Traces on locally compact groups

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Banach Algebras and Applications
Granada
July 23, 2022

Positive definite functions and traces

G – topological group
$u : G \to \mathbb{C}$ is <u>positive definite</u> if

$$\sum_{i,j=1}^{n} \overline{\lambda_i} u(s_j^{-1} s_i) \lambda_j \geq 0 \quad \forall \lambda_1, \ldots, \lambda_n \in \mathbb{C}, \; s_1, \ldots, s_n \in G, n \in \mathbb{N}$$

i.e. $[u(s_j^{-1} s_i)] \in \mathrm{M}_n(\mathbb{C})$ is positive.

Note: $|u(s)| \leq u(e)$ as $\begin{bmatrix} u(e) & u(s) \\ u(s^{-1}) & u(e) \end{bmatrix} \geq 0$.

Basic spaces

$$\mathrm{P}(G) = \{u \in \mathcal{C}(G) : u \text{ pos. def.}\}$$
$$\mathrm{P}_1(G) = \{u \in \mathrm{P}(G) : u(e) = 1\}$$
$$\mathrm{T}(G) = \{u \in \mathrm{P}_1(G) : u(st) = u(ts), \; \forall s, t \in G\}$$

Locally compact case

G – locally compact group, admits left Haar measure.
Hence $\mathcal{C}_c(G) \subseteq \mathrm{L}^1(G)$ are involutive algebras

$$f * g(t) = \int_G f(s)g(s^{-1}t)\,ds, \quad f^*(t) = \overline{f(t^{-1})}\Delta(t^{-1})$$

Universal C*-algebra:

$$\mathrm{C}^*(G) = \overline{\mathcal{C}_c(G)} = \overline{\mathrm{L}^1(G)}$$

(completion w.r.t. largest C*-norm); $\lambda : \mathrm{L}^1(G) \to \mathcal{B}(\mathrm{L}^2(G))$, $\lambda(f)h = f * h$ insures a norm.

States and tracial states

$$\mathrm{P}_1(G) \cong \mathrm{S}(\mathrm{C}^*(G)), \quad \langle f, u\rangle = \int_G f(s)u(s)\,ds$$

$$\mathrm{T}(G) \cong \mathrm{T}(\mathrm{C}^*(G))$$

Abelian and compact cases

$$G \text{ l.c. abelian } \Rightarrow \mathrm{T}(G) = \mathrm{P}_1(G) \cong \mathrm{Prob}(\widehat{G})$$

$$G \text{ compact } \Rightarrow \mathrm{T}(G) = \overline{\mathrm{conv}}\left\{ \frac{1}{d_\pi}\chi_\pi : \pi \in \widehat{G} \right\}$$

Hence $\dim \mathrm{T}(G) = \infty$ if either

G/G' is infinite, or

G admits infinitely many f.d. (irreducible) rep'ns

Tracial kernel

Proposition (GNS)

If $u \in \mathrm{P}_1(G)$, there is w.o.-continuous $\pi_u : G \to \mathrm{U}(\mathcal{H}_u)$, $\xi \in \mathcal{H}_u$ unit, so $u = \langle \pi_u(\cdot)\xi|\xi\rangle$.

Proposition

(i) $u \in \mathrm{P}_1(G) \Rightarrow N_u = \{s \in G : u(s) = 1\}$ closed subgroup.

(ii) $u \in \mathrm{T}(G) \Rightarrow N_u = \ker \pi_u \lhd G$.

Idea. Use uniform convexity of $\mathcal{H}_u$.

Tracial kernel:

$$N_{\mathrm{Tr}} = N_{\mathrm{Tr}}(G) = \bigcap_{u \in \mathrm{T}(G)} N_u.$$

Then $\mathrm{T}(G) = \mathrm{T}(G/N_{\mathrm{Tr}}) \circ q_{N_{\mathrm{Tr}}}$.

Related kernels

$$N_{\mathrm{MAP}} = \bigcap_{N \in \mathcal{N}_{\mathrm{MAP}}} N, \quad \mathcal{N}_{\mathrm{MAP}} = \{N \lhd G : \text{closed}, \ G/N \in [\mathrm{MAP}]\}$$

– von Neumann kernel, MAP = maximally almost periodic

$$N_{\mathrm{SIN}} = \bigcap_{N \in \mathcal{N}_{\mathrm{SIN}}} N, \quad \mathcal{N}_{\mathrm{SIN}} = \{N \lhd G : \text{closed}, \ G/N \in [\mathrm{SIN}]\}$$

– SIN = small invariant neighbourhood

Easy to show that $G/N_{\mathrm{MAP}} \in [\mathrm{MAP}]$.

Problem: G/N_{SIN} may not live in $[\mathrm{SIN}]$.

If $N \in \mathcal{N}_{\mathrm{SIN}}$ then $(G/N_{\mathrm{SIN}})/(N/N_{\mathrm{SIN}}) \cong G/N \in [\mathrm{SIN}]$.

I.e. G/N_{SIN} is residually-SIN; write $G/N_{\mathrm{SIN}} \in [\mathrm{RSIN}]$.

An inclusion of classes

Theorem

$[\mathrm{MAP}] \subset [\mathrm{RSIN}]$.

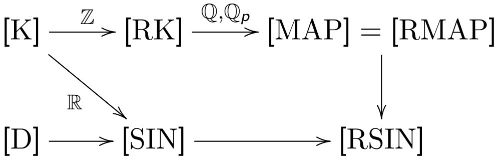
Idea. $G_0 \cong V \times K$. If $\pi : G \to \mathrm{U}(d)$, can show that $\ker \pi \in \mathcal{N}_{\mathrm{SIN}}$.

Corollary

$G \in [\mathrm{MAP}]$ and tot'ly disc'd $\Rightarrow$ G is residually discrete.
$G \in [\mathrm{MAP}]$, tot'ly disc'd, & com. gen. $\Rightarrow$ G is residually finite.

Idea. There is a compact open subgroup $L \subseteq \ker \pi$.
If compactly generated, use [Mal'cev '40].

Inclusions of classes



Examples. $\Gamma \in [\mathrm{D}]$, $\alpha \in F \subseteq \mathrm{Aut}(\Gamma)$ finite, $\{\alpha(s)s^{-1} : s \in \Gamma\}$ infinite

$$G = \underbrace{\Gamma^{\oplus\mathbb{N}}}_{\text{discrete}} \rtimes \underbrace{F^{\mathbb{N}}}_{\text{compact}} \in [\mathrm{RSIN}] \setminus [\mathrm{SIN}].$$

(i) [Murakami '50] $\Gamma = \mathbb{Z}$, $F = \langle \text{inversion} \rangle$: $G \in [\mathrm{RK}] \setminus [\mathrm{SIN}]$

(ii) $\Gamma = \mathrm{SL}_2(\mathbb{Q})$, $F = \left\langle \mathrm{Ad} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$:

$$G \in [\mathrm{RSIN}] \setminus ([\mathrm{SIN}] \cup [\mathrm{MAP}])$$ [von Neumann–Wigner '40]

Back to tracial kernel

Proposition

$N_{\mathrm{Tr}} \subseteq N_{\mathrm{SIN}} \subseteq N_{\mathrm{MAP}}$

Idea. $N_{\mathrm{SIN}} \subseteq N_{\mathrm{MAP}}$ since $[\mathrm{MAP}] \subset [\mathrm{RSIN}]$.
If $N \in \mathcal{N}_{\mathrm{SIN}}$, U rel. com. inv. nbhd. of eN in G/N, then

$$u_U = \frac{1}{m(U)} \langle \lambda_{G/N}(\cdot) 1_U | 1_U \rangle \in \mathrm{T}(G).$$

Remark. If there is non-open $N \in \mathcal{N}_{\mathrm{SIN}}$, or infinite index $N \in \mathcal{N}_{\mathrm{MAP}}$, then $\dim \mathrm{T}(G) = \infty$.

Question. Is there G s.t. $N_{\mathrm{Tr}} \subsetneq N_{\mathrm{SIN}}$?

The role of compact generation

Proposition

Let H be a top'l group, with a family $\mathcal{B}$ of open invariant subsets with $\bigcap_{B\in\mathcal{B}} B = \{e\}$. Suppose G is com. gen., & admits cts. injective hom'm $\varphi : G \to H$. Then $G \in [\mathrm{SIN}]$.

(i) G com. gen. $\Rightarrow N_{\mathrm{Tr}} = N_{\mathrm{SIN}}$.

(ii) G com. gen. in $[\mathrm{RSIN}] \Rightarrow G \in [\mathrm{SIN}]$.

Essential idea in [Hofmann-Mostert '63], used to show com. gen. MAP is SIN.

(i) $u \in \mathrm{T}(G)$, $\varphi : G/N_u \to \pi_u(G) \subseteq \mathrm{U}(\mathcal{H}_u)$, s.o.t.=w.o.t.

(ii) $\varphi(s) = (q_N(s))_{N\in\mathcal{N}_{\mathrm{SIN}}} : G \to \prod_{N\in\mathcal{N}_{\mathrm{SIN}}} G/N$

$$[\mathrm{K}] \xrightarrow{\mathbb{Z},\mathbb{R}} [\mathrm{RK}] \xrightarrow{\mathbb{R}^2\rtimes C_3} [\mathrm{MAP}] \xrightarrow{BS} [\mathrm{SIN}]$$

Connected group G

[Freudenthal '36/Weil '40] G connected: then

$$G \in [\mathrm{SIN}] \quad \Leftrightarrow \quad G \in [\mathrm{MAP}] \quad \Leftrightarrow \quad G = V \times K.$$

Consequence. $N_{\mathrm{Tr}} = N_{\mathrm{MAP}}$

Lie case. R – solvable radical, $S = S_c S_{nc}$ – Levi complement

[Shtern '10] $N_{\mathrm{MAP}} = \overline{[R, G] S_{nc}}$

Consequence. $G/N_{\mathrm{Tr}} = V \times K$, V quot. of max'l vector subgp. of $G/\overline{[R, G]}$, semisimple part of K loc. isom. to S_c.

If G is simply connected, then $V = G/[R, G]$ and $K = S_c$.

Reduced traces

$\lambda : G \to \mathrm{U}(\mathrm{L}^2(G))$ left reg. rep'n,

$$\mathrm{P}_r(G) = \overline{\{\langle \lambda(\cdot)h|h\rangle : h \in \mathrm{L}^2(G)\}}^{uc} = \{\langle \pi(\cdot)\xi|\xi\rangle : \xi \in \mathcal{H}_\pi, \pi \prec \lambda\}$$

('$\prec$'="weakly contained")

$$\mathrm{C}^*_r(G) = \overline{\lambda(\mathrm{L}^1(G))} = \overline{\lambda(\mathcal{C}_c(G))} \subseteq \mathcal{B}(\mathrm{L}^2(G))$$

Reduced traces:

$$\mathrm{T}_r(G) = \mathrm{P}_r \cap \mathrm{T}(G)$$

Existence of reduced traces I

Lemma

G totally disconnected and $\mathrm{SIN} \Leftrightarrow G$ pro-discrete.

Theorem [Forrest–S.–Wiersma '17]

TFAE for G compactly generated:

(i) $\mathrm{T}_r(G) \neq \varnothing$

(ii) $\mathcal{N}_{\mathrm{SIN}}$ admits an amenable element

(iii) G admits an open normal amenable subgroup

(iv) $N^r_{\mathrm{Tr}} = \bigcap_{u \in \mathrm{T}_r(G)} N_u$ ($= G$ if $\mathrm{T}_r(G) = \varnothing$) is amenable

(i) $\Rightarrow$ (ii) $u \in \mathrm{T}_r(G)$, $1 \in \mathrm{P}_r(G)|_{N_u} \Rightarrow N_u$ amen.; $G/N_u \in [\mathrm{SIN}]$.
(ii) $\Rightarrow$ (iii) $N \in \mathcal{N}_{\mathrm{SIN}}$ amenable, $\widetilde{(G/N)/(G/N)_0}$ has open compact normal L, $V \times K \cong (G/N)_0 \to \tilde{L} \to L$, $\tilde{L} \subset G/N$ open normal.

Hence if G connected: $\mathrm{T}_r(G) \neq \varnothing \Leftrightarrow G$ amenable.

Existence of reduced traces II

$AR(G)$ – amenable radical

Theorem [Kennedy–Raum '17]

$\mathrm{T}_r(G) \neq \varnothing \Leftrightarrow AR(G)$ open. In this case

$$\mathrm{T}_r(G) \cong \{u \in \mathrm{T}(AR(G)) : u(srs^{-1}) = u(r)\ \forall s \in G, r \in AR(G)\}.$$

Hence

$$\mathrm{T}_r(G) \cong \left\{ u \in \mathrm{T}(AR(G)/N^r_{\mathrm{Tr}}) : \begin{array}{c} u(srs^{-1}N^r_{\mathrm{Tr}}) = u(rN^r_{\mathrm{Tr}}) \\ \forall s \in G, r \in AR(G) \end{array} \right\}.$$

Groups with(out) unique reduced trace

Theorem

If A is non-discrete abelian, Γ is fin. gen. disc., $\Gamma \curvearrowright A$ irred. then $N^r_{\mathrm{Tr}}(A \rtimes \Gamma) \cap A$ either $\{e\}$ or A; former case $\Leftrightarrow$ $\Gamma \curvearrowright A$ factors through compact $K \curvearrowright A$.

Idea. A either connected or tot. disc'd., $A \rtimes \Gamma$ com. gen. If $N^r_{\mathrm{Tr}}(A \rtimes \Gamma) \cap A = \{e\}$, Ascoli Thm. of [Grosser-Moskowitz '71].

Proposition

K – compact, Γ – disc., $\Gamma \curvearrowright K$
$N^r_{\mathrm{Tr}}(K \rtimes \Gamma) \cap K = K$ unless $X_K = \{\frac{1}{d_\pi}\chi_\pi : \pi \in \widehat{G}\}$ admits non-trivial finite Γ-orbit.

Remark. semi-direct products can be replaced by extensions, e.g. $A \to G \to \Gamma$

Groups with unique reduced trace: Examples

Discrete examples

[Powers '75] $\mathrm{T}_r(\mathrm{F}_2) = \{1_{\{e\}}\}$, i.e. $AR(\mathrm{F}_2) = \{e\}$.

[Pashcke-Salinas '79] free products

[de la Harpe '83, Bekka-Cowling-de la Harpe '94] $\mathrm{PSL}_n(\mathbb{Z})$ etc.

Non-discrete examples.

Below, $G = A \rtimes \Gamma$ or $K \rtimes \Gamma$.

(i) $\Gamma = \mathrm{F}_2$, $q : \mathrm{F}_2 \to \mathbb{Z}^2$, $\eta : \mathbb{Z}^2 \to \mathbb{R}$, $\eta(m,n) = m + \sqrt{2}n$
$M \in \mathrm{GL}_2(\mathbb{R})$, $\sigma(M) \subset \mathbb{C} \setminus (\mathbb{R} \cup \mathbb{T})$.
$s \in \mathrm{F}_2$, $v \in A = \mathbb{R}^2$, $s \cdot v = \exp(iM\eta \circ q(s))v$

(ii) $d \geq 3$ odd, $\Gamma = \mathrm{SL}_d(\mathbb{Z}) = \mathrm{PSL}_d(\mathbb{Z})$, $K = \mathbb{T}^d$; $A = \mathbb{R}^d$.

(iii) $AR(\Gamma) = \{e\}$, $L \supsetneq \{e\}$ compact, $K = L^\Gamma$, shift action.

(iv) $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[1/p])$, $A = \mathrm{M}_2(\mathbb{Q}_p)/\mathbb{Q}_p \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\sigma \cdot x = \sigma x \sigma^T$.

Amenable reduced traces

Fact. $u \in \mathrm{T}(G) \Rightarrow \tilde{u}(s,t) = u(st^{-1})$ is in $\mathrm{P}(G \times G)$.
If $u \in \mathrm{T}_r(G)$ then $\tilde{u} \,\widetilde{\in}\, \mathrm{S}(\mathrm{C}^*_r(G) \otimes_{\max} \mathrm{C}^*_r(G))$.

$$\mathrm{T}_{\mathrm{am},r}(G) = \{u \in \mathrm{T}_r(G) : \tilde{u} \in \mathrm{P}_{1,r}(G \times G) \cong \mathrm{S}(\mathrm{C}^*_r(G) \otimes_{\min} \mathrm{C}^*_r(G))\}$$

[Lance '73] Γ disc.: $\mathrm{C}^*_r(\Gamma)$ nuclear $\Leftrightarrow$ Γ amenable.
[Ng '15] $\mathrm{C}^*_r(G)$ nuclear & $\mathrm{T}_r(G) \neq \varnothing \Leftrightarrow G$ amenable.

Theorem

TFAE

(i) G amenable

(ii) $\mathrm{T}_{\mathrm{am},r}(G) \neq \varnothing$

(iii) $\mathrm{C}^*_r(G)$ nuclear & $\mathrm{T}_r(G) \neq \varnothing$

Idea. (ii) $\Rightarrow$ (i) $u \in \mathrm{T}_{\mathrm{am},r}(G)$, $1 \cong \tilde{u}|_{G_D} \in \mathrm{P}_r(G)$.

AF-embedability and quasi-diagonality

Theorem

If G is 2nd countable then TFAE

(i) $\mathrm{C}^*_r(G)$ embeds into a simple unital AF-algebra;

(ii) $\mathrm{C}^*_r(G)$ admits a faithful amenable trace; and

(iiii) G is amenable and $N_{\mathrm{Tr}} = \{e\}$.

Use AF-embedability theorem of [Schafhauser '20].

Corollary

G amenable, $N_{\mathrm{Tr}} = \{e\}$, then $\mathrm{C}^*_r(G)$ is quasi-diagonal.

Use [Lau-Losert '90] approximation of σ-compact by metrizable.

Certain solvable Lie G have non-QD $\mathrm{C}^*_r(G)$, [Belțiță-Belțiță '21].

Reference and Thank-you

`arXiv:2206.01771` (Just look for Forrest, Spronk, or Wiersma.)

Gracias!

Thanks to Armando, Jerónimo, & José!

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I hope to see you all in Waterloo in 2024.