On certain Hecke algebras arising as deformations of group algebras of Coxeter groups

joint work with Sven Raum

Adam Skalski

IMPAN, Polish Academy of Sciences, Warsaw

Granada, 20th of July 2020

dedicated to the memory of
Henryk Skalski, 1940-2022
Yulia Zdanovska, 2000-2022

Plan of the talk

We will discuss certain operator algebras which can be viewed as multiparameter deformations of operator algebras of right angled Coxeter groups. In particular we will characterise factoriality of the relevant von Neumann algebras.

The algebras in question arise in various natural ways; this means that the factoriality result has several interesting interpretations and consequences, which we will outline.

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Right-angled Coxeter groups

Coxeter system (W, S): a group W generated by a (finite) set of reflections S, with a function $m : S \times S \mapsto \mathbb{N} \cup \{\infty\}$ which determines the relations:

$$(st)^{m_{s,t}} = e, \quad s, t \in S$$

(we have $m_{s,s} = 1$, $s \in S$).

W is right-angled if $m_{s,t} \in \{2, \infty\}$

In the right-angled case we encode m in the graph Γ_W with vertices S and edges

$$E\Gamma_W := \{(s,t) \in S \times S : s \neq t, m_{s,t} = 2\}$$

W is irreducible if the complement of Γ_W is connected; equivalently, W does not decompose as a direct product of two Coxeter groups.

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Basic examples; graphs of groups

The simplest case is given by an empty graph: $W = \mathbb{Z}_2^{\star k}$ (note that $k = 1, 2$ lead to amenable groups).

The graph with three vertices and one edge yields $W = (\mathbb{Z}_2 \times \mathbb{Z}_2) \star \mathbb{Z}_2$

We could keep the finite graph as the way of encoding freeness/commutation, and replace the 'vertex groups' $\mathbb{Z}_2$ by arbitrary groups: this leads to the graph of groups construction. When 'vertex groups' are $\mathbb{Z}$, we obtain right-angled Artin groups.

Right-angled (and not only!) Coxeter groups have strong combinatorial properties, and are related to buildings.

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Group ring and its deformation

(W, S) – Coxeter system of a right-angled Coxeter group.

$$\mathbb{C}[W] = \langle \{ T_s, s \in S : T_s T_t = T_t T_s \text{ if } m_{s,t} = 2 \\ T_s = T_s^* \\ T_s^2 = 1 \} \rangle$$

Let $\mathbf{q} = (q_s)_{s \in S} \in (0,1]^S$.

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Let $\mathbf{q} = (q_s)_{s \in S} \in (0, 1]^S$.

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$\mathbb{C}_{\mathbf{q}}[W]$ – the $\mathbf{q}$-Hecke algebra of W.

Representing the deformed group ring on $\ell^2(W)$

$\mathbb{C}_{\mathbf{q}}[W]$ acts on $\ell^2(W)$, as noted by Dymara: for $s \in S$, $w \in W$ put $p_s = \frac{q_s - 1}{\sqrt{q_s}}$ and

$$\pi_{\mathbf{q}}(T_s^{\mathbf{q}})(\delta_w) = \begin{cases} \delta_{sw} & \text{if } |sw| > |w| \\ \delta_{sw} + p_s \delta_w & \text{if } |sw| < |w| \end{cases}$$

For example

$$\pi_{\mathbf{q}}(T_s^{\mathbf{q}})\pi_{\mathbf{q}}(T_s^{\mathbf{q}})(\delta_e) = \pi_{\mathbf{q}}(T_s^{\mathbf{q}})\delta_s = \delta_e + p_s\delta_s = (1 + p_s\pi_{\mathbf{q}}(T_s^{\mathbf{q}}))\delta_e$$
$$= (1 + (q_s^{\frac{1}{2}} - q_s^{-\frac{1}{2}})\pi_{\mathbf{q}}(T_s^{\mathbf{q}}))\delta_e$$

Definition

Let (W, S) be a right-angled Coxeter system, $\mathbf{q} = (q_s)_{s \in S} \in (0, 1]^S$. The **q-Hecke von Neumann algebra of W** is defined as

$$N_{\mathbf{q}}(W) = \pi_{\mathbf{q}}(\mathbb{C}_{\mathbf{q}}[W])'' \subset B(\ell^2(W)).$$

Note that of course $N_{\mathbf{1}}(W) = L(W)$.

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Hecke von Neumann algebra

Theorem (Dymara)

The vector state $x \mapsto \langle \delta_e, x\delta_e \rangle$ is a faithful normal trace on $N_{\mathbf{q}}(W)$; the commutant of $N_{\mathbf{q}}(W)$ is equal to the natural 'right version' of $N_{\mathbf{q}}(W)$. If we define $\mathbf{q} : W \to (0,1]$ as the 'multiplicative' (with respect to reduced forms) extension of $\mathbf{q} : S \to (0,1]$ and assume that the $\mathbf{q}$-growth series $\sum_{w \in W} \mathbf{q}_w$ converges, then the projection onto the vector $\sum_{w \in W} (\mathbf{q}_w)^{\frac{1}{2}} \delta_w$ belongs to the centre of $N_{\mathbf{q}}(W)$.

We can of course also consider the (reduced) $\mathbf{q}$-Hecke C*-algebra of W, $C^*_{r,\mathbf{q}}(W)$, given by the norm closure of $\pi_{\mathbf{q}}(\mathbb{C}_{\mathbf{q}}[W])$ in $B(\ell^2(W))$.

Results of Dykema show that if $W = (\mathbb{Z}_2)^{\star k}$ (with $k \geq 3$), then $N_{\mathbf{q}}(W)$ are interpolated free group factors.

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Representing the deformed group ring on $\ell^r(W)$

The same formula as before ($s \in S$, $w \in W$):

$$\pi_{\mathbf{q}}^{r}(T_s^{\mathbf{q}})(\delta_w) = \begin{cases} \delta_{sw} & \text{if } |sw| > |w| \\ \delta_{sw} + p_s \delta_w & \text{if } |sw| < |w| \end{cases}$$

yields a representation of $\mathbb{C}_{\mathbf{q}}[W]$ on $\ell^r(W)$ for $r \in [1, \infty)$, and we can consider

$$N_{\mathbf{q}}^{r}(W) = \pi_{\mathbf{q}}^{r}(\mathbb{C}_{\mathbf{q}}[W])'' \subset B(\ell^r(W)).$$

Some geometry – buildings and all that

Buildings – combinatorial structures, special, 'very symmetric' chamber complexes, i.e. collections of combinatorial simplices such that each simplex is contained in a maximal one (chamber)

Coxeter chamber complex: $\Sigma(W, S)$ – simplices given by cosets $w\langle T\rangle$ (with $w \in W,\ T \subset S$), ordered by reverse inclusion.

A building is thick if each facet of a chamber is a facet of at least three chambers (the Coxeter complex is thin!); irreducible, if it is not a non-trivial join of lower-dimensional buildings.

Every thick building X determines a Coxeter system (W, S); given a choice of a fundamental chamber, its facets are labelled by elements of S, $\Sigma(W, S)$ becomes an apartment of X; we also obtain numbers $d_s \geq 3$ counting the chambers to which facets of the fundamental chamber belong. X is locally finite if each of these is finite.

For each right-angled Coxeter system (W, S) of rank at least three (i.e. $|S| \geq 3$) there exist such locally finite thick buildings of arbitrarily large thickness (i.e. large d_s).

Groups acting on buildings

Definition

Let X be a right-angled building of rank at least three, with the Coxeter system (W, S) and thickness $(d_s)_{s \in S}$. Denote by $\mathrm{Aut}^+(X)$ the group of all type (labelling) preserving automorphisms of X (a locally compact, totally disconnected group).

Proposition (Iwahori+Matsumoto)

Let X be as above and let $G \subset \mathrm{Aut}^+(X)$ be a strongly transitive (i.e. 'big enough') closed subgroup of $\mathrm{Aut}^+(X)$. Let B denote an Iwahori subgroup of G: the stabiliser subgroup of a chamber (compact open subgroup of G). Let $\mathbb{C}[G; B]$ denote the algebra of B-bi-invariant compactly supported functions on G, equipped with the convolution induced from G. Then

$$\mathbb{C}[G; B] \cong \mathbb{C}_{\mathbf{q}}[W],$$

with $q_s = d_s^{-1}$, $s \in S$.

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Proposition

Let X, G, B and $\mathbf{q}$ be as above. Denote by $L(G; B)$ the von Neumann algebra generated by $\mathbb{C}[G; B]$ in its natural representation on $\ell^2(G/B)$. Then

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Graph product of representations

A right-angled Coxeter group W (with the Coxeter system (W, S)) can be viewed as a graph product of $\mathbb{Z}_2$ over the graph Γ_W.

Caspers+Fima (earlier Fima, Freslon, Germain...) studied graph products of C^*-algebras. This allows us to study graph product of representations.

Let $a_s \in (-1, 1)$. Define a representation of $\mathbb{Z}_2 = \langle s \rangle$ on $\mathbb{C}^2$ putting

$$\lambda_{a_s}(s) = \begin{pmatrix} a_s & \sqrt{1-a_s^2} \\ \sqrt{1-a_s^2} & -a_s \end{pmatrix}.$$

Theorem

Let (W, S) be a right-angled Coxeter system. Let $\mathbf{a} = (a_s)_{s \in S} \in (-1, 1)^s$. Denote by $\tilde{\lambda}_{\mathbf{a}} : W \to B(\mathsf{H})$ the graph product of representations λ_{a_s} defined above. Put $q_s = \frac{1-|a_s|}{\sqrt{1-a_s^2}}$. Then

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$$\lambda_{a_s}(s) = \begin{pmatrix} a_s & \sqrt{1 - a_s^2} \\ \sqrt{1 - a_s^2} & -a_s \end{pmatrix}.$$

Theorem

Let (W, S) be a right-angled Coxeter system. Let $\mathbf{a} = (a_s)_{s \in S} \in (-1, 1)^s$. Denote by $\tilde{\lambda}_{\mathbf{a}} : W \to B(\mathrm{H})$ the graph product of representations λ_{a_s} defined above. Put $q_s = \frac{1-|a_s|}{\sqrt{1-a_s^2}}$. Then

$$\tilde{\lambda}_{\mathbf{a}}(W)'' \cong N_{\mathbf{q}}(W).$$

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Factoriality result

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Let (W, S) be a right-angled irreducible Coxeter system with $|S| \geq 3$ and let $\mathbf{q} = (q_s)_{s\in S} \in (0,1]^S$. Then if the $\mathbf{q}$-growth series $\sum_{w\in W} \mathbf{q}_w$ diverges, then $N_{\mathbf{q}}(W)$ is a factor, and if the $\mathbf{q}$-growth series converges then $N_{\mathbf{q}}(W) \cong \mathbb{C}p_1 \oplus \cdots \oplus p_n \oplus M$, where $p_1, \ldots, p_n$ are certain rank-one projections and M is a factor.

- for the single-parameter case $\mathbf{q} = (q)_{s\in S}$ this was shown by Garncarek in 2016
- for $W = \mathbb{Z}_2^{\star k}$ it can be deduced (in a not-completely-trivial way) from the work of Dykema
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How does the argument work?

The proof follows a very classical method (also used by Garncarek), known from proving factoriality of ICC groups: take $x \in Z(N_{\mathbf{q}}(W))$ and analyse the vector $x\delta_e$, by studying the function f_x:

$$f_x(w) := \langle \delta_w, x\delta_e \rangle, \quad w \in W.$$

An analogous argument gives the following result:

Theorem

Let (W, S) be a right-angled irreducible Coxeter system with $|S| \geq 3$, let $\mathbf{q} = (q_s)_{s \in S} \in (0, 1]^S$ and let $r \in (1, \infty)$. Set $\tilde{r} = \min\{r, r'\}$. Then $N^r_{\mathbf{q}}(W)$ is a factor (has a trivial centre) if and only if the $(\mathbf{q}, \tilde{r})$-growth series $\sum_{w \in W} \mathbf{q}_w^{\frac{\tilde{r}}{2}}$ diverges.

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Consequences for the automorphism groups of buildings

Theorem

Let X be a right-angled thick irreducible building, with the Coxeter system (W, S) with $|S| \geq 3$ and thickness $(d_s)_{s \in S}$. Let $G \subset \mathrm{Aut}^+(X)$ be a strongly transitive closed subgroup of $\mathrm{Aut}^+(X)$. Let B denote an Iwahori subgroup of G. Consider the quasi-regular representation $\lambda_{G,B}$ of G on $\ell^2(G/B)$. Define $d : W \to \mathbb{N}$ as the 'multiplicative' (with respect to reduced forms) extension of $d : S \to \mathbb{N}$. Then

- if $\sum_{w \in W} d_w^{-1} = \infty$, then $\lambda_{G,B}$ is a type II_∞-factor representation;
- if $\sum_{w \in W} d_w^{-1} < \infty$, then $\lambda_{G,B}$ is the direct sum of a type II_∞-factor representation and of finitely many of so-called Steinberg representations (infinite-dimensional, irreducible).

Further we can also show that if $K \supset B$ is the stabiliser subgroup of a vertex in X, and d_s is fixed (does not depend on s) then $\lambda_{G,K}$ is a type II_∞-factor representation.

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Let (W, S) be a right-angled irreducible Coxeter system with $|S| \geq 3$. Then W admits a natural collection of finite factor representations $\lambda_{\mathbf{a}}$, indexed by $\mathbf{a} = (a_s)_{s \in S} \in (-1, 1)^S$, which are pairwise unitarily inequivalent.

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What else do we know about q-Hecke operator algebras

Let (W, S) be a right-angled irreducible Coxeter system with $|S| \geq 3$, $\mathbf{q} = (q_s)_{s\in S} \in (0, 1]^S$.

- the C^*-algebra $C^*_{\mathbf{q},r}(W)$ is non-nuclear, exact (Caspers+Klisse+Larsen);
- when $\mathbf{q}$ is 'close to 1', the C^*-algebra $C^*_{\mathbf{q},r}(W)$ has unique trace (Caspers+Klisse+Larsen);
- in fact $C^*_{\mathbf{q},r}(W)$ is simple whenever $N_{\mathbf{q}}(W)$ is a factor (Klisse);
- the unordered Elliott invariant can be used to distinguish some $C^*_{\mathbf{q},r}((\mathbb{Z}/2\mathbb{Z})^{\star k})$ for different $\mathbf{q}$, but not all of them (Raum+AS)!
- when the $\mathbf{q}$-growth series diverges, the factors $N_{\mathbf{q}}(W)$ are non-injective, have weak-* completely contractive approximation property and the Haagerup property; sometimes have no Cartan subalgebras (Caspers).

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And what we would like to know

- does $C^*_{\mathbf{q},r}(W)$have unique trace in the whole range of divergence of the **q**-growth series?
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