G. Racher (Salzburg, Austria) 13 slides

# Weak Compactness and Property (T)

1. Theorems A and B for $L^1(G)$
2. Translation invariance and property (T)
3. Proposition A for $C^*(G)$
4. Proposition B for $C^*(G)$
5. Isolated points in the spectrum of a $C^*$-algebra. Corollaries A and B

Keywords: group $C^*$-algebra,
enveloping von Neumann algebra.
Translation invariance.
Weakly compact operators.
Compactness and property (T) of groups:
T = тривиальное представление

## 1. Theorems A and B

$G$ locally compact group, left invariant Haar measure

$L^1(G)$ involutive Banach algebra with convolution $*$

$s \in G$, $f \in L^1(G)$:

$$(L_s f)(x) = f(s^{-1}x) \qquad \text{left translate of } f$$

Theorem A (Sakai, 1964). – $G$ compact iff for some $f \neq 0$ in $L^1(G)$ left multiplication by $f$ is a weakly compact operator on $L^1(G)$,

$$L_f : L^1(G) \longrightarrow L^1(G), \qquad L_f(g) = f * g \qquad g \in L^1(G). \blacksquare$$

$X$ Banach space, $\mathcal{O}X$ unit ball, $T : X \longrightarrow X$ bdd linear operator

$$T \text{ weakly compact} \iff \overline{T(\mathcal{O}X)} \subset X \text{ weakly compact}.$$

$T^{**} : X^{**} \longrightarrow X^{**}$, $\quad X \subset X^{**}$:

$$T \text{ weakly compact} \iff T^{**}(X^{**}) \subset X.$$

$$\dim T(X) < \infty \implies \overline{T(\mathcal{O}X)} \text{ norm compact} \implies \overline{T(\mathcal{O}X)} \text{ weakly compact}$$

Question A. What happens to G if one replaces $L^1(G)$ by $C^*(G)$ ?

Theorem B (Liu - van Rooij, 1974). — There exists a nonzero bounded translation invariant linear mapping $\phi$,

(i) $\phi : L^\infty(G) \longrightarrow L^1(G)^{**}$ iff G amenable,

(ii) $\phi : L^\infty(G) \longrightarrow L^1(G)$ iff G compact.

$$\phi \circ L_s = L_s \circ \phi \qquad \forall s \in G.$$

Question B. What happens to G if one replaces $L^1(G)$ by $C^*(G)$ and $L^\infty(G)$ by $C^*(G)^* = B(G)$, the Fourier-Stieltjes algebra of G ?

## 2. Translation invariance and property (T)

$\omega$ universal continuous unitary representation of $G$

$$\omega : G \longrightarrow \mathcal{L}(h_\omega)$$

$\omega$ direct sum of sufficiently many cont unit rep's of $G$.

$$\omega : L^1(G) \longrightarrow \mathcal{L}(h_\omega), \qquad \omega(f) = \int f(x)\,\omega(x)\,dx \qquad f \in L^1(G)$$

$$\Longrightarrow \quad \omega(L^1(G)) \subset \mathcal{L}(h_\omega) \quad \text{involutive subalgebra}$$

$$C^*(G) = \overline{\overline{\omega(L^1(G))}} \qquad \text{norm closure}$$
$$C^*(G)^\sim = \overline{\omega(L^1(G))} \qquad \text{weak operator closure}$$

$C^*(G)$ $C^*$-algebra of $G$

$C^*(G)^\sim$ enveloping von Neumann algebra of $G$, $1 = 1_{h_\omega} \in C^*(G)^\sim$.

$$C^*(G)^\sim \cong C^*(G)^{**} \qquad \text{isometric isomorph.}$$

$s \in G$, $T \in C^*(G)^\sim$:

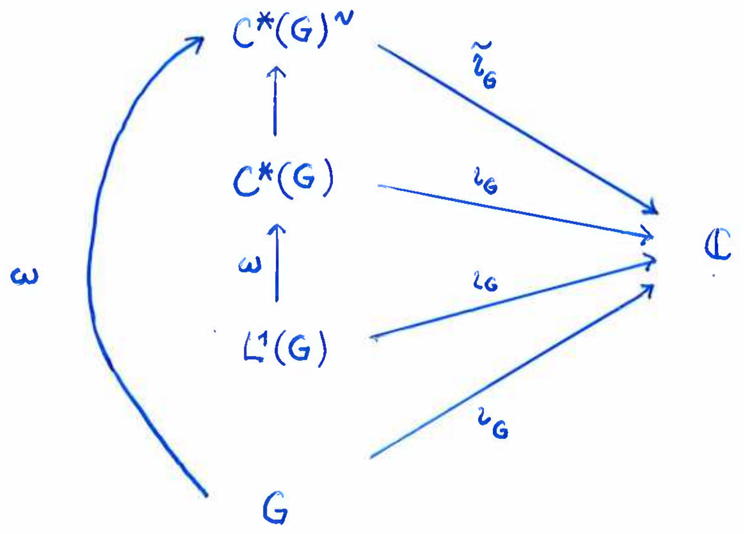
$$L_s T = \omega(s) \circ T \qquad \text{left translate of } T$$
$$R_s T = T \circ \omega(s) \qquad \text{right translate of } T$$

## The trivial representation $\iota_G$ of dimension one

$$\iota_G : G \longrightarrow \mathcal{L}(\mathbb{C}) = \mathbb{C}, \quad \iota_G(x) = 1 \qquad (x \in G)$$

$$\iota_G : L^1(G) \longrightarrow \mathbb{C}, \quad \iota_G(f) = \int_G f(x)\,dx \qquad (f \in L^1(G))$$

$\iota_G$ hermitian multiplicative linear functional on $L^1(G)$, extends to multiplicative linear functionals on $C^*(G)$ and $C^*(G)^\sim$




$$\tilde{\iota}_G(\omega(s)) = \iota_G(s) = 1 \qquad (s \in G)$$

$\tilde{\iota}_G$ σ-weakly (weak*) continuous, multiplicative linear functional

$\Longrightarrow$ $\ker \tilde{\iota}_G = \{ T \in C^*(G)^\sim : \tilde{\iota}_G(T) = 0 \}$ σ-weakly closed twosided ideal.

[Takesaki, vol 1, p. 76]: there exists a unique projection $P_G$ in the center of $C^*(G)^\sim$ s.t.

$$\ker \tilde{\iota}_G = \{ T \in C^*(G)^\sim : \tilde{\iota}_G(T) = 0 \} = C^*(G)^\sim (1 - P_G)$$
$$= \{ T \in C^*(G)^\sim : T P_G = P_G T = 0 \},$$

$$C^*(G)^\sim = C^*(G)^\sim P_G \oplus C^*(G)^\sim (1 - P_G)$$
$$= C^*(G)^\sim P_G \oplus \ker \tilde{\iota}_G .$$

$P_G$ support projection of representation $\tau_G$, [Jak, vol 1, p. 126].

<u>Lemma 1.</u> – The central projection $P_G \in C^*(G)^\sim$ has the following properties:

(i) $\tilde{\tau}_G(P_G) = 1$;

(ii) $P_G C^*(G)^\sim = P_G C^*(G)^\sim P_G = \mathbb{C} P_G$; minimal projection

(iii) $L_s P_G = P_G R_s = P_G \quad (s \in G)$; translat. invariant.

■

<u>Lemma 2.</u> — Any translation invariant operator $T \in C^*(G)^\sim$ has the form

$$T = \tilde{\tau}_G(T) P_G .$$

■

<u>Definition.</u> — G has property (T) if there exists a nonzero translation invariant operator in $C^*(G)$ — rather than in $C^*(G)^\sim = C^*(G)^{**}$.

■

**Theorem** (Každan; Valette, Losert). — The following are equivalent:

a) $\iota_G$ is an isolated point in the spectrum of $C^*(G)$;

b) $P_G \in C^*(G)$;

c) there exists in $C^*(G)$ a translation invariant operator $\neq 0$:

$$T \in C^*(G), \quad T = \omega(s)T \quad \forall\, s \in G.$$

Proof $b \to c$: $P_G \neq 0$, $P_G = \omega(s) P_G$ $(s \in G)$ by Lemma 1;

$c \to b$: $T \in C^*(G)$, $T \neq 0$, $T = \omega(s)T$ $(s \in G)$. By Lemma 2,

$$T = \tilde{\iota}_G(T)\, P_G \implies \tilde{\iota}_G(T) \neq 0 \implies P_G = \frac{1}{\tilde{\iota}_G(T)}\, T \in C^*(G).$$ ∎

**Scholium.** $L^1(G) \subset C^*(G) \subset C^*(G)^{**}$

Translation invariant elements $\neq 0$ in

- $C^*(G)^{**}$ — for all $G$
- $C^*(G)$ $\iff$ $G$ has $(T)$
- $L^1(G)$ $\iff$ $G$ compact

∎

## 3. Proposition A

**Proposition A.** — $G$ has property (T) if and only if there is an element $T \in C^*(G)$ such that $\tau_G(T) \neq 0$ and left multiplication $L_T$ by $T$ is weakly compact on $C^*(G)$:

$$\tau_G(T) \neq 0, \quad L_T : C^*(G) \longrightarrow C^*(G), \quad L_T(S) = TS \quad (S \in C^*(G)), \text{ w.c.}$$

■

Two remarks.

1) $G$ is compact iff for all $T \in C^*(G)$ the operator $L_T$ is w.c. on $C^*(G)$:

$$G \text{ compact} \iff C^*(G) \triangleleft C^*(G)^{**}. \quad \text{Reference?}$$

2) Prop. A does not hold if $L_T$ is w.c. only for some $T \neq 0$.

Example. $G = SL_2(\mathbb{R})$ admits integrable representations $\pi$. By A. Valette, any of those defines a minimal projection $p_\pi \neq 0$ in $L^1(G)$ and hence in $C^*(G)$:

$$p_\pi \in C^*(G), \quad p_\pi C^*(G) p_\pi = \mathbb{C} p_\pi,$$

such that, by K. Ylinen, the operator $L_{p_\pi}$ is w.c. on $C^*(G)$, while $SL_2(\mathbb{R})$ has not (T).

## Proof of prop A.

G has property (T):

$$P_G \in C^*(G),\ \tau_G(P_G) = 1,$$

$$L_{P_G}(C^*(G)) = P_G C^*(G) = P_G C^*(G) P_G = \mathbb{C}.P_G$$

$\Rightarrow L_{P_G}$ one-dimensional $\Rightarrow$ compact $\Rightarrow$ w.c.

Converse:

$$T \in C^*(G),\ \tau_G(T) \neq 0,\ L_T : C^*(G) \longrightarrow C^*(G) \text{ w.c.}$$

$$\Rightarrow (L_T)^{**}(\mathrm{O}\, C^*(G)^{**}) \subset C^*(G) \text{ rel. w.c.}$$

$$\{L_T^{**}(\omega(s)) : s \in G\} = \{T\omega(s) : s \in G\} \text{ rel. w.c.}$$

$$\Rightarrow C_G(T) = \overline{\left\{\sum c_n T\circ\omega(s_n) : c_n \geq 0\ \sum c_n = 1,\ s_n \in G\right\}}$$

weakly compact in $C^*(G)$, convex, invariant under the group of linear isometries

$$R_s = R_{\omega(s)},\ s \in G.$$

Ryll-Nardzewski: there exists

$$T_0 \in C_G(T),\quad T_0 = T_0\,\omega(s) \quad \forall s \in G,$$

$$\tau_G(T_0) = \lim \tau_G\left(\sum c_n T\omega(s_n)\right) = \tau_G(T),$$

$$\tau_G(T) \neq 0 \Rightarrow \tau_G(T_0) \neq 0 \Rightarrow T_0 \neq 0$$

$T_0 \in C^*(G)$, $T_0 \neq 0$, $T_0$ invariant:

G has property (T). $\blacksquare$

## 4. Proposition B

Proposition B. — G has property (T) if and only if there is a bounded translation invariant linear mapping $\phi$ from $B(G)$ into $C^*(G)$ such that $z_G(\phi(u)) \neq 0$ for some $u \in B(G)$:

$$z_G \circ \phi \neq 0, \quad \phi : B(G) \longrightarrow C^*(G), \quad \phi(L_s^* u) = L_s(\phi u), \qquad u \in B(G),\ s \in G.$$

■

Lemma 3. — For any $u \in B(G)$ the sets of left and right translates of $u$ are relatively weakly compact in $B(G)$:

$$\{L_s^* u : s \in G\},\ \{R_s^* u : s \in G\} \text{ rel. weakly compact in } B(G).$$

■

## 5. Isolated points in the spectrum of a $C^*$-algebra

$\mathfrak{A}$ $C^*$-algebra

$\hat{\mathfrak{A}}$ space of equivalence classes of all nonzero irreducible representations of $\mathfrak{A}$ with a certain topology. $\hat{\mathfrak{A}}$ spectrum of $\mathfrak{A}$.

$X$ complex Banach space, $T: X \longrightarrow X$ bounded linear operator, $\lambda$ a point in the spectrum of $T$:

$$\lambda \neq 0, \quad T: X \longrightarrow X \text{ compact} \implies \{\lambda\} \text{ open in spectrum } (T).$$

Proposition C. — Let $\mathfrak{A}$ be any $C^*$-algebra, $\pi$ an element of $\hat{\mathfrak{A}}$, and $T$ an element of $\mathfrak{A}$ such that $\pi(T) \neq 0$. If $L_T$ is weakly compact on $\mathfrak{A}$, then $\{\pi\}$ is open in $\hat{\mathfrak{A}}$:

$$\pi(T) \neq 0, \quad L_T: \mathfrak{A} \to \mathfrak{A} \text{ w.c.} \implies \{\pi\} \text{ open in } \hat{\mathfrak{A}}.$$

Proof. Let $\pi$ be an irred. rep. of class $\pi \in \hat{\mathfrak{A}}$, $T \in \mathfrak{A}$ s.t.

$\pi(T) \neq 0$, $L_T$ weakly compact on $\mathfrak{A}$

(Ylinen) $\implies$ $L_T R_T$ compact on $\mathfrak{A}$.

Ylinen: $\forall \varepsilon > 0$ $\exists\, c_1, \ldots, c_n \in \mathbb{C}$, $\exists\, E_1, \ldots, E_n \in \mathfrak{A}$ projections s.t.

$$\dim E_i \mathfrak{A} E_i < \infty \qquad (1 \leq i \leq n),$$

$$\|\pi(T) - \sum c_i \pi(E_i)\| \leq \|T - \sum c_i E_i\| < \varepsilon.$$

$\pi(T) \neq 0 \implies \exists\, E \in \mathfrak{A}$ proj, $\dim E\mathfrak{A}E < \infty$, $\pi(E) \neq 0$.

Ylinen: $\exists\, F_1, \ldots, F_m \in \mathfrak{A}$ project,

$$\dim F_k \mathfrak{A} F_k = 1 \qquad (1 \leq k \leq m)$$

$$E = F_1 + \ldots + F_m$$

$$\pi(E) = \pi(F_1) + \ldots + \pi(F_m) \neq 0.$$

$\Longrightarrow \exists F \in \mathcal{A}$ project, $\dim F\mathcal{A}F = 1$ (minimal project),
$\pi(F) \neq 0$.

$\rho$ irred rep of $\mathcal{A}$, $\rho(F) \neq 0 \Longrightarrow \rho \simeq \pi$, $\rho = \pi$ in $\hat{\mathcal{A}}$

$\Longrightarrow \hat{\mathcal{A}} \setminus \{\pi\} = \{\sigma \in \hat{\mathcal{A}} : \sigma(F) = 0\}$ closed in $\hat{\mathcal{A}}$

$\Longrightarrow \{\pi\}$ open in $\hat{\mathcal{A}}$ (Barnes, Valette). ■

Corollary A. – $G$ locally compact group, $\pi$ finite dim. irreducible c.u. representation of $G$, and $T \in C^*(G)$:

$\pi(T) \neq 0$, $L_T : C^*(G) \longrightarrow C^*(G)$ w.c. $\Longrightarrow$ $G$ has (T). ■

Proof. Prop C with $\mathcal{A} = C^*(G)$ and a theorem of Wang.

Corollary B. – $G$ locally compact group, $\phi : B(G) \longrightarrow C^*(G)$ bdd translation invariant linear mapping, and $\pi$ finite dim. irreducible c.u. representation of $G$:

$\pi \circ \phi \neq 0$, $\phi : B(G) \longrightarrow C^*(G)$, $\phi(L_s^* u) = L_s(\phi u)$ $(u \in B(G), s \in G)$

$\Longrightarrow$ $G$ has property (T). ■

## Short list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