Spectrally Additive Maps on Banach Algebras

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Joint work with Miles Askes and Rudi Brits

Banach Algebras and Applications 2022
Granada, Spain

Tuesday, 19 July 2022

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$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\}\,;$$

and we use $\sigma'(x) = \sigma(x) \setminus \{0\}$ to denote its **nonzero spectrum**.

Some Terminology

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- A two-sided ideal J of a Banach algebra is said to be **essential** if it has a nonzero intersection with every nonzero ideal in the Banach algebra.
- If the Banach algebra is semisimple, then this is equivalent to saying that the condition $xJ = \{\mathbf{0}\}$ implies $x = \mathbf{0}$.
- A Banach algebra is said to be **prime** if and only if every nonzero two-sided ideal is essential.

Some Reminders about the Socle

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- The **socle** of A, denoted $\mathrm{soc}(A)$, is the collection of all finite sums formed by using elements taken from any of the minimal left (or right) ideals of A.
- If the Banach algebra lacks minimal one-sided ideals, then its socle is trivial i.e. $\{\mathbf{0}\}$.

Introduction

Theorem (Gleason-Kahane-Żelazko, 1967, 1968)

If a linear functional $f : A \to \mathbb{C}$ maps every $x \in A$ into its spectrum $\sigma(x)$, then f is multiplicative.

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Key observation: Since f maps invertible elements of A to invertible elements of $\mathbb{C}$, it is automatically well-behaved with respect to multiplication.

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The GKZ Theorem (among others) motivated I. Kaplansky to formulate the problem of determining the conditions under which a unital linear invertibility preserving map ϕ between two algebras must be a Jordan-homomorphism.

A **Jordan-homomorphism** ϕ is a linear map with the property that

$$\phi(x^2) = \phi(x)^2 \text{ for all } x \text{ in its domain.}$$

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Since $0 \in \sigma(a)$ if and only if $a \notin G(A)$, we can also work our way back to invertibility preservation.

Introduction

As a result of this connection to Kaplansky's problem, over the years there has been a surge of literature on linear maps preserving or compressing various parts of the spectrum, or preserving the spectral radius.

Introduction

Theorem (Jafarian-Sourour, 1986)

Any surjective and linear spectrum preserving map $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ satisfies the following: Either

(a) there exists an invertible bounded linear operator $U : X \to Y$ such that $\phi(T) = UTU^{-1}$ for each $T \in \mathcal{L}(X)$, or

(b) there exists an invertible bounded linear operator $V : X' \to Y$ such that $\phi(T) = VT^*V^{-1}$ for each $T \in \mathcal{L}(X)$.

Hence, ϕ is either an (algebra) isomorphism or anti-isomorphism.

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This result was then extended in two directions which is of relevance here.

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Theorem (Omladič-Šemrl, 1991)

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Corollary (Aupetit-Mouton, 1994)

If $\phi : A \to B$ is a surjective linear spectrum-preserving map between semisimple Banach algebras and and B is a prime Banach algebra with minimal ideals, then ϕ is an isomorphism or an anti-isomorphism.

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Finite Rank Elements and the Socle

- Following B. Aupetit and H. du T. Mouton, we define the **rank** of $a \in A$ by

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- This definition of rank generalizes the classical rank of an operator.
- Moreover, it satisfies all the classical properties such as subadditivity, lower-semicontinuity, and so forth.
- If A is semisimple then $\mathrm{soc}(A) = \{a \in A : \mathrm{rank}^{\sigma}(a) < \infty\}$. (Aupetit-Mouton, 1996)

Trace, Determinant and Multiplicity

- For $a \in \mathrm{soc}(A)$, Aupetit and Mouton define the **trace** of a and the **determinant** of $\mathbf{1} + a$ by

$$\mathrm{tr}(a) = \sum_{\alpha \in \sigma(a)} \alpha m(\alpha, a)$$

and

$$\det(\mathbf{1} + a) = \prod_{\alpha \in \sigma(a)} (1 + \alpha)^{m(\alpha, a)},$$

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- Here $m(\alpha, a)$ is the **multiplicity** of a at α.

Rank One Elements and the Trace

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Theorem (Aupetit-Mouton, 1996)

Let A be semisimple and let $a \in A$. If $\operatorname{tr}(ax) = 0$ for each $x \in \operatorname{soc}(A)$, then $a \operatorname{soc}(A) = \{\mathbf{0}\}$. Moreover, if $a \in \operatorname{soc}(A)$, then $a = \mathbf{0}$.

Spectrally Additive Maps – A Motivation

Theorem (Gleason-Kahane-Żelazko, 1967, 1968)

If a linear functional $f : A \to \mathbb{C}$ maps every $x \in A$ into its spectrum $\sigma(x)$, then f is multiplicative.

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Every functional f on A satisfying $f(x) + f(y) \in \sigma(x + y)$ for each $x, y \in A$ is linear and multiplicative.

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In view of this it seems quite natural to ask if it is possible to do the same with linear spectrum preserving maps?

More precisely, if a surjective map ϕ (with no linearity or even additivity assumed) only has the property that $\sigma(\phi(x) + \phi(y)) = \sigma(x + y)$ for each x, y in the domain of ϕ, is ϕ a Jordan-isomorphism?

Spectrally Additive Group Homomorphisms

Theorem (Askes-Brits-S., 2022)

Let A be semisimple and suppose that $\phi : A \to B$ is a surjective map with the property that $\sigma(x \pm y) = \sigma(\phi(x) \pm \phi(y))$ for all $x, y \in A$. Then:

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(a) $(\phi(\alpha x + \beta y) - \alpha\phi(x) - \beta\phi(y))\,\mathrm{soc}(B) = \{\mathbf{0}\}$ for all $x, y \in A$ and any $\alpha, \beta \in \mathbb{C}$.

(b) $\left(\phi(x^2) - \phi(x)^2\right)\mathrm{soc}(B) = \{\mathbf{0}\}$ for all $x \in A$.

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In particular, if either $\mathrm{soc}(A)$ or $\mathrm{soc}(B)$ are essential, then $\phi : A \to B$ is a continuous Jordan-isomorphism. Moreover, if either A or B is a prime algebra with a nonzero socle, then ϕ is continuous and is either an (algebra) isomorphism or anti-isomorphism.

An Additive Characterization of Finite Rank Elements

Theorem (Askes-Brits-S., 2022)

Suppose that A is semisimple. Let $a \in A$, let $m \in \mathbb{N}$, and let K be any subset of $\mathbb{C}$ with at least $m+1$ nonzero elements. Then the following are equivalent:

(a) $\mathrm{rank}^{\sigma}(a) = \sup_{x \in A} \#\sigma'(xa) = m$.

(b) $\sup_{y \in G(A)} \#\{t \in K : y + ta \notin G(A)\} = m$.

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(b) $\sup_{y \in G(A)} \# \{t \in K : y + ta \notin G(A)\} = m$.

With $K = \{-1, 1\}$ we readily obtain that a spectrally additive group homomorphism preserves rank one elements in both directions.

Spectrally Additive Group Homomorphisms

Indeed, notice that if $\phi : A \to B$ is a spectrally additive group homomorphism, then

$$0 \in \sigma(y \pm a) \iff 0 \in \sigma(\phi(y) \pm \phi(a));$$

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Corollary (Askes-Brits-S., 2022)

Suppose that A is semisimple. If $\phi : A \to B$ is a spectrally additive group homomorphism, then B is semisimple and $\phi(\mathscr{F}_1(A)) = \mathscr{F}_1(B)$.

Is Spectrally Additive enough?

Theorem (Askes-Brits-S., 2022)

Suppose that A is semisimple. Let $a \in A$, let $m \in \mathbb{N}$, and let K be any subset of $\mathbb{C}$ with at least $m+1$ nonzero elements. Then the following are equivalent:

(a) $\mathrm{rank}^{\sigma}(a) = \sup_{x \in A} \#\sigma'(xa) = m$.

(b) $\sup_{y \in G(A)} \# \{t \in K : y + ta \notin G(A)\} = m$.

Since the set K in the theorem must contain at least two nonzero elements to characterize rank one elements, if ϕ is only *spectrally additive*, then this result cannot be used directly to obtain that $\phi(\mathscr{F}_1(A)) = \mathscr{F}_1(B)$.

A New Characterization of Rank One Elements

Theorem (Havlicek-Šemrl, 2006)

Let H be an infinite dimensional Hilbert space. Then an operator $B \in \mathcal{L}(H)$ has rank one if and only if there exists some $R \in \mathcal{L}(H)$, with $R \neq \mathbf{0}$ and $R \neq B$, such that for every $X \in \mathcal{L}(H)$,

$$X + R \in G(\mathcal{L}(H)) \implies X \in G(\mathcal{L}(H)) \text{ or } X + B \in G(\mathcal{L}(H)).$$

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$$X + R \in G(\mathcal{L}(H)) \implies X \in G(\mathcal{L}(H)) \text{ or } X + B \in G(\mathcal{L}(H)).$$

Theorem (S., 2022)

Let A be semisimple and let $b \in A \setminus \{\mathbf{0}\}$. Then $\operatorname{rank}^{\sigma}(b) = 1$ if and only if there exists some $r \in A$ such that for any $x \in A$, we have

(i) $x \in G(A) \implies x + r \in G(A)$ or $x + b \in G(A)$;

(ii) $x + r \in G(A) \implies x \in G(A)$ or $x + b \in G(A)$;

(iii) $x \notin G(A)$ and $x + b \in G(A) \implies x + r \in G(A)$.

Spectrally Additive Maps

Proposition (Askes-Brits-S., 2022)

Suppose that A is semisimple and that $\phi : A \to B$ is a spectrally additive map. Then:

(a) For any $x, y \in A$, $x + y \in G(A) \iff \phi(x) + \phi(y) \in G(B)$.

(b) ϕ is spectrum-preserving and $\phi(\mathbf{0}) = \mathbf{0}$.

(c) $\phi(G(A)) = G(B)$.

(d) ϕ is injective.

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(f) $\phi(\lambda\mathbf{1} + x) = \lambda\mathbf{1} + \phi(x)$ for each $\lambda \in \mathbb{C}$ and $x \in A$.

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Corollary (S., 2022)

Suppose that A is semisimple. If $\phi : A \to B$ is a spectrally additive map, then $\phi(\mathscr{F}_1(A)) = \mathscr{F}_1(B)$.

Trace, Determinant and Multiplicity

(P1) If $a \in \mathrm{soc}(A)$ and $\alpha \in \sigma'(a)$, then $m(\alpha, a) = m(\lambda\alpha, \lambda a)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. (Braatvedt-Brits-S., 2015)

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(P5) For any $a, b \in \text{soc}(A)$ it follows that

$$\det((\mathbf{1} + a)(\mathbf{1} + b)) = \det(\mathbf{1} + a)\det(\mathbf{1} + b).$$

(Aupetit-Mouton, 1996)

Spectrally Additive Maps

Lemma (Askes-Brits-S., 2022)

Let $x \in \mathrm{soc}(A)$ with $\sigma'(x) = \{\alpha\}$ and $m(\alpha, x) = k$. Then $m(\alpha^2, x^2) = k$.

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Let $x \in \mathrm{soc}(A)$ with $\sigma'(x) = \{\alpha, -\alpha\}$ and $m(\alpha, x) = m(-\alpha, x) = k$. Then $m(\alpha^2, x^2) = 2k$.

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From this we are able to show, with a bit of effort, that

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From this we are able to show, with a bit of effort, that

$$\mathrm{tr}((a+b)^2) = \mathrm{tr}((\phi(a)+\phi(b))^2) \text{ for all } a, b \in \mathscr{F}_1(A).$$

Hence, from the linearity and cyclic property of the trace, we obtain that

$$\mathrm{tr}(a^2) + 2\,\mathrm{tr}(ab) + \mathrm{tr}(b^2) = \mathrm{tr}(\phi(a)^2) + 2\,\mathrm{tr}(\phi(a)\phi(b)) + \mathrm{tr}(\phi(b)^2)$$

for all $a, b \in \mathscr{F}_1(A)$.

Spectrally Additive Maps

Theorem (Askes-Brits-S., 2022)

Let A be semisimple and let $\phi : A \to B$ be a spectrally additive map. Then

$$\mathrm{tr}(ab) = \mathrm{tr}(\phi(a)\phi(b)) \text{ for all } a, b \in \mathscr{F}_1(A).$$

Thus, we conclude that ϕ is homogeneous on $\mathscr{F}_1(A)$.

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Lemma (Askes-Brits-S., 2022)

Let A be semisimple and let $\phi : A \to B$ be a spectrally additive map. Then

$$\mathrm{tr}(x^{-1}a) = \mathrm{tr}(\phi(x)^{-1}\phi(a)) \text{ for all } x \in G(A) \text{ and } a \in \mathscr{F}_1(A).$$

Spectrally Additive Maps

Hence,

$$\mathrm{tr}((-\lambda\mathbf{1}+x)^{-1}\,a) = \mathrm{tr}(\phi\,(-\lambda\mathbf{1}+x)^{-1}\,\phi(a)) = \mathrm{tr}((-\lambda\mathbf{1}+\phi(x))^{-1}\,\phi(a))$$

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Lemma (Askes-Brits-S., 2022)

Let A be semisimple and suppose that $\phi : A \to B$ is a spectrally additive map. Then for any $x \in A$ and $a \in \mathscr{F}_1(A)$, we have

$$\mathrm{tr}(x^n a) = \mathrm{tr}(\phi(x)^n \phi(a)) \text{ for all } n \in \mathbb{N}.$$

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$$\operatorname{tr}((-\lambda\mathbf{1}+x)^{-1}\,a)=\operatorname{tr}(\phi\,(-\lambda\mathbf{1}+x)^{-1}\,\phi(a))=\operatorname{tr}((-\lambda\mathbf{1}+\phi(x))^{-1}\,\phi(a))$$

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Lemma (Askes-Brits-S., 2022)

Let A be semisimple and suppose that $\phi : A \to B$ is a spectrally additive map. Then for any $x \in A$ and $a \in \mathscr{F}_1(A)$, we have

$$\operatorname{tr}(x^n a)=\operatorname{tr}(\phi(x)^n\phi(a)) \text{ for all } n \in \mathbb{N}.$$

From this one now deduces that

(a)' $\operatorname{tr}((\phi(\alpha x+\beta y)-\alpha\phi(x)-\beta\phi(y))\,b)=0$ for all $x, y \in A$, $\alpha, \beta \in \mathbb{C}$ and $b \in \operatorname{soc}(B)$;

(b)' $\operatorname{tr}(\left(\phi(x^2)-\phi(x)^2\right)b)=0$ for all $x \in A$ and $b \in \operatorname{soc}(B)$.

Spectrally Additive Maps

Theorem (S., 2022)

Let A be semisimple and suppose that $\phi : A \to B$ is a spectrally additive map. Then:

(a) $(\phi(\alpha x + \beta y) - \alpha\phi(x) - \beta\phi(y))\,\mathrm{soc}(B) = \{\mathbf{0}\}$ for all $x, y \in A$ and any $\alpha, \beta \in \mathbb{C}$.

(b) $\left(\phi(x^2) - \phi(x)^2\right)\mathrm{soc}(B) = \{\mathbf{0}\}$ for all $x \in A$.

In particular, if either $\mathrm{soc}(A)$ or $\mathrm{soc}(B)$ are essential, then $\phi : A \to B$ is a continuous Jordan-isomorphism.

Corollary (S., 2022)

Let A be semisimple and suppose that $\phi : A \to B$ is a spectrally additive map. If either A or B is a prime algebra with a nonzero socle, then ϕ is continuous and is either an (algebra) isomorphism or anti-isomorphism.

Acknowledgment

This work is based on the research supported in part by the National Research Foundation of South Africa (Grant Number: 129692).

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