Specht's Theorem in UHF C^*-algebras

Laurent W. Marcoux

Department of Pure Mathematics
University of Waterloo

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This talk is based on joint work with:

Yuanhang Zhang (Jilin University, China)

- $\mathcal{H}$ - a complex Hilbert space, separable.
- $\mathcal{B}(\mathcal{H})$ - bounded linear operators on $\mathcal{H}$. If $\mathcal{H} \simeq \mathbb{C}^n$, then $\mathcal{B}(\mathcal{H}) \simeq \mathbb{M}_n(\mathbb{C})$.

The two most important notions of *equivalence* of operators are

- **unitary equivalence**: $A \simeq B$ if there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $A = U^* BU$.
- **similarity**: $A \sim B$ if there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $A = S^{-1} BS$.

Both notions make sense inside of any C^*-algebra.

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In $\mathbb{M}_n(\mathbb{C})$, $A \sim B$ if and only if A and B admit a common **Jordan canonical form**. There is no analogue of this for operators acting on an infinite-dimensional space.

Open Question. If $A \oplus A \sim B \oplus B$, is $A \sim B$?

Specht's Theorem. Let $A, B \in \mathbb{M}_n(\mathbb{C})$. The following are equivalent:

(a) $A \simeq B$;

(b) for every word $w(x, y)$ in two non-commuting variables,

$$\mathrm{tr}(w(A, A^*)) = \mathrm{tr}(w(B, B^*)).$$

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Question. To what extent can we extend Specht's Theorem beyond the matrix setting?

In the case of infinite-dimensional, unital C^*-algebras $\mathbb{A}$ (including $\mathcal{B}(\mathcal{H})$ if dim $\mathcal{H} = \infty$), unitary orbits are typically not closed.
A natural extension of unitary equivalence is **approximate unitary equivalence**: given $a, b \in \mathbb{A}$, we write $a \simeq_a b$ if there exists a sequence $(u_n)_n$ of unitary elements of $\mathbb{A}$ such that

$$b = \lim_n u_n^* a u_n.$$

If $\mathbb{A}$ admits a tracial state (i.e. $0 \leq \tau \in \mathbb{A}^*$, $\|\tau\| = 1$ and $\tau(xy) = \tau(yx)$), then $a \simeq_a b$ implies that

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This fails. Spectacularly. Rørdam, Larsen and Laustsen showed that there exists a simple, unital AF algebra $\mathbb{A}$ with

- a unique, faithful tracial state τ, and
- a pair of projections $p, q \in \mathbb{A}$ such that $\tau(p) = \tau(q)$,
- but p is not approximately unitarily equivalent to q.

Note that $\tau(p) = \tau(q)$ implies that $\tau(w(p, p^*)) = \tau(w(q, q^*))$ for all words $w(x, y)$.

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Recall that a **UHF algebra** $\mathbb{A}$ is an inductive limit of finite-dimensional, unital, simple C^*-algebras: in other words, $\mathbb{A}_n \simeq \mathbb{M}_{k_n}(\mathbb{C})$ for all $n \geq 1$,

$$\mathbb{C}I \subseteq \mathbb{A}_1 \subseteq \mathbb{A}_2 \subseteq \cdots \subseteq \mathcal{B}(\mathcal{H})$$

and

$$\mathbb{A} = \overline{\cup_n \mathbb{A}_n}.$$

They are classified (**Glimm's Theorem**) up to *-isomorphism by their **supernatural number**:

$$\alpha(\mathbb{A}) := 2^{\mu_1} 3^{\mu_2} 5^{\mu_3} 7^{\mu_4} \cdots ,$$

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The **universal** UHF algebra $\mathcal{Q}$ is the UHF algebra whose supernatural number is

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Two elements a, b in a C^*-algebra $\mathbb{A}$ are said to be **algebraically equivalent** if there exists a *-isomorphism $\Phi : C^*(a) \to C^*(b)$ such that $\Phi(a) = b$.

For example, given $A \in \mathcal{B}(\mathcal{H})$, A and $A \oplus A \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \simeq \mathcal{B}(\mathcal{H})$ are algebraically equivalent.

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Theorem. *Let $\mathbb{A}$ be a unital C^*-algebra with a faithful tracial state τ, and $a, b \in \mathbb{A}$. Suppose that $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for all words $w(x, y)$. For each polynomial $p(x, y)$ in two non-commuting variables, define $\Phi(p(a, a^*)) = p(b, b^*)$. Then:*

(a) $\|p(a, a^*)\| = \|p(b, b^*)\|$ *for all polynomials* $p(x, y)$.

(b) Φ *is well-defined and extends in a unique way to an isomorphism from* $C^*(a)$ *onto* $C^*(b)$ *which implements the algebraic equivalence of* a *and* b.

(c) $\sigma(a) = \sigma(b)$.

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(d) If $1 \leq k \in \mathbb{N}$ and $[a_{i,j}] \in \mathbb{M}_k(C^*(a))$, then

$$\tau_k([a_{i,j}]) = \tau_k(\Phi^{(k)}([a_{i,j}])).$$

(e) Suppose furthermore that a and b are normal and denote $X := \sigma(a) = \sigma(b)$. If $F \in \mathbb{M}_k(\mathcal{C}(X))$ then

$$\tau_k(\varphi^{(k)}(F)) = \tau_k(\psi^{(k)}(F)),$$

where $\varphi, \psi : \mathcal{C}(X) \to \mathbb{A}$ are defined via $\varphi(f) := f(a)$ and $\psi(f) = f(b)$ for all $f \in \mathcal{C}(X)$.

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Proposition. *Let $\mathbb{A}$ be a UHF algebra, $m, n \in \mathbb{A}$ and suppose that m is normal. The following are equivalent:*

(a) $m \simeq_a n$.

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Proof. You don't want to know.

□

Theorem. [Schafhauser 2020]
If $\mathbb{A}$ is a separable, unital, exact C^*-algebra satisfying the UCT and having a faithful, amenable trace, and if $\mathbb{B}$ is a simple, unital AF algebra with a unique trace and divisible K_0-group, then the unital, trace-preserving $*-$homomorphisms $\mathbb{A} \to \mathbb{B}$ are classified up to unitary equivalence by their behaviour on the K_0 group.

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If $\mathbb{A}$ is a separable, unital, exact C^*-algebra satisfying the UCT and having a faithful, amenable trace, and if $\mathbb{B}$ is a simple, unital AF algebra with a unique trace and divisible K_0-group, then the unital, trace-preserving $*-$homomorphisms $\mathbb{A} \to \mathbb{B}$ are classified up to unitary equivalence by their behaviour on the K_0 group.

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Let $a, b \in \mathcal{Q}$ and suppose that $C^*(a)$ satisfies the UCT. Then

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The bad, and the ugly. One can find $a, b \in \mathbb{M}_{2^\infty}$ such that both $C^*(a)$ and $C^*(b)$ satisfy the UCT, $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for all words $w(x, y)$, but a and b are not unitarily equivalent.

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Definition. *Let $\mathbb{A}$ be a C^* algebra and $a, b \in \mathbb{A}$. We shall say a, b satisfy the* **approximate absolute value condition (AAVC)** *if for any polynomial $p(x, y)$,*

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Note: the sequence $(u_n)_n$ of unitary elements implementing the approximate unitary equivalence of a given pair $|p(a, a^*)|$ and $|p(b, b^*)|$ depends upon the polynomial $p(x, y)$.

Theorem. [LWM, Mastnak and Radjavi]
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The **AAVC** applies to $\mathcal{B}(\mathcal{H})$!

Theorem. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then A and B are approximately unitarily equivalent if A and B satisfy the* **AAVC**.

The proof depends upon Hadwin's formulation of Voiculescu's Weyl-von Neumann Theorem:

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Suppose $A, B \in \mathcal{B}(\mathcal{H})$. Then $A \simeq_a B$ if and only if there is a representation $\pi : C^*(A) \to C^*(B)$ such that $\pi(A) = B$ and $\text{rank}(T) = \text{rank}(\pi(T))$ for every $T \in C^*(A)$.

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This does not, however, extend to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$:

Let $S \in \mathcal{B}(\mathcal{H})$ denote the unilateral forward shift ($Se_n = e_{n+1}, n \geq 1$, where $\{e_n\}_n$ is an **onb** for $\mathcal{H}$).

Then $s := \pi(S)$ and $t := s \oplus s$ satisfy the **AAVC**, since all self-adjoint operators with the same spectrum in the Calkin algebra are unitarily equivalent (**BDF**). They fail to be (approximately) unitarily equivalent because of semi-Fredholm index.

The existence of Fredholm index in the Calkin algebra reflects the fact that the K_1-group of $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is non-trivial (i.e. it is isomorphic to $\mathbb{Z}$). For the Cuntz algebra $\mathcal{O}_2$,

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An unexpected consequence.

Given $A \in \mathbb{M}_n(\mathbb{C})$ and $k \geq 1$, we set

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We saw that there exist $a, b \in \mathbb{M}_{2^\infty}$ such that

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With H. Radjavi, we have shown that if $M, N \in \mathbb{M}_2(\mathbb{C})$ are **normal** matrices, then

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Proof. Because you do want to know: $K_0(\mathcal{C}(\sigma(m)))$ is a free abelian group, so the UCT applies; $K_1(\mathbb{A}) = 0$, so

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If $\varphi(f) := f(m)$, $\psi(f) = f(n)$, $f \in \mathcal{C}(\sigma(m))$, then by (e) above, $K_0(\varphi) = K_0(\psi)$ in $Hom(K_0(\mathcal{C}(\sigma(m)), K_0(\mathbb{A}))$. Thus $KK(\varphi) = KK(\psi)$, and $KL(\mathcal{C}(\sigma(m)), \mathbb{A})$ is a quotient of $KK(\mathcal{C}(\sigma(X), \mathbb{A})$, so $KL(\varphi) = KL(\psi)$. Since traces are preserved, a result of Gong and Lin implies that $\varphi \simeq_a \psi$, whence $m \simeq_a n$. Really, you didn't want to know. □

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