CENTRES OF IDEALS IN βG

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1. PART I

G discrete (and abelian for simplicity).
βG is a compact right topological semigroup

- with the first Arens on $\ell^1(G)^{**}$ restricted to βG, or
- Using the property of βG:
  For each $s \in G$, the continuous mapping
  $$t \mapsto s\,t \;:\; G \to \beta G$$
  extends to a continuous mapping
  $$y \mapsto s\,y \;:\; \beta G \to \beta G.$$
  Then, for each $y \in \beta G$, we extend the mapping $s \mapsto s\,y$ defined from G into βG to a continuous mapping
  $$x \mapsto x\,y \;:\; \beta G \to \beta G,$$
  making βG a compact right topological semigroup.
  (βG has the weak*-topology inherited from $\ell^\infty(G)^*$).

The topological centre of βG is

$$\mathfrak{Z}(\beta G) = \{x \in \beta G;\; y \mapsto yx : \beta G \to \beta G \text{ is continuous}\}.$$

The algebraic centre of βG

$$\mathfrak{Z}_a(\beta G) = \{x \in \beta G;\; xy = yx \text{ for all } y \in \beta G\}.$$

Since we are assuming that G is abelian, $\mathfrak{Z}(\beta G) = \mathfrak{Z}_a(\beta G)$.

If I is a left, right ideal or a subsemigroup of βG,

$\mathfrak{Z}(I) = \{x \in I;\; y \mapsto yx : I \to I \text{ is continuous}\} =$ (when G is abelian)
$\mathfrak{Z}_a(I) = \{x \in I : xy = yx \text{ for all } y \in I\}$.

(Again I is with the weak*-topology inherited from $\ell^\infty(G)^*$.)

For example, $G^* = \beta G \setminus G$ is a closed ideal in βG.

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John's drawing: If $G=\mathbb{Z}$, then βG is

$$\bigcirc \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \bigcirc$$

R and L are closed left ideals in $\beta\mathbb{Z}$, $\mathbb{Z}^* = R \cup L$ is a closed ideal in $\beta\mathbb{Z}$.

$$\mathfrak{Z}(\beta\mathbb{Z}) = \mathbb{Z}, \quad \mathfrak{Z}(\mathbb{Z}^*) = \mathfrak{Z}(R\cup L) = \varnothing$$

$$(-\infty = \lim_n \lim_m (n-m) \neq \lim_m \lim_n (n-m) = \infty).$$

`But how about` $\mathfrak{Z}(R)$ `and` $\mathfrak{Z}(L)$?

Theorem 1.1 (Hindman-Davenport-Strauss)**.** *$\mathfrak{Z}(\beta G) = G$ and $\mathfrak{Z}(G^*) = \varnothing$.*

Sketch: For simplicity assume that G is countable.
van Douwen decomposition $v\mathfrak{D} = \{I\}$ of G^*: Partition G^* into closed left ideals $G^* = \bigcup I$.

$$x \in G^* \Longrightarrow x \in I \text{ for some } I \in v\mathfrak{D} \Longrightarrow yx \in I \text{ but } xy \in J \text{ for } y \in J \in v\mathfrak{D} \text{ with } J\cap I = \varnothing$$
$$\Longrightarrow xy \neq yx \Longrightarrow x \notin \mathfrak{Z}(\beta G) \text{ and } x \notin \mathfrak{Z}(G^*) \Longrightarrow$$
$$\mathfrak{Z}(\beta G) = G \text{ and } \mathfrak{Z}(G^*) = \varnothing.$$

A point p is right (left) cancellable in βG when

$$yp = zp \; (py = pz) \Longleftrightarrow y = z.$$

Theorem 1.2. *If a left (right) ideal L in βG (and so $L \subseteq G^*$) has a right (left) cancellable point p, then $\mathfrak{Z}(L) = \varnothing$.*

Sketch:

$$
\begin{aligned}
x \in \mathcal{Z}(L),\ y \in \beta G &\Longrightarrow (xy)p = x(yp) = (yp)x = y(px) = y(xp) = (yx)p \\
&\Longrightarrow xy = yx \Longrightarrow x \in \mathcal{Z}(\beta G) \Longrightarrow \\
&\mathcal{Z}(L) = \varnothing.
\end{aligned}
$$

Theorem 1.3. *$p \in \beta G$ right (left) cancellable $\Longrightarrow \mathcal{Z}(\beta G p) = \mathcal{Z}(G^* p) = \varnothing$ and $\mathcal{Z}(p\beta G) = \mathcal{Z}(pG^*) = \varnothing$.*

(Note that $\beta G p$ and $G^* p$ are nowhere dense sets.)

Theorem 1.4. *A left ideal L with a non-empty interior in βG has an empty centre.*

Sketch: A non-empty interior gives $T \subseteq A \subseteq G$ with

$$\overline{T} \subseteq \overline{A} \subseteq L$$

and T thin ($|sT \cap tT| < |G|$ whenever $s \neq t$ in G). Since $\overline{T}$ consists of right cancellable points, the claim follows.

`How about when` $p^2 = p$, `is it true that` $\mathcal{Z}(\beta G p) = \varnothing$?

1.1. **Algebra in βG.**

In a semigroup S, an element p is an *idempotent* if $pp = p^2 = p$.

The left and right preorderings of idempotents in a semigroup S (and so in βG), induced by the inclusion relation on principal left and right ideals, are given by

$$p \leq_L q \iff pq = p \iff Sp \subseteq Sq$$

$$p \leq_R q \iff qp = p \iff pS \subseteq qS.$$

In any compact right topological semigroup S, in particular when $S = \beta G$ or G^*,

- idempotents exist in ZFC (Numakura 1952, Wallace 1952-1953-1955, and Ellis 1969).
- left minimal and right minimal idempotents are the same, and exist in ZFC.
- right maximal idempotents exist in ZFC [Ruppert, 2.7-2.9].
- S has a smallest (2-sided) ideal $K(S)$.
- If $E(K(S))$ is the set of idempotents in $K(S)$, then $p \in E(K(S))$ if and only if it is minimal.
- Each of the families

  $\{Sp : p \in E(K(S))\},\ \{pS : p \in E(K(S))\},\ \{pSp : p \in E(K(S))\}$

  partitions $K(S)$, and they are, respectively, the set of minimal left ideals of S, the set of minimal right ideals of S, and the set of maximal subgroups of $K(S)$.

- There are $2^{2^{|G|}}$ many idempotents in G^* [HS].
- There are $2^{2^{|G|}}$ many minimal idempotents in G^* [HS].
- βG (and so G^*) contains $2^{2^{|G|}}$ minimal left ideals [HS].
- βG (and so G^*) contains $2^{2^{|G|}}$ minimal right ideals [Zelenyuk, 2009] and [Filali-Galindo for $G^{\mathcal{LUC}}$, preprint]. [HS, at least 2^c], [Baker-Milnes for $G^{\mathcal{LUC}}$, at least 2^c].
- Each minimal right ideal and each minimal left ideal contains $2^{2^{|G|}}$ many idempotents [Filali-Galindo for $G^{\mathcal{LUC}}$, preprint]. [HS, at least 2^c].
- When G is countable, there are 2^c non-minimal idempotents in $\overline{K(\beta G)}$ [HS, Theorem 8.65].

- There are 2^c many right maximal idempotents in $\mathbb{N}^*$ [HS, Theorem 9.1].
- Right maximal idempotents are not in $K(\beta G)$, so minimal idempotents cannot be right maximal [HS, Theorem 9.8 or Exercise 9.1.4].
- Left maximal idempotents (which are minimal) exist in βG in ZFC when G is countable [Zelenyuk, 2014].

Theorem 1.5 (HS). *If G is countable (not necessarily abelian) and p is a non-minimal idempotent in βG, then $\mathcal{Z}(p\beta Gp) \subseteq Gp$.*

Sketch: Beautiful long proof, based on

$$\begin{aligned} p \text{ non-minimal } &\Longrightarrow p \notin K(\beta G) \Longrightarrow \exists B \subseteq G \text{ such that} \\ &\qquad \beta Gr_1p \cap \beta Gr_2p = \varnothing \text{ for } r_1, \neq r_2 \in B^* \text{ and} \\ &\qquad rp \text{ is right cancellable in } \beta G \text{ for every } r \in B^*. \end{aligned}$$

Corollary 1.6 (HS). *If G is countable and abelian and $p \in \beta G$ is non-minimal, then*

$$\mathcal{Z}(p\beta Gp) = \mathcal{Z}(pG^*p) = Gp.$$

Sketch: Note first that $p\beta Gp = pG^*p$ since $psp = p(sp)p \in pG^*p$ for any $s \in G$. Now if $s \in G$, then

$$sp = spp = psp \in p\beta Gp,$$

and so for any $y = pxp \in p\beta Gp$,

$$(sp)y = (sp)(pxp) = spxp = sy = ys = (pxp)s = (pxp)ps = (pxp)(sp) = y(sp),$$

i.e., $Gp \subseteq \mathcal{Z}(p\beta Gp) = \mathcal{Z}(pG^*p)$.

Corollary 1.7. *Let G be countable and L a left ideal in βG not contained in $K(\beta G)$. Then $\mathcal{Z}(L) = \varnothing$.*

Sketch: Let $x \in L \setminus K(\beta G)$. By [HS, Theorem 6.56], there exists $r \in G^*$ such that rx is right cancellable. Since $rx \in L$, $\mathcal{Z}(L) = \varnothing$ by Theorem 1.2.

`What happens when the idempotent is minimal?`

Theorem 1.8. $\mathcal{Z}(K(\beta G)) = \varnothing$.

Sketch: If $xp \in K(\beta G)$ for some $x \in \beta G$ and an idempotent $p \in K(\beta G)$, then $(xp)q \in \beta Gq$ and $q(xp) \in \beta Gp$ for any other idempotent $q \in K(\beta G)$ with $\beta Gp \cap \beta Gq = \varnothing$.

In fact, in the same way, the centre of each of the left ideal $\beta Gp \cup \beta Gq$ and the right ideal $p\beta G \cup q\beta G$ is empty whenever p and q are not in the same ideal.

SUMMARY:

Let L be a proper left ideal in βG.

- If $L \nsubseteq K(\beta G) \Longrightarrow \mathcal{Z}(L) = \varnothing$.
- $L \subseteq K(\beta G)$ and $L = \bigcup_{p\in S} \beta Gp$, where $S \subseteq K(\beta G)$ and $|S| > 1$ $\Longrightarrow \mathcal{Z}(L) = \varnothing$.

- `If` $L = \beta Gp$ `for some minimal idempotent` $p \Longrightarrow$??.
- `If` $R = p\beta G$ `for some minimal idempotent` $p \Longrightarrow$??.
- `If` $M = p\beta Gp$ `for some minimal idempotent` $p \Longrightarrow$??.

It is known that each maximal group in βG, namely $p\beta Gp$ (and so each βGp and $p\beta G$) for p a minimal idempotent contains a free group on 2^c generators. So these are very non-commutative subsemigroups of βG.

(The proof works for discrete commutative semigroups)

Theorem 1.9. *Let G be abelian and p be an idempotent in G^*. Let Gp has the topology induced by βG. Then Gp is an extremely disconnected, Hausdorff, non-locally compact semitopological group, and*

(J.W. Baker, 1979) $$\beta(Gp) = \overline{Gp} = (\overline{G})p = (\beta G)p.$$

Proof. That Gp is a group is clear. To prove that Gp is extremely disconnected, let $A \cap Gp$ and $B \cap Gp$ be two disjoint open sets in Gp with A and B open in βG. Define f on G by $f(s) = 1$ is $sp \in A$, $f(sp) = -1$ if $sp \in B$, and $f(s) = 0$ otherwise. Extend f to a continuous function $\tilde{f}$ on βG. Note now that if $x \in A \cap Gp$, then $x = xp$ and $\tilde{f}(x) = 1$, and so $\tilde{f}(x) = 1$ for every $x \in cl_{Gp}(A \cap Gp) = \overline{A} \cap Gp$. Similarly $\tilde{f}(x) = -1$ if $x \in cl_{Gp}(B \cap Gp$. Therefore, $cl_{Gp}(A \cap Gp) \cap cl_{Gp}(B \cap Gp) = \varnothing$.

To show that Gp is not locally compact, we claim first that a subset Ep in Gp is closed in βG if and only if E is finite. Suppose otherwise that E is infinite and $C \subseteq E$ be countable. If Ep were closed, we would get

$$Ep = Cp \cup (E \setminus C) = \overline{Cp} \cup \overline{(E \setminus C)p},$$

where Cp and $(E \setminus C)p$ are disjoint by Veech's Theorem (Ellis Theoem since G is discrte). So, arguing as previously, we see that $\overline{Cp}$ and $\overline{(E \setminus C)p}$ are also disjoint. Therefore, $\overline{Cp} = Cp$ and $\overline{(E \setminus C)p} = (E \setminus C)p$. In particular, Cp is closed in βG. This is not possible since the cardinality of a closed set in βG must be at least 2^c by [GJ76] Gillman and Jersion. 9.12, or see [4, Theorem 3.59], while by Veech's Theorem, Cp is countable.

It is now straightforward that a basic (closed) neighbourhood $\overline{E} \cap Gp$ of p in Gp is not closed in βG and so it cannot be compact (in either βG or Gp). To see this, use the fact that p is an idempotent and pick $F \subseteq G$ with $p \in \overline{F}$ and $Fp \subseteq \overline{E} \cap Gp$. Use Veech's Theorem to see that $\overline{E} \cap Gp = E'p$ for some infinite subset E' in G, and apply the above.

Consider now $\overline{P} \cap Gp$, where $\overline{P}$ is any neighbourhood of p in βG. Pick $Q \subseteq G$ with $p \in \overline{Q}$ and $Qp \subseteq \overline{P}$. Then $|Qp| = |Q|$ by Veech's Theorem (Ellis Theorem since G is discrete) and $Qp \subseteq \overline{P} \cap Gp$.

We claim that Gp is properly contained in G^*p and βG....

Baker's argument:

For a given continuous bounded function f on Gp, define g on G by $g(s) = f(sp)$. Then extend g to a continuous function $\tilde{g}$ on βG. The functions $\tilde{g}$ and f agree on Gp since

$$\tilde{g}(sp) = \lim_\alpha g(sp_\alpha) = \lim_\alpha f(sp_\alpha p) = f(spp) = f(sp) \quad \text{for every} \quad s \in G.$$

Since every continuous bounded function on Gp extends continuously to $\overline{Gp} = (\overline{G})p = (\beta G)p$, we see that $\beta(Gp)$ and the left ideal $(\beta G)p$ in βG are the same. □

A topology on G induced by idempotents in G^*

Let G be an infinite group with identity e, and let p be an idempotent in G^*. We put

$$\tau_p = \{P_e = P \cup \{e\} \subseteq G : p \in \overline{P}\}.$$

Then (G, τ_p) is a Hausdorff (due to Veech-Ellis Theorem, or apply directly the 3-set lemma) left topological group. We denote (G, τ_p) by $G(p)$. If G is abelian, then $G(p)$ is a semitopological group, but not necessarily a *topological* group.

Proposition 1.10. *Let G be discrete with identity e. Let $p \in \beta G$ be an idempotent. Then the map $r_p\colon s \mapsto sp$ from $G(p)$ onto Gp is a continuous isomorphism.*

Proof. We prove the continuity at the identity e. The continuity at any other point in G will follow from $G \subseteq \mathcal{Z}(\beta G)$. Let $P \subseteq G$ such that $p \in \overline{P}$ (i.e., $\overline{P}$ is a neighbourhood of p in βG). Since $p = pp \in \overline{A}$ and the mapping $x \mapsto xp\colon \beta G \to \beta G$ is continuous, pick $Q \subseteq G$ such that $p \in \overline{Q}$ and $\overline{Q}p \subseteq \overline{P}$. Then

$$r_p(Q_e) = Q_e p \subseteq \overline{Q}p \subseteq \overline{P},$$

as wanted.

By Veech's Theorem (see e.g. [**?**, Theorem 4.8.9]), the mapping $r_p\colon s \mapsto sp$ is injective. Hence, the mapping r_p is a continuous isomorphism from $G(p)$ onto Gp.

□

Definition 1.11. *An idempotent $p \in G^*$ is*

(i) *strongly right maximal when the equation $xp = p$ is satisfied only for $x = p$ in βG.*
(ii) *strongly left maximal when the equation $px = p$ is satisfied only for $x = p$ in βG.*

- Strongly right maximal idempotents exist in ZFC [Protasov].
- Strongly right maximal are not in $K(\beta G)$, so minimal idempotents cannot be right maximal [HS].
- Minimal idempotents can be left maximal [Zelenyuk, 2014].

Theorem 1.12 (HS, Theorem 9.15 for example)**.** *Let G be an abelian discrete group with identity e and let p be an idempotent in G^*. Then TFAE*

(i) *τ_p is regular.*
(ii) *p is strongly right maximal.*
(iii) *$G(p)$ and Gp are isomorphic and homeomorphic.*

So here we have $G(p)$ as a semitopological group such that $\beta(G(p))$ identified with the left ideal βGp of βG and

$$\mathcal{Z}(\beta(G(p)) = \mathcal{Z}(\beta Gp) = \varnothing.$$

Let G be abelian and for a subset X of G, let

$$FP(X) = \{\prod_{s\in F} s : F \subseteq X, \text{ finite}\}.$$

It is well known that, if p is an idempotent, then every $P \subseteq G$ with $p \in \overline{P}$, contains a set of the form $FP(X)$ for some infinite subset X of G. However, we do not normally expect that $p \in \overline{FP(X)}$. So the following definition states itself:

Definition 1.13. *Let G be abelian. An element $p \in \beta G$ is strongly summable when for every $P \subseteq G$ with $p \in \overline{P}$, there exists $X \subseteq G$ such that $p \in \overline{FP(X)} \subseteq \overline{P}$.*

- Strongly summable elements in βG are idempotents.
- Strongly summable are not in $\overline{K(\beta\mathbb{N})}$ [HS, Theorem 12.21].
- Their existence is established under Martin's Axiom.
- their existence cannot be established in ZFC.
- Strongly summable $\Longrightarrow$ [HS, Theorem 12.39 (in $\beta\mathbb{N}$)] Strongly right maximal $\Longrightarrow$ Right maximal.
- Right maximal $\not\Rightarrow$ Strongly right maximal [Zelenyuk, 2016].

Theorem 1.14 (Protasov)**.** *Let G be a countable Boolean group, and $p \in \beta G$ be strongly summable. Then $G(p)$ is a (maximal) topological group.*

Now under MA, we have $G(p)$ as a topological group such that $\beta(G(p))$ identified with the left ideal βGp of βG and

$$\mathcal{Z}(\beta(G(p)) = \mathcal{Z}(\beta Gp) = \varnothing.$$

Theorem 1.15 (F-Vedenjuoksu 2010)**.** *Let G be a topological group which is not a P-group. The Stone Čech compactification βG of G is a right topological semigroup with $G \subseteq \mathcal{Z}(\beta G)$ if and only if G is pseudocompact.*

By Protasov, $G(p)$ is not totally bounded unless p is minimal.

2. PART II

G a locally compact group.
$G^{\mathcal{LUC}}$ is a compact right topological semigroup.

3. PART III

G a locally compact group.
$L^1(G)^{**}$ and $\mathcal{LUC}(G)^*$ are Banach algebras with the first Arens product.

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