Gleason-Kahane-Żelazko and Kowalski-Słodkowski theorems

Antonio M. Peralta, Instituto de Matemáticas
IMAG, Universidad de Granada.

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Applications

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Local (linear) maps

Let $\mathcal{S}$ be a subset of the space $L(X, Y)$ of all linear maps between Banach spaces X and Y (or more generally a class of maps from X to Y). A linear map $T : X \to Y$ is called a *local $\mathcal{S}$ map* if for each $x \in X$ there exists $S_x \in \mathcal{S}$, depending on x, such that $T(x) = S_x(x)$.

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2-local maps

A (non-necessarily linear nor continuous) mapping $\Delta : X \to Y$ is said to be a *2-local $\mathcal{S}$ map* if for every $x, y \in X$ there exists $T_{x,y} \in \mathcal{S}$, depending on x and y, such that $T_{x,y}(x) = \Delta(x)$, and $T_{x,y}(y) = \Delta(y)$.

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The origins go back to works of Kadison, Larson and Sourour and Šemrl

During this talk we shall be mainly interested in the case in which $\mathcal{S}$ is the set $\text{Iso}(X, Y)$ of all surjective linear isometries from X onto Y (respectively, the class of all non-necessarily linear surjective isometries from X onto Y), in this case local and 2-local $\mathcal{S}$ maps are called *local isometries* and *2-local isometries*, respectively (respectively, *local non-necessarily-linear isometries* and *2-local non-necessarily-linear isometries*). We can similarly define *local* and *2-local automorphisms, derivations, Lie derivations, Jordan derivations, etc...* on a Banach algebra.

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The game:

Finding conditions on $\mathcal{S}$ to assure that every local $\mathcal{S}$ map lies in $\mathcal{S}$, respectively, each 2-local $\mathcal{S}$ map is linear and lies in $\mathcal{S}$.

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Be careful!!

If we take $\mathcal{S} = K(X, Y)$ (the class of compact linear operators) every 1-homogeneous mapping $\Delta : X \to Y$ (i.e. $\Delta(\lambda x) = \lambda \Delta(x)$) is a 2-local $\mathcal{S}$ map.

[Classic results:]

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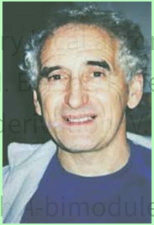

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How to apply a similar argument to more general examples?

Suppose K is a compact Hausdorff space. Any norm closed subalgebra of $C(K)$ **containing the constant functions and separating the points of** K is called a *uniform algebra*.

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Abstract characterization

By the Gelfand theory and the Gelfand-Beurling formula, if A is a unital commutative complex Banach algebra such that $\|a^2\| = \|a\|^2$ for all a in A, then there is a compact Hausdorff space K such that A is isomorphic, as Banach algebra, to a uniform subalgebra of $C(K)$.

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Nice concrete examples:

Suppose K is a compact subset of $\mathbb{C}^n$, the algebra $A(K)$ of all complex valued continuous functions on K which are holomorphic on the interior of K, is an example of *uniform algebra*. When $K = \mathbb{D}$ is the closed unit ball of $\mathbb{C}$, $A(\mathbb{D})$ is precisely the disc algebra.

We recall that every surjective linear isometry of a uniform algebra A is an algebra automorphism of A multiplied by an element of A whose spectrum is contained in $\mathbb{T} = S(\mathbb{C})$ (see, for example, Fleming and Jamison's book on *Isometries on Banach Spaces*).

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Let $K \subseteq \mathbb{C}$ be a compact set whose complement has finitely many components. Then every local isometry (respectively, every local automorphism) on $A(K)$ is a surjective isometry (respectively, an automorphism). This applies, in particular, to the disc algebra.

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Before dealing with the next result we recall some additional notions.

Let E and F denote two metric spaces. A mapping $f : E \to F$ is called *Lipschitzian* if

$$L(f) := \sup \left\{ \frac{d_F(f(x), f(y))}{d_E(x, y)} : x, y \in E,\ x \neq y \right\} < \infty.$$

When $F = Y$ is a Banach space, the symbol $\mathrm{Lip}(E, Y)$ will denote the space of all bounded Lipschitz functions from E into Y. The space $\mathrm{Lip}(E, Y)$ is a Banach space with respect to the following (equivalent) complete norms

$$\|f\|_L := \max\{L(f), \|f\|_\infty\}, \text{ and } \|f\|_s := L(f) + \|f\|_\infty.$$

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For every metric space E, $(\mathrm{Lip}(E), \|.\|_s)$ is a unital commutative complex Banach algebra with respect to pointwise multiplication. However, the norm $\|.\|_L$ does not satisfy the usual hypothesis of Banach algebras that $\|fg\| \leq \|f\|\|g\|$.

If E is a compact metric space, Lip(E) is self-adjoint and separates the points of E, so it is dense in $C(E)$ with respect to the sup norm (Stone-Weierstrass theorem). There exist continuous functions which are not Lipschitz. **Lip(E) is not, in general, a uniform algebra**.

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Surjective linear isometries

Given a surjective isometry $\varphi : F \to E$ between two metric spaces F and E, and an element $\tau \in S_{\mathbb{F}}$, the mapping

$$T_{\tau,\varphi} : \mathrm{Lip}(E) \to \mathrm{Lip}(F), \;\; T_{\tau,\varphi}(f)(s) = \tau f(\varphi(s)), \;\; (f \in \mathrm{Lip}(E)),$$

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Fortunately or not, there exist elements in $\mathrm{Iso}(\mathrm{Lip}(E), \mathrm{Lip}(F))$ which cannot be written as weighted composition operator via a surjective isometry φ and $\tau \in S_{\mathbb{F}}$.

The elements in $\mathrm{Iso}(\mathrm{Lip}(E), \mathrm{Lip}(F))$ which can be written as weighted composition operators via a surjective isometry $\varphi : F \to E$ and $\tau \in S_{\mathbb{F}}$ as above are called *canonical*. The set $\mathrm{Iso}(\mathrm{Lip}(E), \mathrm{Lip}(F))$ is called *canonical* if every element in this set is canonical.

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This property is related to the own nature of the metric spaces [N. Weaver, *Canad. Math. Bull.*'1995]

Let $E = \{t_1, t_2\}$ be the metric space formed by two points with distance $d(t_1, t_2) = 1$. Then $(Lip(E), \|\cdot\|_L)$ is (isometrically isomorphic to) $\mathbb{F}^2$ with norm

$$\|(\alpha_1, \alpha_2)\| = \max\{|\alpha_1|, |\alpha_2|, |\alpha_1 - \alpha_2|\},$$

and the linear mapping $T : (Lip(E), \|\cdot\|_L) \to (Lip(E), \|\cdot\|_L)$, $T(\alpha_1, \alpha_2) = (\alpha_1, \alpha_1 - \alpha_2)$ is an isometric isomorphism which does not arise by composition with an isometry of E. The problem is the composition map, not the weight!

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- (✓) E and F are complete metric spaces of diameter ≤ 2 and 1-connected (i.e. they cannot be decomposed into two nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ such that $d(t, s) \geq 1$ for every $t \in \mathcal{A}$ and $s \in \mathcal{B}$) then Iso((Lip(E), $\|.\|_L$), (Lip(F), $\|.\|_L$)) is canonical (N. Weaver, *Canad. Math. Bull.*'1995). Actually we can restrict our study to the class of complete metric spaces of diameter ≤ 2.

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Let E be a compact metric space. Then every local isometry on $(\mathrm{Lip}(E), \|.\|_s)$ is a surjective isometry, and hence a uni-modular weighted composition operator via a surjective isometry on E.

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The conclusions on 2-local isometries are much more limited.

Some known results for other 2-local maps.

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✓ $H \to$ an infinite-dimensional separable Hilbert space. Then every 2-local automorphism (respectively, every 2-local derivation) $T : B(H) \to B(H)$ is an automorphism (respectively, a derivation) (Šemrl, *Proc. Amer. Math. Soc.*'1997).

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For each uniform algebra A, every 2-local automorphism Δ on A is an isometric isomorphism from A onto $\Delta(A)$. Furthermore, if the group of all automorphisms on A is algebraically reflexive (i.e., if every local automorphism on A is an automorphism), then every 2-local automorphism on A is an automorphism.

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The same authors also showed the existence of non-surjective 2-local automorphisms on $C(K)$ spaces.

Another interesting conclusion is the following:

[Hatori, Miura, Oka and Takagi, *Int. Math. Forum*'2010]

Let $K \subseteq \mathbb{C}$ be a compact subset such that $\text{int}(K)$ is connected and $\overline{\text{int}(K)} = K$. Then every local isometry (respectively, every local automorphism) on $A(K)$ is a surjective isometry (respectively, is an automorphism).

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Furthermore, *under certain topological restrictions* on a compact set $K \subset \mathbb{C}$ or $K \subset \mathbb{C}^2$, every 2-local isometry (respectively, every 2-local automorphism) on $A(K)$ is a surjective linear isometry (respectively, an automorphism).

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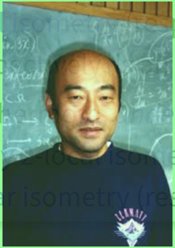

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Hatori, Miura, Oka and Takagi posed the following problem:

Problem:

Is every 2-local isometry on a uniform algebra linear?

To study this problem we introduced and considered a more general class of mappings.

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[Essaleh, Pe., Ramirez, *Linear Multilinear Algebra*'2016]

Let X and Y be a Banach spaces, and let $\mathcal{S}$ be a subset in $L(X, Y)$ (or more generally, a subset of maps from X to Y). A linear mapping $T : X \to Y$ is called a weak-local $\mathcal{S}$ map if for each x in X and each $\phi \in Y^*$ there exists $T_{x,\phi} \in \mathcal{S}$, depending on x and ϕ, such that

$$\phi(T(x) - T_{x,\phi}(x)) = 0.$$

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A mapping $\Delta : X \to Y$ will be called a weak-2-local $\mathcal{S}$ map if

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Let X and Y be Banach spaces such that Y is infinite dimensional. Suppose F is a proper norm-dense subspace of Y. Let $\mathcal{S}$ be the set of all finite rank operators S in $L(X, Y)$ such that $S(X) \subset F$.

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Even in the setting of $C(K)$ spaces, weak-local and weak-2-local isometries cannot be studied via Gleason-Kahane-Żelazko and Kowalski-Słodkowski theorems. For this purpose, we developed appropriate spherical variants of these theorems.

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Spherical versions

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By the Banach–Stone theorem every surjective isometry $\Phi : C(K) \to C(K)$ is of the form $\Phi(a)(t) = u(t)a(\varphi(t))$, where $\varphi : K \to K$ is a homeomorphism on K and u is a unitary element in $C(K)$.

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We state first the spherical variant of the Gleason-Kahane-Żelazko theorem.

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Let $F : A \to \mathbb{C}$ be a linear map, where A is a unital complex Banach algebra. Suppose that $F(a) \in \mathbb{T}\,\sigma(a)$, for every $a \in A$. Then the mapping $\overline{F(1)}F$ is multiplicative.

The proof follows classical methods based on tools of holomorphic functions like Hadamard's Factorization theorem.

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The new tool can be also applied in the setting of Lipschitz algebras.

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(a) Suppose that the set $\mathrm{Iso}((\mathrm{Lip}(E), \|.\|_s), (\mathrm{Lip}(F), \|.\|_s))$ is canonical. Then every weak-local isometry $T : (\mathrm{Lip}(E), \|.\|_s) \to (\mathrm{Lip}(F), \|.\|_s)$ is almost canonical, i.e., it can be written in the form $T(f) = \tau\, \psi(f)$, for all $f \in \mathrm{Lip}(E)$, where $\tau \in \mathrm{Lip}(F)$ is unimodular, and $\psi : \mathrm{Lip}(E) \to \mathrm{Lip}(F)$ is an algebra homomorphism;

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$$\Delta(f)(s) = \tau f(\varphi(s)), \text{ for all } f \in \mathrm{Lip}(E), s \in F_0.$$

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This new result provides the tool to consider a new variant of our problem, which was already posed by L. Molnár. Namely, let $\mathcal{S}$ denote in this case the set of all (non-necessarily linear) surjective isometries between two Banach spaces X and Y. 2-local $\mathcal{S}$-maps are called *2-local non-necessarily linear surjective isometries*.

The setting of semisimple commutative Banach algebras offers a good framework to play.

$B \to$ a unital semisimple commutative Banach algebra, $\mathcal{M} \to$ maximal ideal space of B. The Gelfand transform is a continuous isomorphism identifying B with its image inside $C(\mathcal{M})$. On $C(\mathcal{M})$ we have well-known evaluation functionals δ_t with $t \in \mathcal{M}$.

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Pointwise 2-locality as a particular case of weak 2-locality

Let $\mathcal{S}$ be a class of maps from B_1 to B_2, where B_1, B_2 are unital semisimple commutative Banach algebras. A mapping $\Delta : B_1 \to B_2$ is pointwise 2-local in $\mathcal{S}$ if for every trio $f, g \in B_1$ and $t \in \mathcal{M}_2$ there exists $T_{f,g,t} \in \mathcal{S}$, depending on f, g, t, such that

$$\Delta(f)(t) = T_{f,g,t}(f)(f), \text{ and } \Delta(g)(t) = T_{f,g,t}(g)(t).$$

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There are examples of pointwise 2-local isometries which fail to be surjective or an isometry.

We shall focus on a very interesting class of maps

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$$\mathcal{S} = GWC := \left\{ T : B_1 \to B_2 : \begin{bmatrix} \text{there exists } b, a \in B_2 \text{ with } |a| = 1 \text{ on } M_2, \\ \pi : M_2 \to M_1,\ \varepsilon : M_2 \to \{\pm 1\} \text{ continuous} \\ \text{such that } T(f) = b + a[f \circ \pi]^{\epsilon} \text{ for every } f \in B_1 \end{bmatrix} \right\},$$

where for each $f \in B_1$ and $\varepsilon : M_2 \to \{\pm 1\}$ we set $[f]^{\varepsilon}(t) := f(t)$ if $\varepsilon(t) = 1$ and $[f]^{\varepsilon}(t) := \overline{f(t)}$ if $\varepsilon(t) = -1$ ($t \in M_1$).

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Why is this class so interesting?

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Let A_1, A_2 be uniform algebras on compact Hausdorff spaces X_1, X_2, respectively. It is known that every (non-necessarily linear) surjective isometry $\Delta : A_1 \to A_2$ is in the class GWC.

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Let A_1, A_2 be uniform algebras on compact Hausdorff spaces X_1, X_2, respectively. Then every pointwise 2-local non-necessarily linear isometry from A_1 to A_2 is a map in the class GWC.

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Assuming that X_j is first countable compact Hausdorff space for $j = 1, 2$, then every 2-local non-necessarily linear surjective isometry is a surjective isometry.

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Additional applications of the spherical KS theorem have been found for the study of 2-local isometries on

1. The Banach space $AC(X)$ of all absolutely continuous complex-valued functions on a compact subset X of the real line with at least two points (Hosseini, Jiménez-Vargas, *Results Math.*'2021).

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2. The algebra $Lip(X, C(Y))$ of all $C(Y)$-valued Lipschitz maps on a compact metric space X equipped with the sum norm, where Y is a compact Hausdorff space (Cabrera-Padilla, Jiménez-Vargas, *J. Math. Anal. Appl.*'2022).

Open problem:

✓ Can we extend the spherical versions of the GZK and KS theorems to the case of non-unital complex Banach algebras?

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[Essaleh, Pe., Ramírez, *Linear Multilinear Algebra*'2015]

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Thanks for spending part of your time listening this talk!!!