M = b^2/6$ for some $b \in \mathbb{R}$. In this situation we know that $f(u) = \frac{b^2}{2}\tanh^2(bu) - b^2/3$, upto a translation of the $u$ coordinate, is the solution of (**??**), thus if we translate it and apply the uniqueness of solution in $r_m < f < r$ we conclude the result. ∎

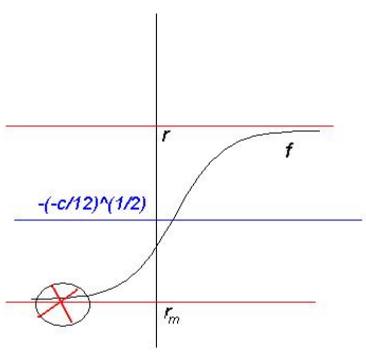


Figure 4

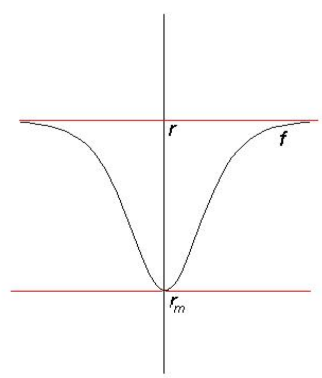


Figure 5

**Theorem 6** *Let $U$ be a global solution of*

$$\frac{\partial U}{\partial u} = -\frac{3}{2}U^2 + f,$$
$$(\frac{\partial U}{\partial v})^2 = 2Uf' + \frac{27}{4}U^4 - 9U^2 f + 3f^2 - 1/3f''.$$

*Then either $U$ is constant, $U = \lambda\ \tanh(3\lambda(\frac{1}{2}u \pm \frac{\sqrt{3}}{2}v)$, $\lambda \neq 0$, or $U = \lambda \tanh(3\lambda u)$.*

**Proof:** If $f$ is constant, then we can easily integrate the first equation of the above system of PDE. A global solution can only be obtained if $f = -\frac{3}{2}\lambda^2$. In this case, the global solution of it can be written as $U(u,v) = \lambda \tanh(\frac{3}{2}\lambda(u \pm h(v))$, where $h(v)$ is a function depending only on $v$. Substituting this into

$$(\frac{\partial U}{\partial v})^2 = 2Uf' + \frac{27}{4}U^4 - 9U^2 f + 3f^2 - \frac{1}{3}f''.$$

it follows that $g'(v)^2 = 3$. Hence, we obtain that

$$U = \lambda\ \tanh(3\lambda(\frac{1}{2}u \pm \frac{\sqrt{3}}{2}v).$$

If $f$ is non constant, then from Proposition 2 only have two possibilities:

1. $f(u) = \frac{b^2}{2}\tanh^2(bu) - b^2/3$ and in this case

$$(g')^2 = p_2(g) = \frac{27}{4}g^4 + .....$$
$$g'' = p_3(g) = \frac{27}{2}g^3 - 9gf + f'$$

   where $g(v) = U(u,v)$. Then, from the last equation $g$ must be bounded above from the maximum zero of $p_3$. But, in this case, $p_2$ has only a zero $g = -\frac{1}{3}b\ \tanh[b\ u]$ which has multiplicity 2. Consequently as in Lemma 1, $g$ must be constant and $U$ is only a function of $u$, we know the solutions in this case.

2. or $-\sqrt{\frac{-c}{12}} < r < \sqrt{\frac{-c}{12}}$ and there exist $u_0$ and $u_1$, such that $f(u_0) = r_m$ and $f(u_1) = r$. In this case we know from (**??**) and Lemma 1 that $f''(u_1) = 12r^2 + c < 0$. Then by taking $g_1(v) = U(u_1, v)$, we have that $g_1$ is a global solution of

$$(g_1')^2 = p_4(g_1) = \frac{27}{4}g_1^4 - 9g_1^2 r - r^2 - \frac{1}{3}c,$$
$$g_1'' = \frac{27}{2}g_1^3 - 9g_1 r.$$

   As above, from the last equation, by using the result of [CY], $g_1$ must be bounded from above. Using that $12r^2 + c < 0$, it follows that the polynomial $p_4(x) = \frac{27}{4}x^4 - 9x^2 r - r^2 - \frac{1}{3}c$ has no real roots. Consequently as in Lemma 1, $g_1$ must be constant. However as $p_4(x)$ does not have any real roots a contradiction follows.

∎

It is clear that $U = \lambda\ \tanh(3\lambda(\frac{1}{2}u \pm \frac{\sqrt{3}}{2}v)$, gives the same surface as $U = \lambda \tanh(3\lambda u)$, as our frame and our coordinates were only determined uniquely up to rotation by $\frac{2\pi}{3}$.

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*F. Milan, A. Martinez, Departemento de Geometria y Topologia, Facultad de Ciencias , Universidad de Granada, Granada, Spain. E-mail: amartine@ugr.es*

*L. Vrancken, LAMATH, ISTV2, Université de Valenciennes, Campus du Mont Houy, 59313 Valenciennes Cedex 9, France. E-mail: luc.vrancken@univ-valenciennes.fr*