Relative cohomology for operator modules over operator algebras

Martin Mathieu
(Queen's University Belfast)

Banach Algebras in Granada

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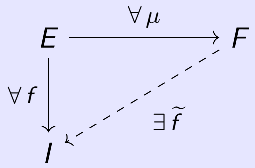
joint work with
Michael Rosbotham
(now at Univ Maine)

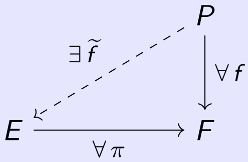
Exact structures for operator modules,
Canadian Journal of Mathematics, 2022.

Let A be a unital algebra and let $\mathscr{M}od_A$ be the category of all unital right A-modules.

$I \in \mathscr{M}od_A$ is *injective* if any morphism whose codomain is I can be extended along every monomorphism μ



$P \in \mathscr{M}od_A$ is *projective* if any morphism whose domain is P can be lifted over every epimorphism π



A complex algebra is said to be classically semisimple if it is a direct sum of minimal right ideals and if it is finitely generated, finitely many minimal right ideals suffice.

A unital complex Banach algebra

$$\text{classically semisimple} \Longleftrightarrow A \cong \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$$

by Artin–Wedderburn together with Gelfand–Mazur;
in particular, it is finite dimensional.

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equivalently, *every* $E \in \mathscr{Mod}_A$ is injective;
equivalently, *every* $E \in \mathscr{Mod}_A$ is projective.

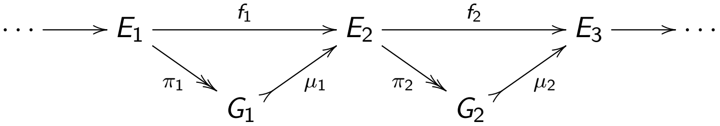
key concept: exact sequences

short exact sequence

$$E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$$

means: μ is mono, π is epi and $\ker \pi = \operatorname{im} \mu$ (so $G \cong F/\operatorname{im} \mu$)

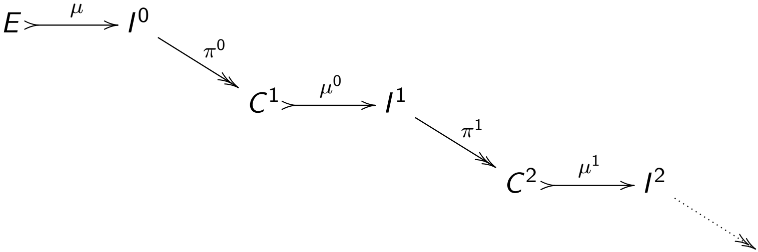
long exact sequence



means: $\operatorname{im} f_1 = \operatorname{im} \mu_1 = \ker \pi_2 = \ker f_2$

Enough injectives and injective resolutions

“every module can be embedded into an injective one” "every module can be embedded into an injective one"



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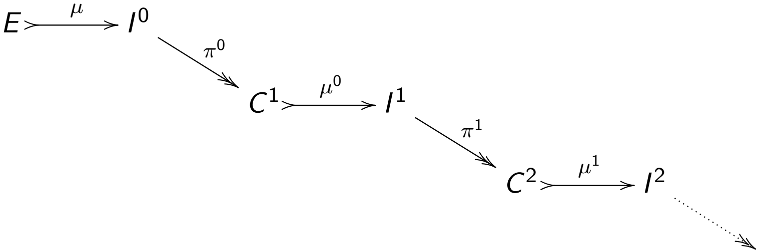


injective resolution

$$E \rightarrowtail I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \dots$$

Our kind of categories

A, unital complex algebra, operator algebra if it is a Banach algebra, an operator space and completely isometric to a closed subalgebra of some $B(H)$;

equivalently, the multiplication on A is (multiplicatively) completely contractive, that is, is linearisation

$$A \otimes_h A \to A \text{ is completely contractive.}$$

e.g., every C^*-algebra; the disk algebra $A(\mathbb{D})$; ...

E, unital right A-module, operator module (over A) if it is an operator space and the linearisation of the module multiplication

$$E \otimes_h A \to E \text{ is completely contractive.}$$

$\mathscr{OMod}_A^\infty$ is the category of all such modules together with completely bounded A-module maps.

Our kind of categories

$\mathscr{OMod}_A^\infty$ is a full subcategory of $\mathscr{mnMod}_A^\infty$ whose objects are the matrix normed A-modules:
the linearisation of the module multiplication

$$E \mathbin{\widehat{\otimes}} A \to E \quad \text{is completely contractive.}$$

e.g., $CB(A, F)$, the completely bounded maps from A into an operator space F is a matrix normed module but not necessarily an operator module.

both categories are additive but not abelian:
monomorphism: injective cb A-module maps
epimorphism: cb A-module maps with dense range
isomorphism: bijective cb A-module maps with cb inverse

what to do?

what to do? exact categories!

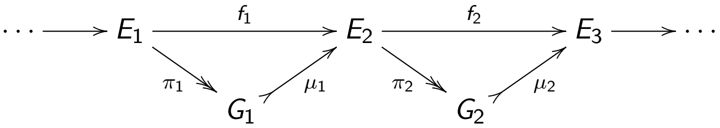
key concept: kernel-cokernel pairs

short exact sequence = kernel-cokernel pair

$$E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$$

means: μ is a kernel of π and π is a cokernel of μ

long exact sequence



means: (μ_1, π_2) is a kernel-cokernel pair (a.s.o.)

where we assume that each f_i is admissible, i.e., can be factorised as $f_i = \mu_i \pi_i$.

Our kind of categories

in both $\mathscr{OMod}_A^\infty$ and $\mathscr{mnMod}_A^\infty$,

kernel: cb A-module isomorphism onto its image

cokernel: completely open cb A-module map

Let $\mathscr{A}$ be an additive category and let $(\mathscr{M}, \mathscr{P})$ be a class of kernel-cokernel pairs which is closed under isomorphisms.

$\mathscr{E}x = (\mathscr{M}, \mathscr{P})$ is an exact structure on $\mathscr{A}$ (in the sense of Quillen) if:

- [E_0] For all $E \in \mathscr{A}$, $\mathrm{id}_E \in \mathscr{M}$.
- [${E_0}^{\mathrm{op}}$] For all $E \in \mathscr{A}$, $\mathrm{id}_E \in \mathscr{P}$.
- [E_1] $\mathscr{M}$ is closed under composition.
- [${E_1}^{\mathrm{op}}$] $\mathscr{P}$ is closed under composition.
- [E_2] The pushout of a morphism in $\mathscr{M}$ along an arbitrary morphism exists and yields a morphism in $\mathscr{M}$.
- [${E_2}^{\mathrm{op}}$] The pullback of a morphism in $\mathscr{P}$ along an arbitrary morphism exists and yields a morphism in $\mathscr{P}$.

In this case we say $(\mathscr{A}, \mathscr{E}x)$ is an exact category.

T. Bühler. *Exact categories.* Expo. Math., 28(1) 1-69, 2010.

Theorem

Let A be an operator algebra. The class $\mathscr{E}x_{\max}$ of **all** *kernel-cokernel pairs forms an exact structure on $\mathscr{OMod}_A^\infty$ and on $\mathscr{mnMod}_A^\infty$.*

Exact categories

Theorem

Let A be an operator algebra. The class $\mathscr{E}x_{\max}$ of **all** *kernel-cokernel pairs forms an exact structure on $\mathscr{OMod}_A^\infty$ and on $\mathscr{mnMod}_A^\infty$.*

this is the largest exact structure any additive category can be endowed with;

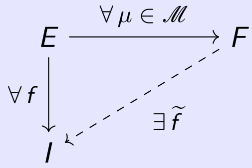
at the opposite end, the minimal exact structure $\mathscr{E}x_{\min}$ always consists of the split kernel-cokernel pairs

Split: $$E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G, \qquad \exists\, \nu : F \dashrightarrow E, \quad \exists\, \theta : G \dashrightarrow F$$

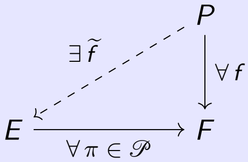
such that $\nu\mu = \mathrm{id}_E$, $\pi\theta = \mathrm{id}_G$ and $\mu\nu + \theta\pi = \mathrm{id}_F$.

Let $\mathscr{A}$ be a category, $\mathscr{M}$ a class of monomorphisms, and $\mathscr{P}$ a class of epimorphisms.

$I \in \mathscr{A}$ is *$\mathscr{M}$-injective* if any morphism whose codomain is I can be extended along morphisms in $\mathscr{M}$



$P \in \mathscr{A}$ is *$\mathscr{P}$-projective* if any morphism whose domain is P can be lifted over morphisms in $\mathscr{P}$



Cohomological dimension in an exact category

Let $(\mathscr{A}, \mathscr{E}x)$ be an exact category with $\mathscr{E}x = (\mathscr{M}, \mathscr{P})$.

"dimension" of $E \in \mathscr{A}$ = shortest length of an injective resolution

cohomological dimension of $(\mathscr{A}, \mathscr{E}x)$:

$$\operatorname{cohomdim}(\mathscr{A}, \mathscr{E}x) := \sup \{ \operatorname{Inj}_{\mathscr{M}}\text{-}\dim(E) \mid E \in \mathscr{A} \}$$

Theorem (Rosbotham 2021)

Let A be a unital C-algebra. Let*

$$\operatorname{dg}_{C*}(A) = \operatorname{cohomdim}(\mathscr{OMod}_A^{\infty}, \mathscr{E}x_{\max})$$

be the global C-dimension of A. Then $\operatorname{dg}_{C*}(A) \geq 2$.*

Definition

An additive functor $\mathsf{F}\colon (\mathscr{A}, \mathscr{E}x_1) \to (\mathscr{B}, \mathscr{E}x_2)$ between two exact categories is *exact* if $\mathsf{F}(\mathscr{E}x_1) \subseteq \mathscr{E}x_2$.

Proposition

Let $\mathsf{F}\colon (\mathscr{A}, \mathscr{E}x_1) \to (\mathscr{B}, \mathscr{E}x_2)$ *be an exact functor between exact categories. If there is another exact structure* $\mathscr{E}x_2'$ *on* $\mathscr{B}$ *then*

$$\mathscr{E}x_1' = \{(\mu, \pi) \in \mathscr{E}x_1 \mid (\mathsf{F}\mu, \mathsf{F}\pi) \in \mathscr{E}x_2'\}$$

forms an exact structure on $\mathscr{A}$.

Applied to our categories

Let $\mathsf{F}\colon (\mathscr{mnMod}_A^\infty, \mathscr{E}x_{\mathsf{max}}) \longrightarrow (\mathscr{Op}^\infty, \mathscr{E}x_{\mathsf{max}})$ be the forgetful functor (where $\mathscr{Op}^\infty = \mathscr{OMod}_{\mathbb{C}}^\infty$). Then

$$\mathscr{E}x_{\mathsf{rel}} := \{(\mu, \pi) \in \mathscr{E}x \mid (\mathsf{F}\mu, \mathsf{F}\pi) \in \mathscr{E}x_{\mathsf{min}}\}$$

forms the relative exact structure on $\mathscr{mnMod}_A^\infty$.

Enough injectives

four exact categories:

$$(\mathscr{OMod}_A^\infty, \mathscr{Ex}_{\mathsf{max}});\ (\mathscr{OMod}_A^\infty, \mathscr{Ex}_{\mathsf{rel}});\ (\mathscr{mnMod}_A^\infty, \mathscr{Ex}_{\mathsf{max}});\ (\mathscr{mnMod}_A^\infty, \mathscr{Ex}_{\mathsf{rel}}) \quad .$$

for instance, $CB(A, I)$ is injective in $(\mathscr{mnMod}_A^\infty, \mathscr{Ex}_{\mathsf{max}})$ for every $I \in \mathscr{Op}^\infty$ injective

and $CB(A, F)$ is injective in $(\mathscr{mnMod}_A^\infty, \mathscr{Ex}_{\mathsf{rel}})$ for every $F \in \mathscr{Op}^\infty$;

since $E \cong CB_A(A, E) \hookrightarrow CB(A, E) \hookrightarrow CB(A, B(H))$ for every $E \in \mathscr{mnMod}_A^\infty$, where $E \subseteq B(H)$ as an operator space,

$\mathscr{mnMod}_A^\infty$ has enough injectives.

Theorem

Let A be a unital operator algebra. The following are equivalent:

- *A is classically semisimple;*
- $\mathrm{cohomdim}(\mathscr{O}\mathscr{M}od_A^\infty, \mathscr{E}x_{\mathsf{rel}}) = 0$;
- $\mathrm{cohomdim}(mn\mathscr{M}od_A^\infty, \mathscr{E}x_{\mathsf{rel}}) = 0$.

Theorem

Let A be a unital operator algebra. The following are equivalent:

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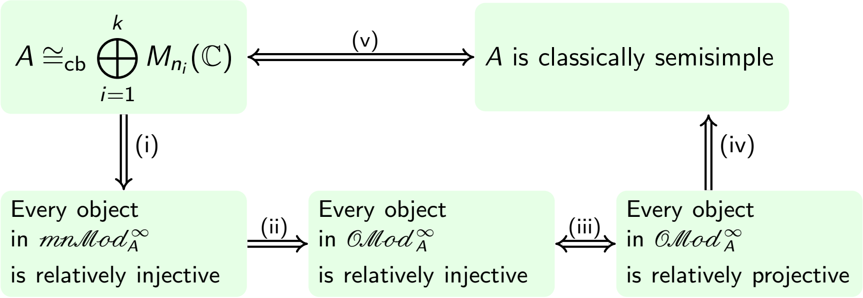
Proof:



Outline of the proof

(i) we show that, for every $E \in mn\mathscr{M}od_A^\infty$, there exist $r \in CB_A(CB(A, E),\, E)$ and $s \in CB_A(E,\, CB(A, E))$ such that $rs = \mathrm{id}_E$. This is achieved by using the explicit structure of A and systems of matrix units.

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(ii) this follows from the fact that $\mathscr{OMod}_A^\infty$ is a full exact subcategory of $\mathscr{mnMod}_A^\infty$ so the admissible monomorphisms are the same and hence any object in $\mathscr{OMod}_A^\infty$ which is injective in $\mathscr{mnMod}_A^\infty$ is injective in $\mathscr{OMod}_A^\infty$.

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(iv) this is the main work in the theorem; it relies on the fact that, for every operator space E, $E \otimes_h A$ is relatively projective in $\mathscr{OMod}_A^\infty$ and that A is classically semisimple if and only if each of its maximal submodules is a direct summand.

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(v) this was already discussed. $\square$