Flat Surfaces in $\mathbb{L}^4$

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Abstract

In this paper we give a global conformal representation for flat surfaces with flat normal bundle in the standard flat Lorentzian space form $\mathbb{L}^4$. Particularly, flat surfaces in the hyperbolic 3-space, the de Sitter 3-space, the null cone and other numerous examples are described.

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1 Introduction

This paper grew out of the authors's investigations into theory of surfaces which could be represented in terms of holomorphic data on Riemann surfaces. Famous examples are given by minimal surfaces in the euclidean 3-space, (see [6]), surfaces of mean curvature one and flat surfaces in the hyperbolic 3-space (see [2], [5]). In all these cases, the discovery of new examples and the recent progress in the study of global properties has been motivated by the existence of a conformal representation like the Weierstrass representation.

When the codimension is two, the problem seems to be more difficult than one could expect. Here, we shall consider surfaces in the standard flat Lorentzian space form $\mathbb{L}^4$.

Our main goal in this paper is to study (globally) flat surfaces Σ in $\mathbb{L}^4$ with flat normal bundle. This particular kind of surfaces have also been studied in a local way (see [4]). We show there exists a canonical conformal structure on Σ such that its Gauss map (see Definition 1) is conformal. Then we prove a conformal representation result that lets us recover our immersion by using a pair (f, ω) consisting of a holomorphic function f and a holomorphic 1-form ω, $\omega \neq 0$ everywhere and a closed 1-form θ.

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By using our representation we describe numerous examples of complete flat surfaces with flat normal bundle and characterize those who lies in some hyperquadrics of $\mathbb{L}^4$.

2 Main Result

Let $\mathbb{L}^4$ be the Minkowski 4-space endowed with linear coordinates (x_0, x_1, x_2, x_3) and the scalar product, $\langle .,. \rangle$ given by the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2$, and denote by $\mathbb{N}^3_+$ its positive null cone. Then $\mathbb{N}^3_+/\mathbb{R}^+$ inherits a natural conformal structure and it can be regarded as the boundary $\mathbb{S}^2_\infty$ of the hyperbolic 3-space $\mathbb{H}^3$ in $\mathbb{L}^4$.
We also consider $\mathbb{L}^4$ identified with the space of 2×2 Hermitian matrices, Herm(2), in the standard way (see [2], [5]). Under this identification, one has $\langle m, m\rangle = -\det(m)$, for all $m \in$ Herm(2), and the complex Lie group $\mathbf{SL}(2,\mathbb{C})$ of 2×2 complex matrices with determinant 1 acts naturally on $\mathbb{L}^4$ by the representation

$$g \cdot m = gmg^*,$$

where $g \in \mathbf{SL}(2,\mathbb{C})$, $g^* = {}^t\overline{g}$ and $m \in$ Herm(2). Consequently, $\mathbf{SL}(2,\mathbb{C})$ preserves the scalar product and orientations.
The space $\mathbb{N}^3_+$ is seen as the space of positive semi-definite 2×2 Hermitian matrices of determinant 0 and its elements can be written as $a\,{}^t\overline{a}$, where ${}^ta = (a_1, a_2)$ is a non-zero vector in $\mathbb{C}^2$ uniquely defined up to multiplication by an unimodular complex number. The map $a\,{}^t\overline{a} \longrightarrow [(a_1, a_2)] \in \mathbb{C}\mathbf{P}^1$ becomes the quotient map of $\mathbb{N}^3_+$ on $\mathbb{S}^2_\infty$ and identifies $\mathbb{S}^2_\infty$ with $\mathbb{C}\mathbf{P}^1$. So the natural action of $\mathbf{SL}(2,\mathbb{C})$ on $\mathbb{S}^2_\infty$ is the action of $\mathbf{SL}(2,\mathbb{C})$ on $\mathbb{C}\mathbf{P}^1$ by Möbius transformations.

Now, we denote by Σ a simply connected surface and $\psi : \Sigma \longrightarrow \mathbb{L}^4$ an immersion with flat induced metric $ds^2 =< d\psi, d\psi >$. Then, there exists an isothermal coordinate immersion $x + iy : \Sigma \longrightarrow \mathbb{C}$ such that

$$ds^2 = dx^2 + dy^2. \tag{1}$$

Lemma 1 *Assume that ψ has flat normal bundle (e.g. $R^\perp \equiv 0$). Then there exist an oriented orthonormal frame $\{\xi, \widetilde{\xi}\}$ of $T^\perp\Sigma$ and functions $\phi, \widetilde{\phi} : \Sigma \longrightarrow \mathbb{R}$, such that the following structure equations hold,*

$$\left\{\begin{array}{l} \psi_{xx} = \phi_{xx}\xi + \widetilde{\phi}_{xx}\widetilde{\xi}, \\ \psi_{xy} = \phi_{xy}\xi + \widetilde{\phi}_{xy}\widetilde{\xi}, \\ \psi_{yy} = \phi_{yy}\xi + \widetilde{\phi}_{yy}\widetilde{\xi}, \end{array}\right. \qquad \left\{\begin{array}{l} \xi_x = -\phi_{xx}\psi_x - \phi_{xy}\psi_y, \\ \xi_y = -\phi_{xy}\psi_x - \phi_{yy}\psi_y, \end{array}\right. \qquad \left\{\begin{array}{l} \widetilde{\xi}_x = \widetilde{\phi}_{xx}\psi_x + \widetilde{\phi}_{xy}\psi_y, \\ \widetilde{\xi}_y = \widetilde{\phi}_{xy}\psi_x + \widetilde{\phi}_{yy}\psi_y. \end{array}\right. \tag{2}$$

Moreover the integrability conditions for this system are

$$det(\nabla^2\phi) = det(\nabla^2\widetilde{\phi}), \qquad \widetilde{\phi}_{xy}(\phi_{xx} - \phi_{yy}) = \phi_{xy}(\widetilde{\phi}_{xx} - \widetilde{\phi}_{yy}), \tag{3}$$

where by $(\nabla^2\cdot)$, $(\cdot)_x$ and $(\cdot)_y$ we shall denote the hessian matrix and the usual partial derivatives with respect to x and y, respectively.

Proof: We can choose an orthonormal frame $\{\eta,\widetilde{\eta}\}$ for the Lorentzian normal bundle which satisfies

$$<\eta,\eta>=-<\widetilde{\eta},\widetilde{\eta}>=1, \qquad <\eta,\widetilde{\eta}>=0.$$

Then from (1), the structure equations of the immersion are given by

$$(4)\quad \left\{\begin{array}{l}\psi_{xx}=E\eta+\widetilde{E}\widetilde{\eta},\\ \psi_{xy}=F\eta+\widetilde{F}\widetilde{\eta},\\ \psi_{yy}=G\eta+\widetilde{G}\widetilde{\eta},\end{array}\right. \quad \left\{\begin{array}{l}\eta_x=-E\psi_x-F\psi_y-A\widetilde{\eta},\\ \eta_y=-F\psi_x-G\psi_y-B\widetilde{\eta},\end{array}\right. \quad \left\{\begin{array}{l}\widetilde{\eta}_x=\widetilde{E}\psi_x+\widetilde{F}\psi_y-A\eta,\\ \widetilde{\eta}_y=\widetilde{F}\psi_x+\widetilde{G}\psi_y-B\eta,\end{array}\right.$$

for some smooth functions E, F, G, $\widetilde{E}$, $\widetilde{F}$, $\widetilde{G}$, A and B on Σ. From the Ricci and Codazzi-Mainardi's equations and using that $R^\perp\equiv 0$, we have

$$(5)\qquad B_x-A_y=\widetilde{F}(E-G)-F(\widetilde{E}-\widetilde{G})=0.$$

Thus, there exists $\mu:\Sigma\longrightarrow\mathbb{R}$ such that $d\mu=Adx+Bdy$. The lemma follows from (5) and the integrability conditions of the structure equations after rewritting (4) with the new orthonormal frame $\{\xi,\widetilde{\xi}\}$, where

$$\xi=\cosh(\mu)\eta+\sinh(\mu)\widetilde{\eta}, \qquad \widetilde{\xi}=\sinh(\mu)\eta+\cosh(\mu)\widetilde{\eta}.$$

□

Definition 1 *For every oriented orthonormal frame $\{\eta,\widetilde{\eta}\}$ of $T^\perp\Sigma$, the map $\mathcal{G}=(\mathcal{G}^+,\mathcal{G}^-):\Sigma\longrightarrow\mathbb{S}^2_\infty\times\mathbb{S}^2_\infty$, where $\mathcal{G}^+=[\eta+\widetilde{\eta}]$ and $\mathcal{G}^-=[\eta-\widetilde{\eta}]$ is well defined and it is called Gauss map of the immersion.*

Lemma 2 *If $\mathcal{G}$ is regular (e.g. $d\mathcal{G}\neq 0$ everywhere), then there exists a canonical conformal structure on Σ such that $\mathcal{G}$, $\mathcal{G}^+$ and $\mathcal{G}^-$ are conformal maps.*

Proof: Consider the complex functions, $z,\widetilde{z},\varphi:\Sigma\longrightarrow\mathbb{C}$ given by

$$(6)\qquad \begin{array}{rcl} z &=& u+iv=(\phi+\widetilde{\phi})_x+i(\phi+\widetilde{\phi})_y,\\ \widetilde{z} &=& \tilde{u}+i\tilde{v}=(-\phi+\widetilde{\phi})_x+i(\phi-\widetilde{\phi})_y,\\ \varphi &=& \varphi_1+i\varphi_2=\widetilde{\phi}_{xx}\phi_{yy}-\phi_{xx}\widetilde{\phi}_{yy}+2i(\widetilde{\phi}_{yy}\phi_{xy}-\phi_{yy}\widetilde{\phi}_{xy}).\end{array}$$

Then, from (3) and (6), we have

$$(7)\qquad \Delta^+d\widetilde{z}-\varphi dz=0, \qquad \Delta^-dz+\overline{\varphi}d\widetilde{z}=0, \qquad \Delta^+\Delta^-+\varphi\overline{\varphi}=0,$$

where $\Delta^+ = det(\nabla^2(\phi + \widetilde{\phi}))$ and $\Delta^- = det(\nabla^2(\phi - \widetilde{\phi}))$.
Let denote by U (respectively, $\tilde{U}$), the set of points where z (respectively, $\widetilde{z}$) is a local diffeomorphism. Combining again (3) and (6) we obtain

$$\frac{\partial \tilde{u}}{\partial u} = \frac{\partial \tilde{v}}{\partial v} = \frac{\varphi_1}{\Delta^+}, \qquad -\frac{\partial \tilde{u}}{\partial v} = \frac{\partial \tilde{v}}{\partial u} = \frac{\varphi_2}{\Delta^+}$$

on $U \cap \tilde{U}$. Since $\mathcal{G}$ is regular, $U \cup \tilde{U} = \Sigma$ and hence $\{(U, z), (\tilde{U}, \widetilde{z})\}$ induces a canonical conformal structure on Σ. Now, from (7), there exists a meromorphic function $f : \Sigma \longrightarrow \mathbb{C}$, such that,

$$d\widetilde{z} = f dz, \qquad \text{with} \quad f = \varphi/\Delta^+ = -\Delta^-/\,\overline{\varphi}. \tag{8}$$

Moreover, from (2), (6) and (8), we have

$$\begin{aligned} 2(\widetilde{\xi} - \xi)_z &= \psi_x - i\psi_y, & 2(\widetilde{\xi} - \xi)_{\widetilde{z}} &= \frac{1}{f}(\psi_x - i\psi_y), \\ 2(\widetilde{\xi} + \xi)_{\widetilde{z}} &= \psi_x + i\psi_y, & 2(\widetilde{\xi} + \xi)_z &= f(\psi_x + i\psi_y), \\ 4(\widetilde{\xi} - \xi)_{\overline{z}z} &= \widetilde{\xi} + \xi, & 4(\widetilde{\xi} + \xi)_{\overline{\widetilde{z}}\widetilde{z}} &= \widetilde{\xi} - \xi, \end{aligned} \tag{9}$$

which together with (1), let us to conclude that $\mathcal{G}$, $\mathcal{G}^+$ and $\mathcal{G}^-$ are conformal maps. $\square$

Remark 1 *If $f \equiv 0$, then from (8) and (9), $\tilde{z}$ is constant, $U = \Sigma$ and $\nu = \xi + \widetilde{\xi}$ is a constant null vector, that is, $\psi(\Sigma)$ lies on a degenerate hyperplane with normal vector ν. In this case, from (1) and (2) we can write the immersion as*

$$\psi = \phi\nu + xA + y\tilde{A} + C_0, \tag{10}$$

for some constant vectors A, $\tilde{A}$, C_0 in $\mathbb{L}^4$ such that

$$< A, \nu > = < \tilde{A}, \nu > = < A, \tilde{A} > = 0, \qquad |A| = |\tilde{A}| = 1.$$

Theorem (Conformal representation)

i) Let Σ be a simply connected surface and $\psi : \Sigma \longrightarrow \mathbb{L}^4$ a flat immersion with flat normal bundle. If the Gauss map $\mathcal{G}$ is regular and on Σ we consider the conformal structure of Lemma 2, then either $\psi(\Sigma)$ lies on a degenerate hyperplane and ψ is as in (10) or there exists a holomorphic curve $g : \Sigma \longrightarrow \mathbf{SL}(2, \mathbb{C})$, a closed 1-form θ and a pair (f, ω) consisting of a holomorphic function f, which does not vanish identically, and a holomorphic 1-form ω on Σ, $\omega \neq 0$ everywhere, satisfying

$$g^{-1}dg = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} \omega, \tag{11}$$

$$\theta = bf\omega + a\overline{\omega}, \qquad |bf\omega| \neq |a\omega| \quad \text{on } \Sigma, \tag{12}$$

for some smooth real functions $a, b : \Sigma \longrightarrow \mathbb{R}$.
Moreover, the immersion can be recovered as

$$\psi = g\Omega g^* + \tilde{\Omega}, \tag{13}$$

where

$$\Omega = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \text{and} \quad d\tilde{\Omega} + gd\Omega g^* = 0. \tag{14}$$

ii) Conversely, let Σ *be a Riemann surface and* (f, ω) *a pair as above. If* $g : \Sigma \longrightarrow \mathbf{SL}(2, \mathbb{C})$ *is a holomorphic curve and* θ *a closed 1-form such that* $g^{-1}dg$ *and* θ *are as in (11) and (12), then* $\psi = g\Omega g^* + \tilde{\Omega} : \Sigma \longrightarrow \mathbb{L}^4$, *(*$\Omega$ *and* $\tilde{\Omega}$ *as in (14)), is a flat immersion with flat normal bundle and regular Gauss map.*

Proof: It is clear from (8) that $\Sigma \setminus U$ (respectively, $\Sigma \setminus \tilde{U}$) is the set of poles (respectively, of zeros) of f. Then, it is not restriction to assume that U is dense in Σ and $\tilde{z}$ is not a global coordinate immersion. (Otherwise, $\tilde{U} = \Sigma$ and the proof follows with a similar argument by taking $\tilde{f} = 1/f$ instead of f and $2\tilde{\omega} = d\tilde{z}$ instead of 2ω).

Since Σ is simply connected we have a global conformal parameter ζ. Thus, from (8) and (9), there exists a holomorphic function $h : \Sigma \longrightarrow \mathbb{C}$, $h \neq 0$ on U, such that

$$\begin{array}{ll} dz = 2hd\zeta, & d\tilde{z} = 2fhd\zeta, \\ (\tilde{\xi} - \xi)_\zeta = h(\psi_x - i\psi_y), & (\tilde{\xi} + \xi)_\zeta = fh(\psi_x + i\psi_y), \\ (\tilde{\xi} - \xi)_{\zeta\bar{\zeta}} = |h|^2(\tilde{\xi} + \xi), & (\tilde{\xi} + \xi)_{\zeta\bar{\zeta}} = |fh|^2(\tilde{\xi} - \xi). \end{array} \tag{15}$$

Using that $\mathcal{G}^-$ and $\mathcal{G}^+$ are conformal maps, (15) and arguing as in [5] we see that there exists a holomorphic curve $g : \Sigma \longrightarrow \mathbf{SL}(2, \mathbb{C})$ given by

$$g = \begin{pmatrix} C & \frac{1}{h}C_\zeta \\ D & \frac{1}{h}D_\zeta \end{pmatrix}, \tag{16}$$

for some holomorphic linear independent solutions, C, D, of the following ordinary linear differential equation

$$LX = X_{\zeta\zeta} - \frac{h_\zeta}{h}X_\zeta - fh^2X \;=\; 0, \tag{17}$$

or equivalently, g satisfies,

$$g^{-1}dg = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} hd\zeta, \tag{18}$$

such that

(19) $$\widetilde{\xi} = gg^*, \quad \xi = ge_3g^*, \qquad \psi_x = ge_1g^*, \qquad \psi_y = ge_2g^*,$$

where

(20) $$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By writting the immersion as

(21) $$\psi = m_x\psi_x + m_y\psi_y + \alpha\xi - \beta\widetilde{\xi},$$

where $2m =< \psi, \psi >$, $\alpha =< \psi, \xi >$, $\beta =< \psi, \widetilde{\xi} >$ and using (15) we have

(22) $$\begin{aligned} &(\beta-\alpha)_\zeta = h(m_x - im_y), \\ &(\beta+\alpha)_\zeta = fh(m_x + im_y), \\ &(\overline{h}(\beta+\alpha))_\zeta = (fh(\beta-\alpha))_{\overline{\zeta}}. \end{aligned}$$

From (19), (20), (21) and (22) we can recover the immersion as

(23) $$\psi = g \begin{pmatrix} -(\alpha+\beta) & \frac{1}{h}(\beta-\alpha)_\zeta \\ \frac{1}{\overline{h}}(\beta-\alpha)_{\overline{\zeta}} & (\alpha-\beta) \end{pmatrix} g^*.$$

If f vanishes identically on Σ, then from Remark 1, ψ is as in (10).
In other case, we observe that from (22) the zeros of h (respectively, fh) are also zeros of $(\beta-\alpha)_{\zeta\overline{\zeta}}$ (respectively, $(\beta+\alpha)_{\zeta\overline{\zeta}}$) and its partial derivatives respect to $\overline{\zeta}$. Thus, there exist smooth real functions $a, b : \Sigma \longrightarrow \mathbb{R}$ given by

(24) $$|fh|^2 b = (\beta+\alpha)_{\zeta\overline{\zeta}} - |fh|^2(\beta-\alpha), \qquad |h|^2 a = (\beta-\alpha)_{\zeta\overline{\zeta}} - |h|^2(\beta+\alpha),$$

such that $\theta = bf\omega + a\overline{\omega}$ is closed, where $\omega = hd\zeta = \frac{1}{2}dz$. Combining, (18), (23) and (24) we see that (11), (13) and (14) hold.
It remains to prove that ω never vanishes on Σ. To see that, from (11), (13) and (14) we have

(25) $$ds^2 = -det(d\psi) = -det \begin{pmatrix} 0 & \theta \\ \overline{\theta} & 0 \end{pmatrix} = \theta\overline{\theta}$$

and then, using the above expression of θ, the induced volume element of ds^2 is given by

(26) $$\sqrt{ds^2}\, d\zeta d\overline{\zeta} \;=\; \left| a^2|h|^2 - b^2|fh|^2 \right| d\zeta d\overline{\zeta}.$$

Since $\tilde{z}$ is not a global coordinate immersion, (26) gives $|ah| > |bfh| \geq 0$, that is, (12) holds, $\omega \neq 0$ everywhere, $U = \Sigma$ and f is holomorphic, which concludes the proof of i).

The converse follows by straight calculations from our hypothesis. $\square$

Definition 2 *The pair (f, ω) and the 1-form θ in the above Theorem will be called* **Weierstrass data** *and* **metric 1-form** *associated to the immersion ψ.*

3 Completeness and Some Examples

We show in this section how the complex representation given in §2 can be used for constructing examples of complete flat surfaces in $\mathbb{L}^4$.
Recall from (26) that we can assume $|bf| < a$. Moreover, using (12) and (25) we have that the induced metric ds^2 satisfies

$$(a - |bf|)^2|\omega|^2 \leq ds^2 \leq (a + |bf|)^2|\omega|^2 < 4a^2|\omega|^2. \tag{27}$$

3.1 Flat surfaces in hyperquadrics

Consider $M^3(c) = \{v \in \mathbb{L}^4 \mid \ < v, v >= c\}$, that is, the hyperbolic 3-space $\mathbb{H}^3$ for $c = -1$, the de Sitter 3-space $\mathbb{S}^3_1$ for $c = 1$ and the null cone $\mathbb{N}^3$ for $c = 0$.
Let Σ be a simply connected surface and $\tilde{\psi} : \Sigma \longrightarrow M^3(c)$ an immersion with flat induced Riemannian metric. If $i : M^3(c) \longrightarrow \mathbb{L}^4$ denotes the usual inclusion, then it is not difficult to prove that $\psi = i \circ \tilde{\psi}$ has flat normal bundle and regular Gauss map $\mathcal{G}$.

Since we can take $\tilde{\xi} = \tilde{\psi}$ for $c = -1$, $\xi = -\tilde{\psi}$ for $c = 1$ and $\tilde{\xi} - \xi = \tilde{\psi}$ for $c = 0$, from (2) and (9) we see that the conformal structure given in Lemma 2 is the induced by the second fundamental form of $\tilde{\psi}$. Moreover, from (19) and (20), there exists a holomorphic curve $g : \Sigma \longrightarrow \mathbf{SL}(2, \mathbb{C})$ satisfying (11) such that

$$\tilde{\psi} = g \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} g^*, \tag{28}$$

where $a = b = 1$ if $c = -1$, $a = -b = 1$ if $c = 1$ and $a = 2,\ b = 0$ if $c = 0$.

Observe that from (23), the above expression also holds when f vanishes identically on Σ. Thus, our representation extends the one given in [5] for flat surfaces in $\mathbb{H}^3$.

Complete flat surfaces in $M^3(c)$ can be described in an easy way. In fact, if $\tilde{\psi}$ is a complete immersion, then from (27) and (28), $ds^2 \leq 4|\omega|^2$. Therefore, Σ is conformally equivalent to $\mathbb{C}$ (see [6]) and we can assume that $\omega = d\zeta$, for some global parameter ζ. Since $|bf| < a$, one gets either $b = 0$ or $f = d_0$ for some constant $d_0 \in \mathbb{C}$.

1. When b=0, $c = 0$ and $ds^2 = 4|\omega|^2$ is complete if and only if f is an entire holomorphic function.

2. If $b \neq 0$ and $d_0 = 0$, one solution of (11) is given by

$$g(\zeta) = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}$$

and the flat immersions are, up to isometries,

$$\psi(\zeta) = \begin{pmatrix} a & a\bar{\zeta} \\ a\zeta & a|\zeta|^2 + b \end{pmatrix}.$$

3. If b, $d_0 \neq 0$, we consider the new parameter $z = d_1\zeta$, with $d_1^2 = d_0$. In this case, a solution of (11) is

$$g(z) = \frac{1}{\sqrt{2d_1}} \begin{pmatrix} e^{-z} & -d_1 e^{-z} \\ e^{z} & d_1 e^{z} \end{pmatrix}.$$

And the immersion is given, up to isometries, by

$$\psi(z) = \frac{1}{2\sqrt{|d_0|}} \begin{pmatrix} (a + b|d_0|)\, e^{-(z+\bar{z})} & (a - b|d_0|)\, e^{-z+\bar{z}} \\ (a - b|d_0|)\, e^{z-\bar{z}} & (a + b|d_0|)\, e^{z+\bar{z}} \end{pmatrix}.$$

Proposition 1 *Let Σ be a simply connected surface and $\psi : \Sigma \longrightarrow \mathbb{L}^4$ a flat immersion with flat normal bundle and a regular Gauss map. Let us denote by H and $\sigma_H(.,.) = < -dH(.),. >$ the mean curvature vector of the immersion and its induced bilinear form. Then $c_0\,\mathrm{trace}(\sigma_H) + c_1 \det(\sigma_H) = 0$, for some constants c_0 and c_1 if and only if, up to a translation in $\mathbb{L}^4$, $\psi(\Sigma)$ lies in a degenerate hyperplane of $\mathbb{L}^4$ or in $M^3(c)$ for some c.*

Proof: Let (f, ω) be the Weierstrass data associated to ψ, then from (11), (13), (14), (19) and (20) we have

$$\xi_\zeta = \frac{|f|^2 b + a}{b^2|f|^2 - a^2}\psi_\zeta - \frac{(a+b)fh^2}{(b^2|f|^2 - a^2)|h|^2}\psi_{\bar{\zeta}}, \tag{29}$$

$$\widetilde{\xi}_\zeta = \frac{|f|^2 b - a}{b^2|f|^2 - a^2}\psi_\zeta + \frac{(b-a)fh^2}{(b^2|f|^2 - a^2)|h|^2}\psi_{\bar{\zeta}}, \tag{30}$$

and the mean curvature vector of the immersion is given by

$$H = -\frac{|f|^2 b + a}{b^2|f|^2 - a^2}\xi - \frac{-|f|^2 b + a}{b^2|f|^2 - a^2}\widetilde{\xi}. \tag{31}$$

Now, combining the above equations with (11), (13), (14), (19) and (20) we also see that

$$\det(\sigma_H) = -\frac{|f|^2}{(b^2|f|^2 - a^2)^2}, \qquad \mathrm{trace}(\sigma_H) = -\frac{|f|^2 ab}{(b^2|f|^2 - a^2)^2}. \tag{32}$$

Therefore, $c_0\,\mathrm{trace}(\sigma_H) + c_1 \det(\sigma_H) = 0$ if and only if $f \equiv 0$ or ab is constant.

In the first case, from Remark 1, ψ lies in a degenerate hyperplane and it is given as in (10). If ab is constant, using that θ is closed one sees easily that a and b are also constant and the result follows from (13) and (14). $\square$

3.2 Other examples

By using (27) we can also describe other families of complete flat surfaces which are not contained in $M^3(c)$. For instance, since θ is closed, when f is constant, $f \equiv 1$, one has $b(\zeta, \overline{\zeta}) = p(\zeta + \overline{\zeta}) - q(\zeta - \overline{\zeta})$ and $a(\zeta, \overline{\zeta}) = p(\zeta + \overline{\zeta}) + q(\zeta - \overline{\zeta}) + c_3 > 0$, for some constant c_3. Thus, for any p and q bounded from below, we can choose c_3 in order to obtain a complete flat surface.

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