A $C(K)$-space with few operators

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Banach Algebras and Applications

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$$U(A, F) = \{x_n : n \in A \setminus F\} \cup \{y_A\}, \quad \text{where} \quad F \in [\mathbb{N}]^{<\omega},$$

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