A generalization of the Spectral Rank in Banach Algebras to Rings

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An additive characterization of finite rank elements in a Banach algebra

Let A be semisimple Banach Algebra, with multiplicative identity $\mathbf{1}$ and group of multiplicatively invertible elements $G(A)$. For $x \in A$ we denote by

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\},$$

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\} \text{ and } \sigma'(x) = \sigma(x) \setminus \{0\}$$

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Following Aupetit and Mouton, we define the *spectral rank* of an element $a \in A$ as

$$\mathrm{rank}^{\sigma}(a) = \sup_{x \in A} \#\sigma'(xa) = \sup_{x \in A} \#\{\lambda \in \mathbb{C} \setminus \{0\} : \lambda\mathbf{1} - xa \notin G(A)\}$$

if the supremum exists; otherwise $\mathrm{rank}^{\sigma}(a) = \infty$.

Theorem (Holomorphic Functional Calculus)

Let A be a Banach algebra and let $x \in A$. Suppose that Ω is an open set containing $\sigma(x)$ and that Γ is an arbitrary smooth contour included in Ω and surrounding $\sigma(x)$. Then the following mapping

$$f \to f(x) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda \mathbf{1} - x)^{-1} \, d\lambda$$

from $H(\Omega)$, the algebra of holomorphic functions on Γ, into A has the properties:

1. $(f_1 + f_2)(x) = f_1(x) + f_2(x)$,
2. $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = f_2(x) \cdot f_1(x)$,
3. $1(x) = \mathbf{1}$ *and* $I(x) = x$ *(where* $I(\lambda) = \lambda$*)*,
4. *if* (f_n) *converges to* f *uniformly on compact subsets of* Ω*, then*

$$f(x) = \lim f_n(x),$$

5. $\sigma(f(x)) = f(\sigma(x))$.

By making use of the holomorphic functional calculus, we obtain the following result:

Lemma

Let $a \in A$ and $m \in \mathbb{N}$. If $\#\sigma'(xa) \geq m$ for some $x \in A$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{C} - \{0\}$, then there exists some $y \in A$ such that $\alpha_1, \ldots, \alpha_m \in \sigma'(ya)$. Moreover, if $x \in G(A)$, then y can be chosen in such a way that $y \in G(A)$ as well.

Lemma (Aupetit's Scarcity of Elements with Finite Spectrum)

Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Then either the set of $\lambda \in D$ such that $\sigma(f(\lambda))$ is finite is a Borel set having zero capacity, or there exits an integer $n \geq 1$ and a closed discrete subset E of D such that $\#\sigma(f(\lambda)) = n$ for $\lambda \in D \backslash E$ and $\#\sigma(f(\lambda)) < n$ for $\lambda \in E$. In that case the n points of $\sigma(f(\lambda))$ are locally holomorphic functions of $D \backslash E$.

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The notion of *capacity* of a Borel subset of $\mathbb{C}$ is essentially a measure of the size of the set. Note that any set containing an open ball $B(z_0, r)$ $(z_0 \in \mathbb{C}, r > 0)$ has nonzero capacity.

By noting that $\lambda\mathbf{1} + x \in G(A)$ for all $\lambda \in \mathbb{C}$ with $|\lambda| > \rho(-x)$, and that the set $\{\lambda \in \mathbb{C} : |\lambda| > \rho(-x)\}$ has non-zero capacity. Aupetit's Scarcity Lemma gives us the following result:

Lemma

Let $a \in A$ and $n \in \mathbb{N}$. Suppose that $\#\sigma(ya) \leq n$ for all $y \in G(A)$. Then $\#\sigma(xa) \leq n$ for all $x \in A$.

In their paper *Trace and determinant in Banach algebras*, Aupetit and Mouton showed that the following properties are equivalent.

Theorem

For any $a \in A$ and integer $m \geq 0$, where A is semisimple:

(a) $\#\sigma'(xa) \leq m$ *for every* $x \in A$.

(b) $\#\{t \in \mathbb{C} : 0 \in \sigma(y + ta)\} \leq m$ *for every* $y \in G(A)$.

Theorem

Let $a \in A$, let $m \in \mathbb{N}$, and let K be any subset of $\mathbb{C}$ with at least $m+1$ nonzero elements. Then the following are equivalent:

(a) $\mathrm{rank}^{\sigma}(a) = \sup_{x \in A} \#\sigma'(xa) = m$.

(b) $\sup_{y \in G(A)} \# \{t \in K : y + ta \notin G(A)\} = m$.

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(b) $\sup_{y \in G(A)} \#\{t \in K : y + ta \notin G(A)\} = m$.

Proof:

Suppose first that (a) holds. By the density of the set

$$E(a) = \left\{u \in A : \#\sigma'(ua) = \operatorname{rank}^{\sigma}(a)\right\} \tag{1}$$

in A, there exists some $x \in G(A)$ such that $\#\sigma'(xa) = m$.

Thus, if we let $\gamma_1, \ldots, \gamma_m \in K - \{0\}$ be distinct, then by our first Lemma, there exists some $v \in G(A)$ such that

$$\frac{1}{\gamma_1}, \ldots, \frac{1}{\gamma_m} \in \sigma'(va).$$

Consequently, for any $j \in \{1, \ldots, m\}$, we have

$$\frac{1}{\gamma_j}\mathbf{1} - va \notin G(A) \implies -v^{-1} + \gamma_j a \notin G(A).$$

Thus,

$$\#\left\{t \in K : -v^{-1} + ta \notin G(A)\right\} \geq m. \tag{2}$$

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On the other hand, from (a) and the previous theorem (by Aupetit and Mouton, stated before the current result), we have

$$\#\{t \in K : y + ta \notin G(A)\} \leq m \text{ for all } y \in G(A). \tag{3}$$

Thus, from (2) and (3) we therefore have

$$\sup_{y \in G(A)} \# t \in K : y + ta \notin G(A) = m,$$

establishing (b).

Now assume that (b) holds. By definition of the supremum we may infer the existence of some $x \in G(A)$ such that

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Since $0 \notin \{t \in K : x + ta \notin G(A)\}$, there are distinct complex numbers

$$\lambda_1, \ldots, \lambda_m \in K - \{0\}$$

such that for each $j \in \{1, \ldots, m\}$, we have

$$x + \lambda_j a \notin G(A) \implies -\frac{1}{\lambda_j}\mathbf{1} - x^{-1}a \notin G(A).$$

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such that for each $j \in \{1, \ldots, m\}$, we have

$$x + \lambda_j a \notin G(A) \implies -\frac{1}{\lambda_j}\mathbf{1} - x^{-1}a \notin G(A).$$

Thus, we conclude that $\#\sigma'(x^{-1}a) \geq m$, and so, $\mathrm{rank}^{\sigma}(a) \geq m$.

Assume now, for a contradiction, that $\mathrm{rank}^{\sigma}(a) > m$. Then $\#\sigma'(ua) > m$ for some $u \in A$.

We claim that $\#\sigma'(ua) > m$ for some $u \in G(A)$.

If $\#\sigma'(ya) \leq m$ for all $y \in G(A)$, then it follows from our second Lemma that $\#\sigma(xa) \leq m + 1$ for all $x \in A$.

Thus, $\mathrm{rank}^{\sigma}(a) \leq m + 1$. However, since $\mathrm{rank}^{\sigma}(a) > m$, it forces $\mathrm{rank}^{\sigma}(a) = m + 1$.

But then, since $G(A)$ is an open set and we have assumed that $\#\sigma'(ya) \leq m$ for all $y \in G(A)$, the density of the set $E(a)$ defined in (1) produces a contradiction.

It therefore follows that $\#\sigma'(ua) > m$ for some $u \in G(A)$ as claimed.

But then, since $G(A)$ is an open set and we have assumed that $\#\sigma'(ya) \leq m$ for all $y \in G(A)$, the density of the set $E(a)$ defined in (1) produces a contradiction.

It therefore follows that $\#\sigma'(ua) > m$ for some $u \in G(A)$ as claimed. From our first Lemma we now infer the existence of some $v \in G(A)$ such that

$$\#\{t \in K : v + ta \notin G(A)\} \geq m + 1 > m.$$

But this contradicts (b).

We have therefore established that $\mathrm{rank}^{\sigma}(a) > m$ is impossible.

Hence, since we know that $\mathrm{rank}^{\sigma}(a) \geq m$, we can conclude that (a) holds.

Let $a \in A$. Then for any $y \in G(A)$ and $t \in \mathbb{C}$, we have

$$y + ta \notin G(A) \iff \mathbf{1} + ty^{-1}a \notin G(A).$$

Hence, the previous theorem readily gives the following:

Corollary

Let K be any infinite subset of $\mathbb{C}$. Then

$$\operatorname{rank}^{\sigma}(a) = \sup_{y \in G(A)} \#\{t \in K : \mathbf{1} + tya \notin G(A)\}$$

for any $a \in A$.

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It is often possible to obtain global spectral conditions from local ones via results which depend on subharmonic function theory. However, in the general setting of a ring, one no longer has this luxury. It will therefore be useful to obtain a formula for the rank where the subset K still replaces $\mathbb{C}$, but the supremum is taken over all of A.

By making use of the previous Corollary, without too much difficulty we can obtain the following result:

Proposition

Let K be any infinite subset of $\mathbb{C}$. Then

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Let K be any infinite subset of $\mathbb{C}$. Then

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for any $a \in A$.

By taking $K = \mathbb{Z}$ in the Proposition above, we arrive at a formula for the spectral rank which can be considered in the setting of a ring.

Rank in Rings

R will denote an associative ring with additive identity $\mathbf{0}$, multiplicative identity $\mathbf{1}$, group of units $\mathcal{U}(R)$.

By extending on the work by Brešar and Šemrl, Stopar provided an *algebraic* definition of Rank in Rings as follows:

With the convention that the sum of zero minimal right ideals is $\{\mathbf{0}\}$, we can define the *right rank* of $a \in R$ as the least nonnegative integer n such that a is contained in the sum of n minimal right ideals of R. If such an integer does not exist, then the right rank of a is infinite.

The *right socle* of R is defined as the sum of all minimal right ideals of R and is a two sided ideal of R. In particular, if R lacks minimal right ideals then its right socle is $\{\mathbf{0}\}$.

By definition of the right rank we see that the right socle of R is precisely the collection of elements of R with finite right rank.

Analogously, one can also define the *left rank* of $a \in R$ and the *left socle* of R via minimal left ideals of R.

However, if R is a *semiprime* ring then its left and right socle are identical. In this situation we will simply refer to it as the *socle* of R and denote it by $\mathrm{soc}(R)$. Moreover, we also have that the left and right rank of an element a in a semiprime ring R must be equal, which we can then call the *algebraic rank* of a in R and denote it as $\mathrm{rank}_R^{\pi}(a)$.

We however extend on the work of Aupetit and Mouton as follows:

Definition

Let R be a ring with multiplicative identity $\mathbf{1}$ and group of units $\mathcal{U}(R)$. We define the spectral rank of $a \in R$ by

$$\mathrm{rank}_R^{\sigma}(a) = \sup_{x \in R} \# \{t \in \mathbb{Z} : \mathbf{1} + txa \notin \mathcal{U}(R)\}$$

if the supremum exists; otherwise $\mathrm{rank}_R^{\sigma}(a) = \infty$.

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if the supremum exists; otherwise $\mathrm{rank}_R^{\sigma}(a) = \infty$.

Note that any element in the Jacobson radical of a ring has a spectral rank of zero. So, in order to ensure that only the additive identity $\mathbf{0}$ has a spectral rank of 0, we should restrict our attention to J-semisimple rings.

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Property

For any $a, b \in R$ we have that

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Property

Suppose that $\phi : R \to S$ is a ring isomorphism. Then

$$\operatorname{rank}_R^\sigma(a) = \operatorname{rank}_S^\sigma(\phi(a)) \text{ for all } a \in R.$$

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Suppose that $\phi : R \to S$ *is a ring isomorphism. Then*

$$\operatorname{rank}^{\sigma}_{R}(a) = \operatorname{rank}^{\sigma}_{S}(\phi(a)) \text{ for all } a \in R.$$

Property

Let $R_1, \ldots, R_k$ *be rings, and let* $R = R_1 \times \cdots \times R_k$ *be their direct product. Then for any* $a = (a_1, \ldots, a_k) \in R$, *we have*

$$\operatorname{rank}^{\sigma}_{R}(a) = \sum_{j=1}^{k} \operatorname{rank}^{\sigma}_{R_j}(a_j).$$

Recall that if A is a complex Banach algebra, then for any nonzero element $x \in A$ and scalar λ, $\lambda x = \mathbf{0}$ forces $\lambda = 0$. In particular, one then observes that complex Banach algebras do not contain any cyclic additive subgroups of finite order.

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Example

In the J-semisimple ring $\mathbb{Z}_6$, all nonzero elements have infinite spectral ranks. In particular, we point out that 3 is a minimal idempotent with infinite spectral rank. The latter follows from the observation that

$$\mathbf{1} + t3 = 4 \notin \mathcal{U}(\mathbb{Z}_6)$$

for all odd integers t; so $\# \{t \in \mathbb{Z} : \mathbf{1} + t3 \notin \mathcal{U}(\mathbb{Z}_6)\} = \infty$.

Definition

We say that a ring R is $\mathbb{Z}_n$-free if it does not contain any non-trivial cyclic additive subgroups of finite order.

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Theorem

Let R be a J-semisimple and $\mathbb{Z}_n$-free ring. Then, for any nonzero idempotent $e \in R$, we have that $\operatorname{rank}^{\sigma}(e) = 1$ if and only if eRe is a division ring (e is a minimal idempotent).

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The forward implication relies on the J-semisimple property of the ring, whereas the converse requires the $\mathbb{Z}_n$-free property.

Definition

Let $a \in R$. We say that a is left (respectively, right) semipotent if every nonzero left (respectively, right) ideal of R contained in Ra (respectively, aR) contains a nonzero idempotent.

For our purposes, we shall restrict our attention to left semipotent and simply refer to it as semipotent. Notice that $\mathbf{0}$ is vacuously semipotent. We now fix the following notation:

$$\mathcal{F} = \{a \in R : \mathrm{rank}^{\sigma}(a) < \infty \text{ and } a \text{ is semipotent}\}.$$

We are able to obtain the following connection between the spectral and algebraic rank in rings.

Theorem

Let R be a J-semisimple and $\mathbb{Z}_n$-free ring. Then $\mathcal{F} = \mathrm{soc}(R)$ and $\mathrm{rank}^{\sigma}(a) = \mathrm{rank}^{\pi}(a)$ for each $a \in \mathcal{F}$.

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As a consequence of this theorem, we note that if a has a finite algebraic rank in a J-semisimple and $\mathbb{Z}_n$-free ring R, then a has the exact same spectral rank. On the other hand, if a has a finite spectral rank *and* a is semipotent, then a has the exact same algebraic rank.

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$$\mathrm{rank}^{\pi}(ab) + \mathrm{rank}^{\pi}(bc) \leq \mathrm{rank}^{\pi}(abc) + \mathrm{rank}^{\pi}(b)$$

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Corollary

Let R be a J-semisimple and $\mathbb{Z}_n$-free ring. Then

$$\mathrm{rank}^{\sigma}(ab) + \mathrm{rank}^{\sigma}(bc) \leq \mathrm{rank}^{\sigma}(abc) + \mathrm{rank}^{\sigma}(b)$$

for all $b \in \mathcal{F}$ and $a, c \in R$.

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