A new inequality for Schur multipliers

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—Joint with JM Conde-Alonso, AM González-Pérez and E Tablate—

Introduction

Fourier and Schur multipliers

Hörmander-Mikhlin multipliers

Given $m : \mathbf{R}^n \to \mathbf{C}$, let

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A central problem in harmonic analysis:

Given $1 \leq p \leq \infty$, for which m's is T_m L_p-bounded?

(Well-understood: $p = 1, 2, \infty$ / The general problem is out of reach)

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Hörmander-Mikhlin theorem (1956/1960)

If $1 < p < \infty$

$$\Big\| T_m : L_p(\mathbf{R}^n) \to L_p(\mathbf{R}^n) \Big\| \le C_p \sum_{|\gamma| \le [\frac{n}{2}]+1} \Big\| |\xi|^{|\gamma|} \big| \partial_\xi^\gamma m(\xi) \big| \Big\|_\infty .$$

⋆ Locally: Key/optimal singularity at $0 \rightsquigarrow$ Asymptotic behavior.

⋆ HM up to order $(n-1)/2$: Necessary for radial multipliers and $p < \infty$!

Fourier multipliers: Group algebras

Let (G, μ) be a unimodular group with

$$\lambda : \mathrm{G} \to \mathcal{U}(L_2(\mathrm{G}, \mu)) \quad \text{given by} \quad \big[\lambda(g)\varphi\big](h) = \varphi(g^{-1}h).$$

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Define its group von Neumann algebra as follows

$$vN(\mathrm{G}) := \overline{\mathrm{span}\Big\{ f = \int_{\mathrm{G}} \widehat{f}(g)\lambda(g)\, d\mu(g) : \widehat{f} \in \mathcal{C}_{\mathrm{c}}(\mathrm{G}) \Big\}}^{\mathrm{w}} \subset \mathcal{B}(L_2(\mathrm{G}, \mu)).$$

If e is the unit in G, the Haar trace τ is then determined by $\tau(f) = \widehat{f}(e)$.

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Given $m : \mathrm{G} \to \mathbf{C}$, its **Fourier multiplier** is the map

$$\widehat{T_m f}(g) = \tau(T_m f \lambda(g)^*) = m(g)\tau(f\lambda(g)^*) = m(g)\widehat{f}(g).$$

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- ⋆ Pioneering work of Haagerup '79 + coauthors.
- ⋆ L_p-**theory**: Very strong efforts in the last 10 years
  Lafforgue-de la Salle, Junge-Mei-P, Mei-Ricard, P-Ricard-de la Salle...
- ⋆ **Approximation properties** ≈ Fourier L_p-summability
  Geometric group theory + Group vNa classification theory

The relation between Fourier and Schur multipliers plays a key role...

Given $m : \mathrm{G} \to \mathbf{C}$, its **Herz-Schur multiplier** is

$$S_m(A) = \Big(m(gh^{-1}) A_{gh} \Big).$$

Herz-Schur multipliers: Matrix algebras

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Let

$$S_p(\mathrm{G}) = \text{Schatten } p\text{-class over } L_2(\mathrm{G}) \rightsquigarrow \|A\|_p = \mathrm{tr}(|A|^p)^{\frac{1}{p}},$$

$$L_p(vN(\mathrm{G})) = \text{NC } L_p\text{-space over } (vN(\mathrm{G}), \tau) \rightsquigarrow \|f\|_p = \tau(|f|^p)^{\frac{1}{p}}.$$

Fourier-Schur transference [Neuwirth/Ricard + Caspers/de la Salle]

If $1 \le p \le \infty$ and G is **amenable**

$$\Big\| S_m : S_p(\mathrm{G}) \to S_p(\mathrm{G}) \Big\|_{\mathsf{cb}} = \Big\| T_m : L_p(vN(\mathrm{G})) \to L_p(vN(\mathrm{G})) \Big\|_{\mathsf{cb}}.$$

Moreover, the upper bound holds for nonamenable l.c. groups as well.

Combining FS transference with HM theorem

$$\underbrace{\Big\| S_m \colon S_p(\mathbf{R}^n) \to S_p(\mathbf{R}^n) \Big\|_{\mathrm{cb}} \lesssim \frac{p^2}{p-1} \sum_{|\gamma| \le [\frac{n}{2}]+1} \Big\| |\xi|^{|\gamma|} \big| \partial_\xi^\gamma m(\xi) \big| \Big\|_\infty .}_{\textbf{(HMS)}}$$

m is **constant on secondary diagonals**
and admits a singularity in the main diagonal

NonToeplitz Schur multipliers

Arbitrary Schur multipliers in $\mathbf{R}^n \rightsquigarrow M(x,y) \neq m(x-y)$...

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The Grothendieck-Haagerup characterization

S_M is bounded on $\mathcal{B}(L_2(\mathrm{X}))$ iff S_M is cb-bounded iff there exists a Hilbert space $\mathcal{K}$ and uniformly bounded families (u_x) and (w_y) in $\mathcal{K}$ satisfying the identity

$$M(x,y) = \langle u_x, w_y \rangle_{\mathcal{K}} \quad \text{for all} \quad x,y \in \mathrm{X}.$$

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Rather limited literature regarding L_p-**boundedness** $(1 < p < \infty)$:

⋆ The **Arazy conjecture** = Submatrices of

$$M(x,y) = \frac{f(x) - f(y)}{x - y} \quad \text{for } f \in \mathrm{Lip}(\mathbf{R}).$$

Conjectured by Arazy '82 and solved by Potapov/Sukochev '11.

⋆ **Marcinkiewicz type conditions**: Bded variation columns/rows.

⋆ **Unconditionality in S_p and matrix $\Lambda(p)$-sets**: Harcharras '99.

Mikael de la Salle's problem

During the École d'automne "Fourier Multipliers on Group Algebras" at Besançon (2019), Mikael de la Salle formulated the problem below (also at UCLA Functional Analysis Seminar in 2020):

M. de la Salle's question. Let $M\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ be smooth outside the diagonal, with compact support for simplicity. Is there a controlled explosion on the diagonal which gives $S_M : S_p \to S_p$ for $1 < p < \infty$?

This conjecture for (nonToeplitz) multipliers is beyond FS transference.

The main result
Hörmander-Mikhlin-Schur multipliers

An easy remark

In $\mathbf{Z}$, every Toeplitz symbol $M(j,k) = m(j-k)$ is identified with the Fourier multiplier T_m on the torus $\mathbf{T}$. Moreover, setting $M_\alpha(j,k)=\alpha(j)$ and $M_\beta(j,k) = \beta(k)$ we note that

$$S_{M_\alpha}(A) = \mathrm{diag}(\alpha) \cdot A \quad \text{and} \quad S_{M_\beta}(A) = A \cdot \mathrm{diag}(\beta).$$

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Then, recalling that M can be rewritten as

$$\begin{aligned} M(j,k) &= M_r(k-j,k) &= \sum\nolimits_\ell m_{r\ell}(j-k)\beta_\ell(k), \\ M(j,k) &= M_c(j,j-k) &= \sum\nolimits_\ell \alpha_\ell(j) m_{c\ell}(j-k), \end{aligned}$$

S_M is a combination of **Fourier** and left/right **pointwise** multipliers...

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Schur multipliers are conceivably **matrix pseudodifferential operators**.

Expected: Regularity conditions in terms of infinitely many mixed $\partial_x \partial_y$.

Main result: Finite many unmixed ∂_x, ∂_y + Diagonal singularity (Mikael).

Euclidean HMS multipliers

Theorem A

If $1 < p < \infty$

$$\big\| S_M : S_p(\mathbf{R}^n) \to S_p(\mathbf{R}^n) \big\|_{\mathrm{cb}} \lesssim \frac{p^2}{p-1} |||M|||_{\mathrm{HMS}},$$

$$|||M|||_{\mathrm{HMS}} := \sum_{|\gamma| \le [\frac{n}{2}]+1} \Big\| |x-y|^{|\gamma|} \Big\{ \big|\partial_x^\gamma M(x,y)\big| + \big|\partial_y^\gamma M(x,y)\big| \Big\} \Big\|_\infty .$$

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Remark. Theorem A is a strict generalisation of Mikhlin's theorem.

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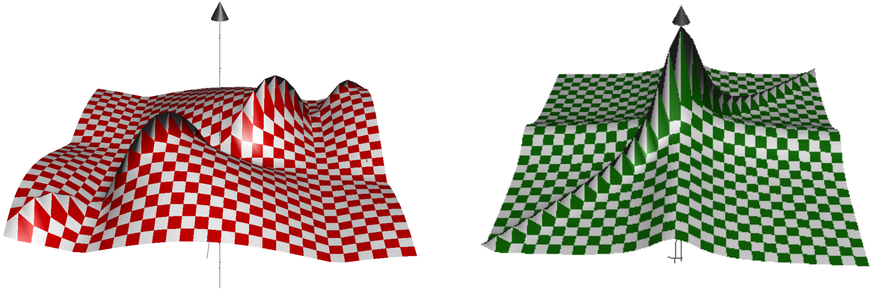
More general statements:

⋆ Replace $\mathbf{R}^n$ by G.

⋆ Replace $[n/2]+1$ by $n/2+\varepsilon$.

⋆ Replace $|\xi|$ by anisotropic metrics.

Euclidean HMS multipliers



NonToeplitz Hörmander-Mikhlin-Schur multipliers in $\mathbf{R} \times \mathbf{R}$
Any Toeplitz symbol would be forced to be constant at $x = y + \alpha$ for all $\alpha \in \mathbf{R}$, unlike above

Applications
Matrix algebras

1. Less regularity near L_2

Define

$$\left| M \right|_{q\sigma} := \sup_{\substack{j \in \mathbf{Z} \\ x,y \in \mathbf{R}^n}} \left\| \psi(\cdot - y) M(2^j \cdot, 2^j y) \right\|_{W_{q\sigma}} + \left\| \psi(x - \cdot) M(2^j x, 2^j \cdot) \right\|_{W_{q\sigma}}.$$

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Corollary A1 (HMS p-conditions)

If $\big|1/p - 1/2\big| < \delta/n$ and $n/q < \delta < n/2$

$$\big\| S_M : S_p(\mathbf{R}^n) \to S_p(\mathbf{R}^n) \big\|_{\mathrm{cb}} \leq C_p \, \big| M \big|_{q\delta}.$$

This is a Schur multiplier extension of the Calderón-Torchinsky theorem.

Corollary A2 (Arazy's conjecture)

If $M(x,y) = \big(f(x) - f(y)\big)/(x-y)$ for $x \neq y$, then

$$\big\| S_M : S_p(\mathbf{R}) \to S_p(\mathbf{R}) \big\|_{\mathrm{cb}} \leq C \frac{p^2}{p-1} \|f\|_{\mathrm{Lip}} \quad \text{for} \quad 1 < p < \infty.$$

Immediate! Potapov/Sukochev confirmed it in 2011 (sophisticated).

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If $0 < \alpha < 1$, set

$$M_{\alpha f}(x,y) = \frac{f(x) - f(y)}{|x-y|^\alpha} \quad \text{for} \quad f : \mathbf{R} \to \mathbf{C} \quad \alpha\text{-Hölder} \quad \text{and} \quad x \neq y.$$

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Corollary A2+ (Beyond Arazy's conjecture)

If $|1/p - 1/2| < \min\{\alpha, 1/2\}$, then

$$\big\| S_{M_{\alpha f}} : S_p(\mathbf{R}) \to S_p(\mathbf{R}) \big\|_{\mathrm{cb}} \leq C_p \|f\|_{\Lambda_\alpha}.$$

We get complete S_p-boundedness for $1 < p < \infty$ as long as $\alpha \geq 1/2$.

3. Matrix LP partitions

Let $\Psi : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+$ be smooth st:

a) $\sum_{j \in \mathbf{Z}} \Psi_j = 1$ a.e. with $\Psi_j(x, y) = \Psi(2^j x, 2^j y)$.

b) The supports of $\{\Psi_j : j \in \mathbf{Z}\}$ have finite overlapping.

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Corollary A3 (Matrix LP theorem)

Let $1 < p < \infty$:

i) If $p \leq 2$

$$\|A\|_{S_p} \asymp_{\mathrm{cb}} \inf_{S_{\Psi_j}(A) = A_j + B_j} \Big\| \Big(\sum_{j\in\mathbf{Z}} A_j A_j^* + B_j^* B_j \Big)^{\frac{1}{2}} \Big\|_{S_p}.$$

ii) If $p \geq 2$

$$\|A\|_{S_p} \asymp_{\mathrm{cb}} \Big\| \Big(\sum_{j\in\mathbf{Z}} S_{\Psi_j}(A) S_{\Psi_j}(A)^* + S_{\Psi_j}(A)^* S_{\Psi_j}(A) \Big)^{\frac{1}{2}} \Big\|_{S_p}.$$

NonToeplitz example — Pick radial $\Psi_j(x,y) = \psi_j\big[\big(|x|^2 + |y|^2\big)^{\frac{1}{2}}\big]$.

Corollary A4 (A discrete formulation)

If $\Delta\varphi(k) := \varphi(k+1) - \varphi(k)$ and $1 < p < \infty$

$$\big\| S_M : S_p(\mathbf{Z}) \to S_p(\mathbf{Z}) \big\|_{\mathrm{cb}} \lesssim \frac{p^2}{p-1} |||M|||_{\mathrm{HMS}_\Delta},$$

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Corollary A5 (Herz-Schur multipliers)

If $m(g) = \widetilde{m}(\beta(g)) = m'(\beta(g^{-1}))$ and $M(g,h) = m(gh^{-1})$

$$\big\| S_M : S_p(\mathrm{G}) \to S_p(\mathrm{G}) \big\|_{\mathrm{cb}}$$

$$\lesssim_p \sup_{g \in \mathrm{G}} \sum_{|\gamma| \le [\frac{n}{2}]+1} \Big\| |\xi|^{|\gamma|} \Big\{ \big| \partial_\xi^\gamma (\widetilde{m} \circ \alpha_g)(\xi) \big| + \big| \partial_\xi^\gamma (m' \circ \alpha_g)(\xi) \big| \Big\} \Big\|_\infty.$$

This mostly recovers the work of Junge/Mei/P in group vN algebras.

Back to Fourier multipliers

Lie group algebras

Cocycle approach [JMP]:

⋆ Finite-dimensional orthogonal cocycles.

⋆ Auxiliary differential structures less natural for Lie groups.

A local HM theorem

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⋆ Local Mikhlin conditions for Lie derivatives in $SL_n(\mathbf{R})$.

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Theorem C1 (Local HM criterion)

Let G be a n-dimensional unimodular Lie group with its Riemannian metric $L_{\mathrm{R}} : \mathrm{G} \times \mathrm{G} \to \mathbf{R}_+$. Let $1 < p < \infty$ and let $m : \mathrm{G} \to \mathbf{C}$ be a Fourier symbol supported by a sufficiently small neighborhood of the identity. Then, the following inequality holds

$$\big\| T_m \big\|_{\mathrm{cb}(L_p(vN(\mathrm{G})))} \lesssim \frac{p^2}{p-1} \sum_{|\gamma| \le [\frac{n}{2}]+1} \big\| L_{\mathrm{R}}(g,e)^{|\gamma|} d_g^\gamma m(g) \big\|_\infty .$$

Theorem C2 (HM for stratified Lie groups)

Let G be a n-dimensional stratified Lie group. Let $L_{\mathrm{SR}} : \mathrm{G} \times \mathrm{G} \to \mathbf{R}_+$ be the subRiemannian metric wrt its homogeneous dilation. Then, the following inequality holds for any $m : \mathrm{G} \to \mathbf{C}$ and $1 < p < \infty$

$$\big\| T_m \big\|_{\mathrm{cb}(L_p(vN(\mathrm{G})))} \lesssim \frac{p^2}{p-1} \sum_{|\gamma| \le [\frac{n}{2}]+1} \big\| L_{\mathrm{SR}}(g,e)^{\{\gamma\}} d_g^\gamma m(g) \big\|_\infty.$$

Stratified Lie groups

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Stratified Mikhlin condition for all Fourier multipliers:

$$\{\gamma\} = \sum_{k=1}^{n} \ell_k \big| \{ s : j_s = k \} \big| = \sum_{s=1}^{|\gamma|} \ell_{j_s} .$$

(A derivative in the k-th stratum is dealt with as a k-th order derivative)

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Stratified Lie groups

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- ⋆ Thm C2 gives **new forms of noncommutative Riesz transforms**.

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Let G be a n-dimensional simple Lie group with $n \geq 2/\tau_{\mathrm{G}}$. Then, the following inequality holds for any $m : \mathrm{G} \to \mathbf{C}$ and every $1 < p < \infty$

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⋆ Then, the weight L_{G} is locally Euclidean and

$$L_{\mathrm{G}}(g) \approx \|\mathrm{Ad}_g\|^{\tau_{\mathrm{G}}} \quad \text{assymptotically.}$$

⋆ $\tau_{SL_n(\mathbf{R})} = \dfrac{1}{2} \Rightarrow$ Thm C3 improves and generalizes [PRS] for $SL_n(\mathbf{R})$.

Thank you!!

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New NCCZ ideas: RHS $\leq |||M|||_{\mathrm{HMS}}$ (Noncommuting CZ kernels!).