Influence of global cosmological expansion on local dynamics and kinematics

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(Published 28 January 2010)

Attempts to estimate the influence of global cosmological expansion on local systems are reviewed. Here "local" is taken to mean that the sizes of the considered systems are much smaller than cosmologically relevant scales. For example, such influences can affect orbital motions as well as configurations of compact objects, like black holes. Also discussed are how measurements based on the exchange of electromagnetic signals of distances, velocities, etc. of moving objects are influenced. As an application, orders of magnitude of such effects are compared with the scale set by the apparently anomalous acceleration of the Pioneer 10 and 11 spacecrafts, which is 10^{-9} m/s^2. There is no reason to believe that the latter is of cosmological origin. However, the general problem of gaining a qualitative and quantitative understanding of how the cosmological dynamics influences local systems remains challenging, with only partial clues being so far provided by exact solutions to the field equations of general relativity.

PACS number(s): 95.30.Sf, 04.20.Cv, 04.20.Jb, 98.80.Jk

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I. INTRODUCTION

There is by now ample evidence that our Universe is expanding on average. This means that on the largest scales one observes redshifts from structures that are interpreted as recessional motion, also called the Hubble flow. To first approximation, the relative velocity

between two structures grows linearly with their mutual distance. The constant of proportionality is the so-called Hubble constant H_0 whose value is now fairly accurately measured as being close to 70 km s^{-1} Mpc^{-1}; see, e.g., Komatsu *et al.* (2009). This means that for any additional megaparsec (Mpc=3.262×10^6 lightyears =3.086×10^{19} km) the recessional velocity picks up an extra 70 km per second. Clearly, typical peculiar velocities superimpose on the global Hubble flow. For galaxies they can be up to 1000 km per second, so that the Hubble flow definitely dominates at distances above 200 Mpc, i.e., above supercluster scale. In this respect it is remarkable that Hubble's classic paper (Hubble, 1929) of 1929 plots the velocity-distance relation of extragalactic nebulae only up to 2 Mpc, though it has to be added that in those days distances where generally underestimated, sometimes up to a factor of 10.

For pedagogical purposes the global expansion is sometimes represented by the two-dimensional balloon model, in which three-dimensional space corresponds to the two-dimensional surface of an inflating rubber balloon; see, e.g., Misner *et al.* (1973). At each point attached to the rubber material an observer sees other points attached in a state of radial recessional motion, the faster the further they are away. This picture is used to stress that each point is locally (i.e., with respect to the local rubber material) at rest but receding from all other points because space in-between is itself expanding. However, this global expansion does not affect all structures: Local overdensities in the matter distribution may inhibit space from expanding. In the balloon model of Misner *et al.* (1973) this is represented by little pennies being glued onto the balloon. The rubber material underneath the coins does not expand due to the stiff glue which holds it in place. The question arises. What, in reality, are the structures corresponding to the coin and what dynamical mechanism provides the glue? It is often heard that "bound systems" do not participate in the global expansion, or that systems below the scale of galaxy clusters "break away" from the Hubble flow. But what does bound and break away really mean?[1] For example, is it obvious that the astronomical unit is not affected by global expansion [compare Krasinsky and Brumberg (2004) and Standish (2004)] or can it even be, as suggested by Fahr and Siewert (2008), that our Universe is contracting on small scales while it expands in the large? If so, what precisely would rule the relation between contracting and expanding scales?

The purpose of this paper is to review and discuss attempts that aim to make precise and answer some of these fundamental questions, taking due account of the dynamical laws and the kinematical framework of general relativity. We emphasize the changes in kinematical relations within time-dependent spacetime geometries, which seem to be widely neglected in related discussions.

Next to being a question of fundamental interest, the raised issue also needs to be clarified quantitatively in connection with more practical aims, like, e.g., the modeling of celestial reference frames (Klioner and Soffel, 2005). The specific question of whether the global expansion has any influence on the local dynamics and kinematics within the Solar System has recently also attracted increasing attention in connection with the so-called "Pioneer Anomaly" (Anderson *et al.*, 1998, 2002; Markwardt, 2002; Nieto and Turyshev, 2004; Turyshev *et al.*, 2005a, 2005b), henceforth abbreviated by PA. Here frequency measurements in Doppler tracking are translated into standard kinematical quantities, like velocity and acceleration. The result shows an anomalous acceleration of the Pioneer satellites directed towards the center of the Solar System. Markwardt (2002) reported the magnitude of this acceleration to be $a=8.6\pm1.34\times10^{-10}$ m s^{-2}. Note that such an apparently small acceleration amounts to variations in spatial localization of nearly 500 km after 10 years. It so happens that the magnitude of this acceleration is very close to the product of the current value of the Hubble constant H_0 and the velocity of light in vacuum,

$$H_0 c \approx (70 \text{ km s}^{-1} \text{ Mpc}^{-1})(3\times10^5 \text{ km s}^{-1})$$

$$= 7\times10^{-10} \text{ m s}^{-2}. \tag{1}$$

Whether this "almost coincidence" of numbers does indeed have any deeper significance can and should only be decided on the basis of reliable estimates within the dynamical framework of general relativity. There already exist various speculations and claims in the literature that try to attribute the PA to either simple kinematical [see, e.g., Rosales and Sanchez-Gomez (1998)] or dynamical [see, e.g., Fahr and Siewert (2008)] effects of a time varying background geometry, though none of them does justice to the requirements posed by general relativity[2] This is clearly a difficult task: There is little analytical knowledge of how to model in terms of exact solutions, or at least in terms of controlled approximations to exact solutions, the hierarchy of mutually embedded systems: Solar System→Galaxy→Local Group →Cluster→Supercluster→Standard-Cosmological Solution. Usually we expect each such system to define a typical length scale beyond which we may consider it as quasi-isolated (Cox, 2007). But, clearly, whether this is a valid assumption or not can only be decided on the basis of a self-consistent dynamical consideration. In our context all this suggests to first study the influence of cosmic expansion on the most simple systems immersed in an otherwise homogeneous cosmological background. We will see that this already poses a number of nontrivial analytical as well as conceptual problems.

In this article we derive upper bounds for various effects of global expansion on local systems in the context of such simple models. The idea here is that the upper

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[1] For a discussion on the meaning of "joining the Hubble flow," see Barnes *et al.* (2006).

[2] The interested reader will soon find out that we disagree with all such claims.

bounds so derived will *a fortiori* be upper bounds in more realistic models, since a further embedding of the system we consider into a higher structure of local over-densities will further suppress the influence of cosmological expansion. This is evidently true in situations in which the spherically symmetric Einstein-Straus model applies, but can also be argued for as a result of taking into account small-scale anisotropies in the matter distribution, as has been done from first-order perturbations of the Newtonian equations (Dominguez and Gaite, 2001). We conclude from this that if we find the relevant upper bounds to be outside current experimental reach, this will maintain to be the case in more realistic contexts.

II. STRATEGIC OUTLINE AND RESULTS

A. Improved Newtonian equations

The strategies that so far have been followed are twofold: Either one studies *modified* Newtonian or special relativistic equations of motions for two point-particles with a force of mutual attraction (gravitational or electromagnetic). The modifications are derived from putting the system into a fixed standard-cosmological background (usually spatially flat) without back-reactions being taken into account. We discuss this approach in Secs. III and IV. Our discussion, based on Carrera and Giulini (2005), complements the perturbative analysis in Cooperstock *et al.* (1998), which misses all orbits which are unstable under cosmological expansion (which do exist). In this respect we follow a similar strategy as in Price (2005) [the basic idea of which goes back at least to Pachner (1963, 1964)] and also Adkins *et al.* (2007), though we think that there are also useful differences. We also supply quantitative estimates and clarify that the improved Newtonian equations of motion are written in terms of the right coordinates (nonrotating and metrically normalized). The purpose of this model is to develop a good physical intuition for the qualitative as well as quantitative features of any *dynamical* effects involved.

Eventually the Newtonian model just mentioned has to be understood as a limiting case of a genuinely relativistic treatment. For the gravitational case this is done in Sec. V (an alternative and more geometric derivation is given in Sec. VI.B), where we employ the McVittie metric to model a spherically symmetric mass embedded in a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The geodesic equation is then, in a suitable limit, shown to lead to the improved Newtonian model discussed above [see also Carrera and Giulini (2005)]. The same holds for the electromagnetic case, as shown in Sec. IV. There we take a slight detour to also reconsider a classic argument by Dicke and Peebles (1964), which shows the absence of any relevant dynamical effect of global expansion. Its original form only involved the dynamical action principle together with some simple scaling argument. Since this reference is one of the most frequently cited in this field, and since the simplicity of the argument (which hardly involves any real analysis) is definitely deceptive, we give an independent treatment that makes no use of any hypothetical scaling rules for physical quantities other than spatial lengths and times. Our treatment, which follows Carrera and Giulini (2005), also reveals that the original argument by Dicke and Peebles is insufficient to discuss leading-order effects of cosmological expansion. It is therefore also ineffective in its attempt to contradict Pachner (1963, 1964).

B. Exact solutions

The other approach consists of finding exact solutions to Einstein's field equations for an inhomogeneous situation that, in the most simple case, models a single, quasilocalized, nonrotating, electrically neutral inhomogeneity within a FLRW universe. Using this inhomogeneous solution as background one can then study the motion of test particles (following geodesics in the background geometry) and, in particular, the influence of expansion on this motion.

This approach can be subdivided into two strategies. The first tries to literally construct a new exact solution out of two known ones, so that the new solution contains a connected piece from each of the two old ones as isometric submanifolds. These we refer to as *matched solutions*. This is relaxed in the second, more general strategy, where the new solution is merely required to somehow approximate the relevant part of each of the two old solutions in some region. These we refer to as *melted solutions*. Needless to say that melted solutions offer a much greater variety for construction than matched ones. However, it is also true that often not much is known about the proper physical interpretation of the former. In this respect the matching solutions usually provide a much clearer picture.

According to the above requirements, in both cases we restrict attention to spherically symmetric spacetimes which approximate a FLRW solution of standard cosmology for "large radii" and a noncharged nonrotating compact object characterized by the exterior Schwarzschild solution for "small radii." (Clearly there must be some characteristic radius in terms of which large and small radii are defined.) Also, one often restricts attention to the spatially flat FLRW models for simplicity, which also seems justified in view of current cosmological data which are compatible with spatial flatness.

1. Matched solutions

A first approach to the matching idea was initiated by Einstein and Straus (1945, 1946) and later worked out in more analytical detail by Schücking (1954). Here the matched solution is really such that for radii smaller than a certain matching radius R_v (henceforth called the *vacuole* or *Schücking radius*) it is exactly given by the Schwarzschild solution (exterior for a black hole, exterior plus interior for a star) and for radii above this radius it is exactly given by a FLRW universe for dust

matter without cosmological constant (this can be generalized, see below). The radius R_v is a function of the central gravitational mass M and the cosmological mass density ϱ, through the latter of which it also depends on the cosmological time t. It is determined by

$$\frac{4\pi}{3}R_v^3\varrho = M. \tag{2}$$

This formula holds for flat as well as curved FLRW models if "radius" is taken to mean "areal radius," the definition of which is that a two-sphere of areal radius R has a proper surface area of $4\pi R^2$. In flat space the areal radius coincides with the proper radius (the geodesic distance between the center and any point on the sphere), so that $4\pi R_v^3/3$ is just the proper volume inside the sphere of radius R_v (cf. Sec. V.A). However, in backgrounds of positive (negative) curvature this expression is smaller (larger) than the proper volume [the proper volume grows faster (slower) with areal radius] and hence, for given ϱ, the left-hand side of Eq. (2) is also smaller (larger) than the proper mass of the dust contained within a sphere of areal radius R_v.

Here we recall that the gravitational mass of a lump of matter is not just proportional to the amount of matter (baryons) in that region. For example, the kinetic energy as well as the gravitational binding energy also contributes to the gravitational mass. This is expressed in Eq. (D47) of Appendix D.4, where further explanations will be provided. As is well known, the mathematical characterization of appropriate notions of quasilocal gravitational mass that would apply to general spacetimes is a notoriously difficult problem to which various attempts for solutions exist; see Szabados (2004) for the current status. However, in the spherically symmetric case, to which we restrict attention, the so-called Misner-Sharp energy gives a satisfying and convenient concept of active gravitational mass. Its definition will be given in Sec. V.A and more details, including its equality in value to the Hawking mass, are discussed in Appendixes D.3 and D.4.

The original construction by Einstein and Straus and its analytical completion by Schücking were quite complicated. We give a simpler and conceptually clearer description in Sec. V.C, using a suitable reformulation of the condition for the matching of solutions. However, it is not hard to gain some intuitive understanding for the matching construction and the value of R_v as defined by Eq. (2). For the moment we restrict to the spatially flat case and consider the homogeneous and isotropic dust-filled universe at some moment of time t. The dust within a three-ball of proper radius R_v represents an amount of matter of total mass M as given by Eq. (2). Now compress this amount of matter in a spherically symmetric fashion until it becomes a compact star or a black hole. In Newtonian gravity the gravitational field outside a spherically symmetric mass distribution only depends on the total mass and not on its radial density distribution. This is also true in general relativity, which

is essentially the content of Birkhoff's theorem.[3] Hence the above compression preserves equilibrium (albeit an unstable one, see below) for the dust particles just outside the boundary sphere of radius R_v. For radii smaller than R_v we have the Schwarzschild solution (which is the unique nontrivial spherically symmetric vacuum solution according to Birkhoff's theorem) which therefore matches to the FLRW solution for $R \geqslant R_v$ at the boundary $R = R_v$ where the matter density is discontinuous. The spatial two-sphere $R = R_v$ is comoving with the Hubble flow, meaning that its proper surface area grows in case of expansion. Finally, in case of constant positive (negative) spatial curvature, Eq. (2) tells us that the matched Schwarzschild solution has a smaller (larger) mass than the mass that the amount of dust represents within the ball of areal radius R_v within the FLRW universe.

The Einstein-Straus model can be generalized in several ways. Instead of cutting out one ball, one can cut several nonoverlapping ones and fill in the interiors with Schwarzschild geometries of appropriate masses. For obvious reasons these are sometimes referred to as "swiss-cheese models." These, in turn, can be generalized to the cases of nonvanishing cosmological constant (Balbinot *et al.*, 1988) or nonvanishing pressure (Bona and Stela, 1987). Finally, the Einstein-Straus model can be generalized to spherically symmetric but inhomogeneous Lemaître-Tolman-Bondi (LTB) cosmological backgrounds (Bonnor, 2000).

Since for the Einstein-Straus model the geometry within $R \leqslant R_v$ is exactly Schwarzschild (for vanishing cosmological constant) or Schwarzschild–de Sitter spacetime (for nonvanishing cosmological constant), it is clear that any dynamical system situated in this background geometry (no back reaction) only detects that part of the cosmic expansion that is due to a nonvanishing cosmological constant. In particular, for vanishing cosmological constant, the cosmic expansion that goes on outside the expanding vacuole $R = R_v$ is not felt from within. Hence global expansion due to ordinary (localizable) matter can, in principle, be completely inhibited by local inhomogeneities.

There are, however, several severe problems concerning the Einstein-Straus approach. First, it cannot provide a realistic model for the environment of small structures in our Universe, small meaning below the scales of galaxy clusters or superclusters. To see this, apply Eq. (2) to a spatially flat universe whose background matter density ϱ is given by the critical density

$$\varrho_{\text{crit}} := 3H_0^2/8\pi G, \tag{3}$$

where G is Newton's constant. Then Eq. (2) gives

$$R_v = (R_S R_H^2)^{1/3} \approx (M/M_\odot)^{1/3}400 \text{ ly}, \tag{4}$$

where

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[3] A proof of Birkhoff's theorem will appear as a by-product from our considerations in Appendix D.3.

$$R_S := 2GM/c^2 \approx (M/M_\odot)3 \text{ km}, \quad (5)$$

$$R_H := c/H_0 \approx 4 \text{ Gpc} \approx 1.3 \times 10^{23} \text{ km}, \quad (6)$$

are the *Schwarzschild radius* for the mass M and the *Hubble radius*, respectively. $M_\odot = 2 \times 10^{30}$ kg is the solar mass.

For a single solar mass this gives a vacuole radius of almost 400 lightyears, which is almost two orders of magnitude larger than the average distance of stars in our Galaxy. Therefore, the swiss-cheese model cannot apply at the scale of stars in galaxies. This changes as one goes to larger scales. For example, the Virgo cluster is estimated to have a mass of approximately 10^{15} solar masses (Fouqué *et al.*, 2001),[4] which makes its vacuole radius 10^5 times larger than that for a single solar mass, so that it is approximately given by 10 Mpc. This is just a little smaller than the average distance of groups and clusters of galaxies within the Virgo supercluster. Hence the Einstein-Straus approach might well give viable models above cluster scales. A similar conclusion can be drawn for the vacuole construction in LTB spacetimes (Bonnor, 2000): There it is argued that the vacuole might be as big as the Local Group.

The Einstein-Straus solution (as well as its generalization for LTB spacetimes given by Bonnor) may also be criticized on theoretical grounds. An obvious one is its dynamical instability: slight perturbations of the matching radius to larger radii will let it increase without bound, slight perturbations to smaller radii will let it collapse. This can be proven formally [see, e.g., Krasiński (1998) and Bonnor (2000)] but it is also rather obvious, since R_v is defined by the equal and opposite gravitational pull of the central mass on one side and the cosmological masses on the other. Both pulls increase as one moves towards their side, so that the equilibrium position must correspond to a local maximum of the gravitational potential. Another criticism of the Einstein-Straus solution concerns the severe restrictions under which it may be generalized to non-spherically-symmetric situations; see, e.g., Senovilla and Vera (1997) and Mena *et al.* (2002, 2003, 2005).

2. Melted solutions

The above discussion shows that the Einstein-Straus approach does not give us useful information regarding the dynamical impact of cosmic expansion on structures well below the scales of galaxy clusters. For this reason other exact solutions are sought. In this respect we remind the reader on the following general aspect: In physics we are hardly ever in the position to mathematically rigorously model physically realistic scenarios. Usually we are at best able to provide either approximate solutions for realistic models or exact solutions for approximate models, and in most cases approximations are made on both sides. The art of physics then precisely consists in finding the right mixture in each given case. However, in this process our intuition usually strongly rests on the existence of at least some "nearby" exact solutions. Accordingly, one seeks exact solutions in general relativity that, with some degree of physical approximation, model a spherically symmetric body immersed in an expanding universe. However, it is not as easy as one might think at first to characterize "body" and "immersed."[5] Clearly it is associated with some inhomogeneity in form of a spatial region with an overdense matter distribution, as compared to that of the approximately homogeneous distribution far out. But a body should also be quasi-isolated in order to be distinguishable from a mere local density fluctuation with a smooth transition. Typical exact solutions that models the latter are the LTB solutions, in which matter is represented by pressureless dust that freely falls into the local overdense inhomogeneity. In some sense, these form the other extreme to the Einstein-Straus solutions in that they make the transition as smooth and mild as one wishes. Here we are interested in models that somewhat lie in-between these extremes.

An attempt to combine an interior Schwarzschild solution (representing a star) and a flat FLRW universe was made by Gautreau (1984). Here the matter model consists of two components, a perfect fluid with pressure and equation of state $p = p(\varrho)$ outside the star, and the superposition of this with the star's dust matter inside the star. However, Gautreau also made the assumption that the matter outside the star moves on radially infalling geodesics, which is only consistent if the pressure outside is spatially constant. Thus one is reduced to exact FLRW outside the star (van den Bergh and Wils, 1984) or the LTB model. [Further remarks may be found in Krasiński (1998).] Other solutions, modeling a black hole in a cosmological spacetime, have been given. However, these solutions model objects which are either rotating (Vaidya, 1977, 1984; Ramachandra *et al.*, 2003), charged (Gao and Zhang, 2004), or both (Patel and Trivedi, 1982). Surveys on the subject of cosmological black holes are given in Vishveshwara (2000) and McClure (2006). Further interesting solutions are given in Rajesh Nayak *et al.* (2001) and in Sultana and Dyer (2005) and Faraoni and Jacques (2007). The solutions proposed in the latter two works can be seen as generalizations of McVittie's model (McVittie, 1933), discussed in Sec. V.D. A crucial feature of these solutions is, however, that the strength of the inhomogeneity[6] varies in time, whereas for the McVittie model it remains constant. These solutions are of interest in their own right [for a detailed analysis see Carrera and Giulini

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[4]Their considerations are based on a LTB model for the cluster.

[5]In a linear theory, the "simultaneous presence" of two structures, like a local inhomogeneity in an "otherwise" homogeneous background, naturally corresponds to the mathematical operation of addition of the corresponding individual solutions. In a nonlinear theory, however, no such simple recipe exists.

[6]In Sec. V.D.1 we identify the strength of the inhomogeneity with the Weyl part of the Misner-Sharp energy.

(2009)], but our goal here is to focus on the effects due to cosmological expansion and not on the effects due to a changing strength of the central inhomogeneity. The solution proposed in the former work (Rajesh Nayak *et al.*, 2001) is the melting of a Schwarzschild spacetime in an Einstein's static universe. This is a purely static solution whose properties and geodesics where studied in Ramachandra and Vishveshwara (2002). For our purposes, however, this spacetime is not interesting since it is asymptotically an Einstein universe, and hence not in agreement with the present picture of our universe at large scales.

For these reasons in Sec. V.D we pay attention to the McVittie model. This contains a distinguished central object in the sense that the mass within a sphere centered at the inhomogeneity splits into a piece that comes from the continuously distributed cosmological fluid (with pressure) and a constant piece that does not depend on the radius of the enclosing sphere; see Eq. (82). Moreover, the latter piece is also constant in time, meaning that the strength of the central inhomogeneity remains constant. By the way, McVittie's solutions contain the Schwarzschild–de Sitter one as a special case, which was recently used to estimate the effects of cosmological expansion on local systems (Kagramanova *et al.*, 2006; Hackmann and Lämmerzahl, 2008c). In Sec. V.D.2 we show that in a suitable weak-field and slow-motion approximation the geodesic equation in McVittie spacetime reduces to the improved Newtonian equations discussed earlier. An alternative and more geometric derivation of the improved Newtonian equations for the McVittie case is presented in Sec. VI.B [see Eq. (149)].

C. Kinematical effects

1. Timing and distances

Neither the improved Newtonian model nor other general *dynamical* arguments make any statement about possible *kinematical* effects, i.e., effects in connection with measurements of *spatial distances* and *time durations* in a cosmological environment whose geometry changes with time. This is an important issue if one wants to perform the tracking of a spacecraft, that is a "mapping out" of its trajectory, which basically means to determine its simultaneous spatial distance to the observer at given observer times. But we know from general relativity that the concepts of "simultaneity" and "spatial distance" are not uniquely defined. This fact needs to be taken due care of when analytical expressions for trajectories, e.g., solutions to the equations of motion in some arbitrarily chosen coordinate system, are compared with experimental findings. In those situations it is likely that different kinematical notions of simultaneity and distance are involved which need to be properly transformed into each other before being compared. For example, these transformations can result in additional acceleration terms involving the product (1). Accordingly, there were claims that these kinematical effects could account for the PA; see, e.g., Rosales and Sanchez-Gomez (1998), Rosales (2002), Nottale (2003),

Nieto *et al.* (2005), Palle (2005), and Ranada (2005), and also statements to the contrary (Lämmerzahl *et al.*, 2008). In Sec. VI.A, following Carrera and Giulini (2005), we confirm the existence of kinematical acceleration terms proportional to $H_0 c$, but they are suppressed with additional powers of $\beta = v/c$, which renders them irrelevant as far as the PA is concerned.

2. Doppler tracking

The discussion in Sec. VI.B is based on Carrera and Giulini (2006). We explain in some detail the geometric theory for setting up the kinematical framework in which Doppler tracking should be discussed in order to properly speak of relative velocities and accelerations. This is a nontrivial issue which is, not properly appreciated in the literature on this subject [related general discussions are Bini *et al.* (1995) and Bolós (2007)]. Using this setting, we show how to derive an exact Doppler-tracking formula for a flat FLRW universe. This we use to give reliable upper bounds for kinematical effects caused by cosmic expansion. We also discuss generalizations to McVittie spacetime. Even though such effects exist, they again turn out to be irrelevant for the PA.

III. NEWTONIAN APPROACH

In order to gain intuition we consider a simple bounded system, say an atom or a planetary system, immersed in an expanding cosmos. We ask for the effects of this expansion on our local system. Does our system expand with the cosmos? Does it expand only partially? Or does it not expand at all? Here we shall not be concerned with the far more complex problem of how stable large-scale structures may emerge from unstable local gravitational dynamics in an expanding universe. This has been discussed in Buchert and Dominguez (2005), and references therein.

A. Restricted two-body problem in an expanding universe

We consider the dynamical problem of two bodies attracting each other via a force with $1/R^2$ falloff. For simplicity we may think of one mass as being much smaller than the other one, though this is really inessential. One may think of two galaxies, a star and a planet, a planet and a spacecraft, or a (classical) atom given by an electron orbiting around a proton. The system is placed into an isotropically expanding ambient universe. We wish to know the leading-order influence of the ambient expansion onto the relative two-body dynamics.

To leading order, the global expansion is described by the simple linear *Hubble law* $\dot{R} = HR$, which states that the relative radial velocity of two comoving objects at a mutual distance R grows proportional to that distance. More precisely, the term "distance" is understood here as the geodesic distance in the spacetime hypersurface of constant cosmological time t between its two intersection points with the two worldlines of the objects considered. H denotes the *Hubble parameter*, which gener-

ally depends on t but not on space. It is given in terms of the *scale parameter* $a(t)$ via $H = \dot{a}/a$.

Taking into account $\dot{H} = (\ddot{a}/a) - H^2$, the acceleration that results from the Hubble law is given by

$$\ddot{R}\big|_{\text{cosm.acc.}} = \dot{H}R + H\dot{R} = \ddot{a}R/a = -qH^2R, \tag{7}$$

where

$$q := -\ddot{a}a/\dot{a}^2 = -(\ddot{a}/a)H^{-2} \tag{8}$$

is the dimensionless *deceleration parameter*. To get a feeling for the magnitude, we remark that for the current best estimates for the parameters H and q, $H_0 \approx 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $q_0 \approx -0.6$, respectively, we get $\ddot{a}/a \approx 3 \times 10^{-36} \text{ s}^{-2}$, which even at Pluto's distance of 40 a.u. merely amounts to a tiny outward pointing acceleration of $2 \times 10^{-23} \text{ m s}^{-1}$.

Now note that, in the sense of general relativity, a body that is comoving with the cosmological expansion is moving on an inertial trajectory, i.e., it is force free. On the other hand, according to Newton, a dynamical force is, by definition, the cause for *deviations* from inertial motion. In the present context this would mean that dynamical forces are the causes for deviations from the motions described by Eq. (7), which suggests that in Newton's law, $m\ddot{\vec{x}} = \vec{F}$, we should make the replacement

$$\ddot{R} \mapsto \ddot{R} - (\ddot{a}/a)R \tag{9}$$

in order to apply to the (sufficiently slow) motion of interacting point masses in an expanding universe. Note that this also applies to gravitational interactions in a Newtonian approximation in which gravity is considered to be a force in the above sense.

As we will see, the replacement (9) can be justified rigorously in a variety of contexts, like for gravitationally bound systems, using the equation of geodesic deviation in general relativity. Whenever we attempt to justify the replacement (9) we must not forget that the Newtonian equations of motion (without Coriolis and centrifugal type "forces" in them) necessarily refer to preferred systems of coordinates which are (1) locally nonrotating, (2) whose origin is freely falling, and (3) in which the coordinate values directly refer to (local) inertial time (time coordinate) and spatial geodesic distance (space coordinates), as measured by comoving clocks and rods. This is achieved by using so-called Fermi normal coordinates [see, e.g., Misner *et al.* (1973)] in a neighborhood of a geodesic worldline—e.g., that of the Sun or the proton. This is also the approach followed in Cooperstock *et al.* (1998). Note that a Fermi system of coordinates can be defined for worldlines of arbitrary acceleration and correspond to locally nonrotating frames, which may physically be realized by a system of at least two noncollinear gyros in torque-free suspensions taken along the worldline. Along a geodesic, that is a worldline of zero acceleration, this system corresponds to a local inertial observer and is called Fermi *normal*. The equation of geodesic deviation in these coordinates now gives the variation of the spatial geodesic distance

to a neighboring geodesically moving object, e.g., a planet or spacecraft. It reads[7]

$$d^2x^k/d\tau^2 + R^k{}_{0l0}x^l = 0. \tag{10}$$

Here x^k are the spatial nonrotating normal coordinates whose values directly refer to the proper spatial distance. In these coordinates we further have (Cooperstock *et al.*, 1998)

$$R^k{}_{0l0} = -\delta^k_l \ddot{a}/a \tag{11}$$

on the worldline of the first observer, where the overdot refers to differentiation with respect to the cosmological time, which reduces to the proper time along the observer's worldline.

Equations (10) and (11) simply state that in Fermi normal coordinates around one inertial observer another nearby inertial observer is radially accelerating away at a magnitude $(\ddot{a}/a)R$, just as envisaged before. In linear approximation, this acceleration has to be added to that resulting from the other metric perturbation that is caused by the mass at the position of the first observer. As a result, in the case of purely gravitational interaction, we obtain the equation of motion of a test particle (whose metric perturbation we neglect) in the gravitational field of a heavier object whose metric perturbation away from the FLRW cosmological background we approximate to linear order. In the case of charged objects, we neglect the metric perturbations caused by the masses of both charges as well as their electromagnetic field, and simply take into account their mutual electromagnetic interaction.

Neglecting large velocity effects (i.e., terms quadratic or higher order in v/c) we can now write down the equation of motion for the familiar two-body problem. After specification of a scale function $a(t)$, we get two ordinary differential equations (ODEs) for the variables (R, φ), which describe the position[8] of the orbiting body with respect to the central one,

$$\ddot{R} = L^2/R^3 - C/R^2 + (\ddot{a}/a)R, \tag{12a}$$

$$R^2\dot{\varphi} = L. \tag{12b}$$

These are the $(\ddot{a}/a)$-improved Newtonian equations of motion for the two-body problem, where L represents the (conserved) angular momentum of the planet (or electron) per unit mass and C the strength of the attractive force. In the gravitational case $C = GM$, where M is the mass of the central body, and in the electromagnetic case, for the electron-proton system, $C = e^2/m$ (Gaussian unit), where e and m are the electron's charge and mass, respectively. In Secs. V and IV we show how to obtain

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[7]By construction of the coordinates, the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ vanish along the worldline of the first observer. Since this worldline is geodesic, Fermi-Walker transportation just reduces to parallel transportation.

[8]Recall that "position" refers to Fermi normal coordinates, i.e., R is the radial geodesic distance to the observer at $R=0$.

Eq. (12) in appropriate limits from the full general relativistic treatments.

We now study the effect the $\ddot{a}$ term has on the unperturbed Kepler orbits. We start with the remark that this term results from the *acceleration* and not just the expansion of the universe.

Next we point out that, in the concrete physical cases of interest, the time dependence of this term is negligible to a good approximation. Indeed, putting $f := \ddot{a}/a$, the relative time variation of the coefficient of R in Eq. (7) is $\dot{f}/f$. For an exponential scale function $a(t) \propto \exp(\lambda t)$ (vacuum-energy-dominated universe) this vanishes, and for a power law $a(t) \propto t^\lambda$ (for example, matter- or radiation-dominated universes) this is $-2H/\lambda$, and hence of the order of the inverse age of the universe. If we consider a planet in the Solar System, the relevant time scale of the problem is the period of its orbit around the Sun. The relative error in the disturbance, when treating the factor $\ddot{a}/a$ as constant during an orbit, is hence smaller than 10^{-9}. For atoms it is much smaller, of course. In principle, a time varying $\ddot{a}/a$ causes changes in the semimajor axis and eccentricity of Kepler orbits (Sereno and Jetzer, 2007). But here we neglect the time dependence of Eq. (7) and set $\ddot{a}/a$ equal to a constant A. Because of Eq. (8) we have $A := -q_0 H_0^2$. Then Eq. (12a) can be integrated,

$$\tfrac{1}{2}\dot{R}^2 + U(R) = E, \tag{13}$$

where the effective potential is

$$U(R) = L^2/2R^2 - C/R - AR^2/2. \tag{14}$$

We show below that the three parameters (L, C, A) can be effectively reduced to two.

B. Specifying the initial-value problem

Solutions of Eqs. (13) and (12b) are specified by initial conditions $(R, \dot{R}, \varphi, \dot{\varphi})(t_0) = (R_0, V_0, \varphi_0, \omega_0)$ at the initial time t_0. The discussion of the dynamical behavior of R is most effectively done in terms of the effective potential. Moreover, since perturbations are best discussed in terms of dimensionless parameters, we also introduce a length scale and a time scale that appropriately characterize the dynamical perturbation and the solution to be perturbed.

The length scale is defined as the radius at which the acceleration due to the cosmological expansion has the same magnitude as the two-body attraction. This happens precisely at the critical radius

$$R_c := (C/|A|)^{1/3}. \tag{15}$$

For $R < R_c$ the two-body attraction dominates, whereas for $R > R_c$ the effect of the cosmological expansion is the dominant one.

In order to gain an understanding of the length scales of the critical radius it is instructive to express it in terms of the physical parameters. In the case of gravitational interaction we have $C/|A| = GM/(|q_0|H_0^2)$ and thus

$$R_c = (R_S R_H^2/2|q_0|)^{1/3}. \tag{16}$$

Inserting the approximate value $q_0 = -1/2$ of the present epoch, this reduces to the Schücking radius (4).

In the electromagnetic case, e.g., for an electron-proton system, we have $C/|A| = (e^2/m)/(|q_0|H_0^2)$. Defining, in analogy with Eq. (5), the length scale

$$R_e := 2e^2/mc^2 \approx 5.64 \times 10^{-15} \text{ m}, \tag{17}$$

the critical radius (15) becomes

$$R_c = (R_e R_H^2/2|q_0|)^{1/3} \approx 30 \text{ a.u.}, \tag{18}$$

where in the last step we inserted $q_0 = -1/2$. This is about as large as the Neptune orbit.

From Eqs. (16) and (18) one sees that, in both cases, a larger (smaller) $|q_0|$ implies a smaller (larger) critical radius, according to expectations.

So much for the length scale. The time scale is defined to be the period of the unperturbed Kepler orbit (a solution to the above problem for $A = 0$) of semimajor axis R_0. By Kepler's third law it is given by

$$T_K := 2\pi (R_0^3/C)^{1/2}. \tag{19}$$

It is convenient to introduce two dimensionless parameters which essentially encode the initial conditions R_0 and ω_0,

$$\lambda := \left(\frac{\omega_0}{2\pi/T_K}\right)^2 = \frac{L^2}{CR_0}, \tag{20}$$

$$\alpha := \text{sgn}(A)\left(\frac{R_0}{R_c}\right)^3 = A\frac{R_0^3}{C}. \tag{21}$$

For close to Keplerian orbits λ is close to 1. For reasonably sized orbits α is close to 0. For example, in the Solar System, where $R_0 < 100$ a.u., one has $|\alpha| < 10^{-16}$. For an atom whose radius is smaller than 10^4 Bohr radii we have $|\alpha| < 10^{-57}$.

Now, defining

$$x(t) := R(t)/R_0, \tag{22}$$

Eqs. (13) and (12b) can be written as

$$\tfrac{1}{2}\dot{x}^2 + (2\pi/T_K)^2 u_{\lambda,\alpha}(x) = e, \tag{23}$$

$$x^2\dot{\varphi} = \omega_0, \tag{24}$$

where $e := E/r_0^2$ now plays the role of the energy constant and the reduced two-parameter effective potential $u_{\lambda,\alpha}$ is given by

$$u_{\lambda,\alpha}(x) := \frac{\lambda}{2x^2} - \frac{1}{x} - \frac{\alpha}{2}x^2 \tag{25}$$

(compare Fig. 1). The initial conditions now read

$$(x, \dot{x}, \varphi, \dot{\varphi})(t_0) = (1, V_0/R_0, \varphi_0, \omega_0). \tag{26}$$

The point of introducing the dimensionless variables is that the three initial parameters (L, C, A) of the effective potential could be reduced to two: λ and α. This will be convenient in the discussion of the potential.
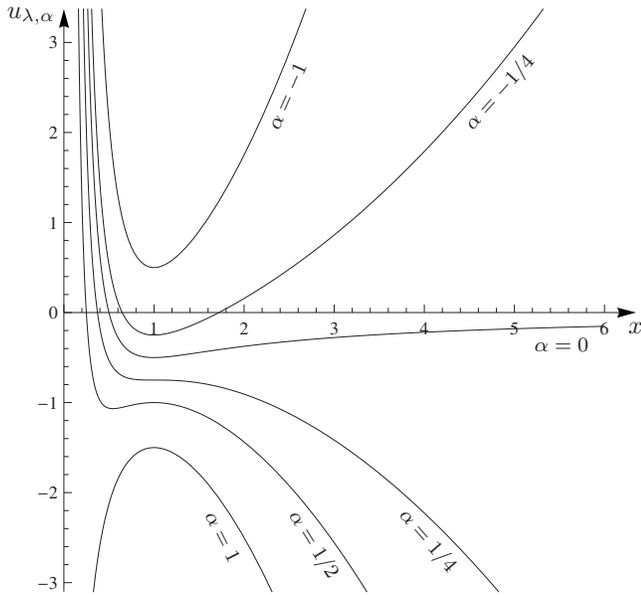
FIG. 1. The effective potential $u_{\lambda,\alpha}$ for circular orbits (for which $\lambda = 1 - \alpha$) for some values of α. The initial conditions are $x = 1$ and $\dot{x} = 0$ [see Eq. (22)]. At $x = 1$ the potential has an extremum, which for $\alpha < 1/4$ is a local minimum corresponding to stable circular orbits. For $1/4 \le \alpha < 1$ these become unstable. The value $\alpha = 0$ corresponds to the Newtonian case.

C. Discussion of the reduced effective potential

Circular orbits correspond to extrema of the effective potential (14). Expressed in terms of the dimensionless variables this is equivalent to $u'_{\lambda,\alpha}(1) = -\lambda + 1 - \alpha = 0$. By its very definition (20), λ is always non-negative, implying $\alpha \le 1$. For negative α (decelerating case) this is always satisfied. On the contrary, for positive α (accelerating case), this implies, in view of Eq. (21), the existence of a critical radius, given by R_c, beyond which no circular orbit exists. These orbits are stable if the considered extremum is a true minimum, i.e., if the second derivative of the potential evaluated at the critical value is positive. Now, $u''_{\lambda,\alpha}(1) = 3\lambda - 2 - \alpha = 1 - 4\alpha$, showing stability for $\alpha < 1/4$ and instability for $\alpha \ge 1/4$. For the accelerating case, in view of Eq. (21), this implies that the circular orbits are stable iff R_0 is smaller than the critical value

$$R_{\text{sco}} := (1/4)^{1/3} R_c \approx 0.63 R_c, \tag{27}$$

where "sco" stands for "stable circular orbits."

Summarizing, we have the following situation: in the decelerating case (i.e., for negative α or, equivalently, for negative A) stable circular orbits exist for every radius R_0; one just has to increase the angular velocity by some amount stated below in Eq. (28). On the contrary, in the accelerating case (i.e., for positive α, or, equivalently, for positive A), we have the following three regions:

- $R_0 < R_{\text{sco}}$, where circular orbits exist and are stable;

- $R_{\text{sco}} \le R_0 \le R_c$, where circular orbits exist but are unstable; and

- $R_0 > R_c$, where no circular orbits exist.

Generally, there exist no bounded orbits that extend beyond the critical radius R_c, the reason being simply that there is no $R > R_c$ where $U'(R) > 0$. Larger systems will just be slowly pulled apart by the cosmological acceleration and approximately move with the Hubble flow at later times.[9] Modifications of this strict qualitative distinction implied by time dependencies of A in Eq. (14) were discussed in Faraoni and Jacques (2007).

Turning back to the case of circular orbits, we now express the condition for an extrema derived above, $\lambda = 1 - \alpha$, in terms of the physical quantities, which leads to

$$\omega_0 = (2\pi/T_K)\sqrt{1 - \text{sgn}(A)(R_0/R_c)^3}. \tag{28}$$

This equation says that, in order to get a circular orbit, our planet, or electron, must have a smaller or bigger angular velocity according to the universe expanding in an accelerating or decelerating fashion, respectively. This is just what one would expect, since the effect of a cosmological "pulling apart" or "pushing together" must be compensated by a smaller or larger centrifugal forces, respectively, as compared to the Keplerian case. Equation (28) represents a modification of the third Kepler law due to the cosmological expansion. In principle this is measurable, but it is an effect of order $(R_0/R_c)^3$ and hence very small indeed; e.g., smaller than 10^{-17} for a planet in the Solar System.

Instead of adjusting the initial angular velocity as in Eq. (28), we can ask how one has to modify r_0 in order to get a circular orbit with the angular velocity $\omega_0 = 2\pi/T_K$. This is equivalent to searching the minimum of the effective potential (25) for $\lambda = 1$. This condition leads to the fourth-order equation $\alpha x^4 - x + 1 = 0$ with respect to x. Its solutions can be exactly written down using Ferrari's formula, though this is not illuminating. For our purposes it is more convenient to solve it approximately, treating α as a small perturbation. Inserting the ansatz $x_{\min} = c_0 + c_1\alpha + \mathcal{O}(\alpha^2)$ we get $c_0 = c_1 = 1$. This is really a minimum since $u''_{1,\alpha}(x_{\min}) = 1 + \mathcal{O}(\alpha) > 0$. Hence we have

$$R_{\min} = R_0\left[1 + \text{sgn}(A)\left(\frac{R_0}{R_c}\right)^3 + \mathcal{O}((R_0/R_c)^6)\right] \tag{29}$$

This tells us that in the accelerating (decelerating) case the radii of the circular orbits with $\omega_0 = 2\pi/T_K$ becomes bigger (smaller), again according to physical expectation. As an example, the deviation in the radius for an hypothetical spacecraft orbiting around the Sun at 100 a.u. would be just of the order of 1 mm. Since it grows with the fourth power of the distance, the deviation at 1000 a.u. would be of the order of 10 m.

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[9]This genuine nonperturbative behavior was not seen in the perturbation analysis performed in Cooperstock et al. (1998).

IV. GENERAL-RELATIVISTIC TREATMENT FOR ELECTROMAGNETICALLY-BOUNDED SYSTEMS

In this section we show how to arrive at Eq. (12) from a relativistic treatment of an electromagnetically bounded two-body system embedded (without back-reaction) into an expanding (spatially flat) universe. This implies solving Maxwell's equations in the cosmological background (30) for an electric point charge (the proton) and then integrate the Lorentz equations for the motion of a particle (electron) in a bound orbit [cf. Bonnor (1999)]. Equation (12) then appears in an appropriate slow-motion limit. However, in order to relate this straightforward method to the argument of Dicke and Peebles, we proceed by taking a slight detour which makes use of the conformal properties of Maxwell's equations.

A. The argument of Dicke and Peebles

Dicke and Peebles (1964) presented an apparently general and elegant argument that purports to show the insignificance of any dynamical effect of cosmological expansion on a local system that is bound by either electromagnetic or gravitational forces and which should hold true at any scale. Their argument involves a rescaling of spacetime coordinates $(t, \vec{x}) \mapsto (\lambda t, \lambda \vec{x})$ and certain assumptions on how other physical quantities, most prominently mass, behave under such scaling transformations. For example, they assume mass to transform like $m \mapsto \lambda^{-1} m$. However, their argument is really independent of such assumptions, as shown below. We work from first principles to clearly display all assumptions made.

We consider the motion of a charged point particle in an electromagnetic field. The whole system, i.e., particle plus electromagnetic field, is placed into a cosmological FLRW space time with flat ($k=0$) spatial geometry. The spacetime metric reads

$$\boldsymbol{g} = c^2 \boldsymbol{dt}^2 - a^2(t)(\boldsymbol{dr}^2 + r^2 \boldsymbol{g}_{S^2}), \tag{30}$$

where

$$\boldsymbol{g}_{S^2} = \boldsymbol{d\theta}^2 + \sin^2 \theta \boldsymbol{d\varphi}^2 \tag{31}$$

denotes the metric on the unit two-sphere in standard coordinates. We introduce conformal time t_c via

$$t_c = f(t) := \int_k^t \frac{dt'}{a(t')}, \tag{32}$$

by means of which we can write Eq. (30) in a conformally flat form

$$\boldsymbol{g} = a_c^2(t_c)(c^2 \boldsymbol{dt}_c^2 - \boldsymbol{dr}^2 - r^2 \boldsymbol{g}_{S^2}) = a_c^2(t_c) \boldsymbol{\eta}, \tag{33}$$

where $\boldsymbol{\eta}$ denotes the flat Minkowski metric. Here we wrote a_c to indicate that we now expressed the expansion parameter a as function of t_c rather than t, i.e.,

$$a_c := a \circ f^{-1}. \tag{34}$$

The electromagnetic field is characterized by the tensor $F_{\mu\nu}$, comprising electric and magnetic fields,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_n/c \\ -E_m/c & -\varepsilon_{mnj}B_j \end{pmatrix}. \tag{35}$$

In terms of the electromagnetic four-vector potential $A_\mu = (\varphi/c, -\vec{A})$ one has

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{36}$$

so that, as usual, $\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}$. The expression for the four-vector of the Lorentz force of a particle of charge e moving in the field $F_{\mu\nu}$ is $eF^\mu_\nu u^\nu$, where u^μ is the particle's four-velocity.

The equations of motion for the system particle + electromagnetic field follow from an action which is the sum of the action of the particle, the action for its interaction with the electromagnetic field, and the action for the free field, all placed in the background (30). Hence we write

$$S = S_P + S_I + S_F, \tag{37}$$

where

$$S_P = -mc^2 \int_z d\tau = -mc \int \sqrt{g(z', z')} d\lambda, \tag{38a}$$

$$S_I = -e \int_z A_\mu dx^\mu = -e \int A_\mu(z(\lambda)) z'^\mu d\lambda$$

$$= -\int d^4 x A_\mu(x) \int d\lambda e \delta^{(4)}(x - z(\lambda)) z'^\mu, \tag{38b}$$

$$S_F = -\frac{1}{4} \int d^4 x \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}$$

$$= -\frac{1}{4} \int d^4 x \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}. \tag{38c}$$

Here λ is an arbitrary parameter along the worldline $z : \lambda \mapsto z(\lambda)$ of the particle, and z' the derivative $dz/d\lambda$. The differential of the proper time along this worldline is

$$d\tau = \sqrt{g(z', z')} d\lambda = \sqrt{g_{\mu\nu}(z(\lambda)) \frac{dz^\mu}{d\lambda} \frac{dz^\nu}{d\lambda}} d\lambda. \tag{39}$$

It is now important to note that (1) the background metric g does not enter Eq. (38b) and that Eq. (38c) is conformally invariant (in four space-time dimensions only). Hence the expansion factor $a(t_c)$ does not enter these two expressions. For this reason we could write Eq. (38c) in terms of the flat Minkowski metric, though it should be kept in mind that the time coordinate is now given by conformal time t_c. This is not the time read by standard clocks that move with the cosmological observers, which rather show the cosmological time t (which is the proper time along the geodesic flow of the observer field $\boldsymbol{\partial}/\boldsymbol{\partial} t$).

The situation is rather different for the action (38a) of the particle. Its variational derivative with respect to $z(\lambda)$ is

$$\frac{\delta S_p}{\delta z^\mu(\lambda)} = -mc\left\{\frac{\frac{1}{2}g_{\alpha\beta,\mu}z'^\alpha z'^\beta}{\sqrt{g(z',z')}} - \frac{d}{d\lambda}\left[\frac{g_{\mu\alpha}z'^\alpha}{\sqrt{g(z',z')}}\right]\right\}. \tag{40}$$

We now introduce the *conformal proper time* τ_c via

$$d\tau_c = (1/c)\sqrt{\eta(z',z')}d\lambda = (1/ca)\sqrt{g(z',z')}d\lambda. \tag{41}$$

We denote differentiation with respect to τ_c by an overdot, so that, $z'/\sqrt{g(z',z')} = \dot{z}/ca$. Using this to replace z' by $\dot{z}\sqrt{g(z',z')}/ca$ and also g by $a^2\eta$ in Eq. (40) gives

$$\frac{\delta S_p}{\delta z^\mu(\lambda)} = \frac{\sqrt{g(z',z')}}{ac}ma\{\eta_{\mu\alpha}\ddot{z}^\alpha - P^\alpha_\mu\phi_{,\alpha}\} \tag{42}$$

where

$$a =: \exp(\phi/c^2) \quad \text{and} \quad P^\alpha_\mu := \delta^\alpha_\mu - \frac{\dot{z}^\alpha\dot{z}^\nu}{c^2}\eta_{\nu\mu}. \tag{43}$$

Recalling that $\delta S_P = \int[\delta S_p/\delta z^\mu(\lambda)]\delta z^\mu d\lambda = \int[\delta S_p/\delta z^\mu(\tau_c)]\delta z^\mu d\tau_c$ and using Eq. (41), Eq. (42) is equivalent to

$$\frac{\delta S_p}{\delta z^\mu(\tau_c)} = ma(\ddot{z}^\alpha - P^\alpha_\mu\phi_{,\alpha}), \tag{44}$$

where from now on we agree to raise and lower indices using the Minkowski metric, i.e., $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$ in Minkowski inertial coordinates.

Writing Eq. (38b) in terms of the conformal proper time and taking the variational derivative with respect to $z(\tau_c)$ leads to $\delta S_I/\delta z^\mu(\tau_c) = -eF_{\mu\alpha}\dot{z}^\alpha$, so that

$$\frac{\delta S}{\delta z^\mu(\tau_c)} = ma(\ddot{z}_\mu - P^\alpha_\mu\phi_{,\alpha}) - eF_{\mu\alpha}\dot{z}^\alpha. \tag{45}$$

The variational derivative of the action with respect to the vector potential A is

$$\frac{\delta S}{\delta A_\mu(x)} = \partial_\alpha F^{\mu\alpha}(x) - e\int d\tau_c\delta(x - z(\tau_c))\dot{z}^\mu(\tau_c). \tag{46}$$

Equations (45) and (46) show that the fully dynamical problem can be treated as if it were situated in static flat space. The field equations that follow from Eq. (46) are just the same as in Minkowski space. Hence we can calculate the Coulomb field as usual. On the other hand, the equations of motion receive two changes from the cosmological expansion term: the first is that the mass m is now multiplied with the (time-dependent) scale factor a, the second is an additional scalar force induced by a. Note that all space time-dependent functions on the right-hand side are to be evaluated at the particle's location $z(\tau_c)$, whose fourth component corresponds to ct_c. Hence, writing out all arguments and taking into account that the time coordinate is t_c, we have for the equation of motion

$$\ddot{z}^\mu = \frac{e}{ma_c(z^0/c)}F^\mu_{\ \alpha}(z)\dot{z}^\alpha$$
$$- (-c^2\eta^{\mu\alpha} + \dot{z}^\mu\dot{z}^\alpha)\partial_\alpha\ln a_c(z^0/c)$$

$$= \frac{e}{ma_c(z^0/c)}F^\mu_{\ \alpha}(z)\dot{z}^\alpha$$
$$- (-c\eta^{\mu 0} + \dot{z}^\mu\dot{z}^0/c)a'_c(z^0/c)/a_c(z^0/c), \tag{47}$$

where a'_c is the derivative of a_c.

So far no approximations were made. Now we write $\dot{z}^\mu = \gamma(c,\vec{v})$, where $\vec{v}$ is the derivative of $\vec{z}$ with respect to the conformal time t_c, henceforth denoted by a prime, and $\gamma = 1/\sqrt{1 - v^2/c^2}$. Then we specialize to slow motions, i.e., neglect effects of quadratic or higher powers in v/c (special relativistic effects). For the spatial part of Eq. (47) we get

$$\vec{z}'' + \vec{z}'(a'_c/a_c) = \frac{e}{ma_c}(\vec{E} + \vec{z}' \times \vec{B}), \tag{48}$$

where we once more recall that the spatial coordinates used here are the comoving (i.e., conformal) ones and the electric and magnetic fields are evaluated at the particle's position $\vec{z}(t_c)$.

From the above equation we see that the effect of cosmological expansion in the conformal coordinates shows up in two ways: first in a time dependence of the mass which scales with a_c, and, second, in the presence a friction term. Dicke and Peebles neglect the friction term and simply conclude as follows: In the adiabatic approximation, which is justified if typical time scales of the problem at hand are short compared to the world age [corresponding to small ε_2 in Eq. (98b)], the time-dependent mass term leads to a time varying radius in comoving (or conformal) coordinates of $r(t_c) \propto 1/a_c(t_c)$. Hence the physical radius (given by the cosmological geodesically spatial distance) $r_* = a_c r$ stays constant in this approximation. Hence, within this approximation, electromagnetically bound systems do not feel any effect of cosmological expansion.

But what does "this approximation" refer to? We will see that it amounts to neglecting precisely the leading-order contributions. This is easy to see if we cast Eq. (48) into physical coordinates, given by the cosmological time t and the cosmological geodesic spatial distance $r_* := a(t)r$. We have $dt_c/dt = 1/a$ and the spatial geodesic coordinates are $\vec{y} := a(t)\vec{z}$. Denoting the time derivative by an overdot with respect to t, the two terms on the left-hand side of Eq. (48) become

$$\vec{z}'' = a(\vec{y}H^2 - \dot{\vec{y}}H) + a(\ddot{\vec{y}} - \vec{y}\ddot{a}/a), \tag{49a}$$

$$\vec{z}'(a'_c/a_c) = -a(\vec{y}H^2 - \dot{\vec{y}}H), \tag{49b}$$

where $H = \dot{a}/a$. This shows that the friction term cancels against the first-order derivative terms in $\vec{y}$ and a that one gets in rewriting the left-hand side of Eq. (48) in

physical coordinates.[10] The only additional term next to $\ddot{\vec{y}}$ that survives is precisely the familiar acceleration term (7). Inserting Eq. (49) into Eq. (48), whose right-hand side we now specialize to a pure electric Coulomb field, $\vec{E}(\vec{z}) = Q\vec{z}/|\vec{z}|^3$ and $\vec{B}(\vec{z}) = 0$, we arrive at

$$\ddot{\vec{y}} - \vec{y}(\ddot{a}/a) = \frac{eQ}{m|\vec{y}|^3}\vec{y}. \tag{50}$$

After introducing polar coordinates in the orbital plane we exactly get Eq. (12).

B. Exact condition for nonexpanding circular orbits

Bonnor (1999) derived a necessary and sufficient condition for the existence of nonexpanding orbits for the electron-proton system in a spatially flat FLRW spacetime. Here "nonexpanding" is defined as of constant areal radius. This condition follows directly from the Lorentz equation of motion for the electron in the external electric field of the proton, the normalization condition of the electron's four-velocity, and the condition of constancy of the areal radius. In our notation, introducing the dimensionless quantities $h(t) := RH(t)/c$, $l := L/Rc$, and $\mu := R_e/2R$, the conditions for the existence of nonexpanding circular orbits reads as follows:

$$\frac{R}{c}\dot{h} = \frac{(1-h^2)^{3/2}}{(1+l^2)^{1/2}}\left(\mu - \frac{l^2 + h^2}{\sqrt{(1+l^2)(1-h^2)}}\right). \tag{51}$$

Recall that R_e is defined in Eq. (17) and $H(t)$ and L denote the Hubble function and, the (conserved) electron's angular momentum per unit mass respectively. The above condition is a first-order autonomous ODE for the function $h(t)$, and hence for the Hubble function $H(t)$. This is the constraint on the spacetime [more precisely, on the scale factor $a(t)$] that one gets by imposing the existence of nonexpanding circular orbits for two oppositely charged point masses. If such orbits exist, Eq. (51) amounts to the generalization of Kepler's third law to FLRW spacetimes, which gives here a relation between the scale function, on the one hand, and the orbital parameters R and L as well as the field-strength parameter R_e, on the other hand. Recall that in Newtonian physics the third Kepler law is, in our notation, simply given by $l^2 = \mu$.

The easiest solutions of Eq. (51) are of course the stationary ones, that is with $h(t) \equiv h_0$, for some constant h_0. This means that the scale factors is exponentially expanding,

$$a(t) = a_0 \exp(H_0 t), \tag{52}$$

where $H_0 := h_0 c/R$ and a_0 is some positive constant. In other words, the spacetime is given by the de Sitter so-

lution (Λ-dominated universe). In this case Eq. (51) reduces to

$$\frac{l^2 + h_0^2}{\sqrt{(1+l^2)(1-h_0^2)}} = \mu. \tag{53}$$

Notice that a larger Hubble parameter, hence a larger h_0, makes the left-hand side larger. Consequently, Eq. (53) tells us that with a larger Hubble parameter we must give to the electron a *smaller* angular velocity (smaller l) in order to keep it on a nonexpanding circular orbit with the *same* radius. This, according to intuition, is in order to compensate the extra cosmological pull with a reduced centrifugal term. In case of Minkowski spacetime ($h_0 = 0$) the above relation reads $l^2/\sqrt{1+l^2} = \mu$; hence one can interpret the factor $1/\sqrt{1+l^2}$ as a special-relativistic correction to the Newtonian relation $l^2 = \mu$. The largest radius at which, in an FLRW spacetime with exponentially growing scale factor, there is a nonexpanding orbit follows from Eq. (53) in the limit $l \to 0$. In this limit the condition reduces to $h_0^2/\sqrt{1-h_0^2} = \mu$, which, for small parameters h_0, simplifies to $h_0^2 \approx \mu$. Solving for R this gives the radius $(R_e R_H^2/2)^{1/3}$, which, taking into account that $q_0 = -1$ because of Eq. (52), exactly corresponds to the critical radius (18).

The other (nonstationary) solutions of Eq. (51) can also be found. After separation of variables and an elementary integration one gets t as function of h in terms of trigonometric functions composed with inverse hyperbolic functions. This exact expression is again not very illuminating and cannot generally be explicitly inverted so as to obtain h in terms of elementary functions of t. However, if we make use of the smallness of the parameters μ, l^2, and h^2, a leading-order expansion in these quantities gives a much simpler expression for $t(h)$ which can be explicitly inverted. In fact, this approximate solution $h(t)$ is obtained much quicker by solving Eq. (51) with the right-hand side being replaced with its leading-order expansion in the mentioned quantities, that is, by solving

$$R\dot{h}/c = \mu - l^2 - h^2. \tag{54}$$

Here $\mu - l^2$ is a constant which depends on the orbit parameters. One must now distinguish between three cases: (a) $\mu - l^2 =: \kappa^2 > 0$ for some positive κ, (b) $\mu - l^2 =: -\nu^2 < 0$ for some positive ν, and (c) $\mu - l^2 = 0$. Recalling the Newtonian relation $l^2 = \mu$, orbits in the three cases have an angular momentum which is, respectively, smaller, bigger, and equal to the Newtonian one. Integrating Eq. (54) we get, putting without loss of generality $t_0 = 0$, $h(t) = \kappa \tanh(\kappa ct/R)$, $h(t) = -\nu \tan(\nu ct/R)$, and $h(t) = R/ct$, for the cases (a), (b), and (c), respectively. Then, integrating once and exponentiating the result, we get the corresponding scale functions:

(a) Case $\mu - l^2 =: \kappa^2 > 0$ (nonexpanding orbits have sub-Newtonian angular momentum)

$$a(t) = a_0 \cosh(\kappa ct/R), \quad t \in [0, \infty). \tag{55a}$$

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[10] Since the friction term cancels, the critical remark made in note [27] of Adkins *et al.* (2007) regarding its magnitude is based on a misunderstanding.

(b) Case $\mu - l^2 =: -\nu^2 < 0$ (nonexpanding orbits have super-Newtonian angular momentum)

$$a(t) = a_0 \cos(\nu ct/R), \quad t \in \left[0, \frac{\pi R}{2\nu c}\right). \quad (55b)$$

(c) Case $\mu - l^2 = 0$ (nonexpanding orbits have Newtonian angular momentum)

$$a(t) = a_0 t, \quad t \in (0, \infty). \quad (55c)$$

In all three cases (a), (b), and (c) a_0 is a positive constant and the acceleration term $\ddot{a}/a$ is a constant which is positive, negative, and zero, respectively. Hence, as one would intuitively expect, the nonexpanding orbits have an angular momentum which is smaller, larger, or equal the Newtonian one, depending on whether the acceleration factor $\ddot{a}/a$ is positive, negative, or zero.

V. GENERAL-RELATIVISTIC TREATMENT FOR GRAVITATIONALLY-BOUNDED SYSTEMS

As discussed in Sec. II.B, we now discuss exact solutions that may represent quasi-isolated spherically symmetric gravitating systems "embedded" into cosmological spacetimes. As regards the meaning of embedded we distinguish between the strategies of "matching" and "melting" as outlined in Sec. II.B.

A. Spherically symmetric matchings

The complexity and nonlinearity of Einstein's equations make it a difficult task to construct a suitable variety of exact solutions which serve as realistic models for actual physical situations. Often exact solutions are only known for highly idealized situations, typically with high degrees of symmetry, in which the field equations sufficiently simplify. One way to construct new solutions (in a suitable sense, see below) from old ones is to glue them across suitably chosen hypersurfaces along which the matter distribution may become singular due to surface layers. This approach was pioneered by Lanczos in the early 1920s (Lanczos, 1924) and put into geometric form by Darmois (1927) and Israel (1966); see Misner *et al.* (1973). In this section, under the assumption of spherically symmetry, we present a new alternative set of conditions which are *equivalent* to the old ones. The new conditions only involve scalar quantities, are easy to verify, and have good physical interpretations. More details are given in Carrera (2010).

Here we restrict to piecewise continuous matter distributions without singular (δ-distribution-like) surface layers, as in the presence of stars with sharply defined surfaces. Einstein's equations can the be satisfied for piecewise twice continuously differentiable fields, if the field equations at the location of the matching hypersurface are replaced by their one-dimensional ε-interval integrals in normal direction to the hypersurface. The condition that two twice continuously differentiable solutions (in the ordinary sense) can be matched into a

piecewise twice continuously differentiable solution (in the re-interpreted sense just explained) is then simply given by the so-called *Darmois junction conditions (DJC)*: For a non-null matching hypersurface Γ, (i) the induced metric $\boldsymbol{g}_\Gamma$ and (ii) the extrinsic curvature $\boldsymbol{K}_\Gamma$ shall be continuous through Γ.

We pause for a moment to say a few more words about the notion of "continuity through Γ." Gluing together two pieces of spacetimes means the following: Initially one has two spacetimes, say $(\mathcal{M}^+, \boldsymbol{g}^+)$ and $(\mathcal{M}^-, \boldsymbol{g}^-)$, with oriented boundaries Γ^+ and Γ^-, respectively. Given a diffeomorphism $\phi : \Gamma^+ \to \Gamma^-$ between the boundaries, the glued spacetime is the quotient of the disjoint union of $\mathcal{M}^+$ and $\mathcal{M}^-$ under the identification of each point of $p \in \Gamma^+$ with $\phi(p) \in \Gamma^-$. The *matching hypersurface* Γ is now the common image of Γ^+ and Γ^- after identification in the quotient spacetime. Now, a tensor field $\boldsymbol{T}$ is said to be continuous through Γ if $\boldsymbol{T}|_{\Gamma^+}$ equals $\boldsymbol{T}|_{\Gamma^-}$ under the push-forward action of the diffeomorphism ϕ, hence if $\phi_*(\boldsymbol{T}|_{\Gamma^+}) = \boldsymbol{T}|_{\Gamma^-}$.

We now return to the DJC and, in particular, their physical interpretation. If $\boldsymbol{n}$ is a continuous choice of unit normal of Γ, it implies that

$$\boldsymbol{T}(\boldsymbol{n}, \cdot) \text{ is continuous through } \Gamma. \quad (56)$$

This follows directly from Eqs. (C8) of the Einstein tensor given in Appendix C. If Γ is timelike and hence $\boldsymbol{n}$ spacelike, Eq. (56) just states the continuity of the normal components for the energy-momentum flux densities, whereas their tangential components together with the energy density may jump across Γ. In the absence of surface layers this continuity condition is just a physically obvious consequence of local energy-momentum conservation, whereas jumps in, say, the energy-density must clearly be allowed for. For completeness we note that for a spacelike matching surface (56) states the continuity of the densities of energy and momentum as measured by an observer moving along $\boldsymbol{n}$ (taken to be future pointing), whereas the corresponding currents may jump.

We now restrict our attention to spherical symmetric spacetimes glued along hypersurfaces of spherical symmetry. This means that the latter are left invariant, as set, under the action of SO(3). We recall that the structure of a spherically symmetric spacetime is that of a warped product $\mathcal{M} = \mathcal{B} \times_R S^2$ of a two-dimensional Lorentzian manifold $\mathcal{B}$ and the two-sphere by means of the warping function, $R : \mathcal{B} \to \mathbb{R}_+$, called the *areal radius*. The matching hypersurfaces are then of the form $\Gamma = \gamma \times_R S^2$, where γ is the projection of Γ into $\mathcal{B}$ and is called the *matching curve*. The DJC should then reduce to appropriate conditions along the curve γ. Indeed, Theorem 1 below shows that, in the spherically symmetric case, the DJC are equivalent to the following:

Spherically symmetric junction conditions (SSJC): Let Γ be a smooth non-null spherically symmetric matching hypersurface between two spherically symmetric spacetimes and $\boldsymbol{n}$ a continuous choice of unit normal vector field on Γ. Denote with γ the projection of Γ onto $\mathcal{B}$.

Moreover, let v the (unique up to a sign) spherically symmetric, unit vector field on Γ orthogonal to n. The following four functions: (i) the arc length of γ, (ii) the extrinsic curvature of γ in $\mathcal{B}$: $g(n, \nabla_v v)$, (iii) the areal radius R, (iv) the Misner-Sharp energy E, shall be continuous through the matching curve γ.

The Misner-Sharp (MS) energy (Misner and Sharp, 1964; Hernandez and Misner, 1966), which we denote by E, is a concept of quasilocal mass that can be defined in presence of spherical symmetry, and which proves useful for computational and interpretational purposes. It is a function defined in purely geometrical manner as follows. Given a point p of spacetime, compute the sectional curvature K of the plane tangent to the two-dimensional SO(3) orbit through p and multiply this with minus[11] one-half of the third power of the areal radius,

$$E := -\frac{1}{2}R^3 K. \tag{57}$$

From Eq. (D9) we immediately read off

$$E = (R/2)(1 + \langle dR, dR \rangle), \tag{58}$$

which provides a convenient expression for the computation of the MS energy. We show in Appendix D that the MS energy is the charge of a conserved current and how it depends on the energy-momentum tensor for the matter. There we also discuss its Newtonian limit. This allows one to interpret it as amount of active gravitational energy contained in the interior of the sphere of symmetry [SO(3) orbit] through p. There we also show that the MS energy at p is equal to the Hawking quasilocal mass of the two-sphere of symmetry [SO(3) orbit] through p and hence converges to the Bondi mass at null infinity and, in an asymptotically flat spacetime, to the Arnowitt-Deser-Misner (ADM) mass at spatial infinity [for the latter two issues see Szabados (2004) and also Hayward (1996)]. Moreover, we give the decomposition of the MS energy in its Ricci and Weyl parts; see Eq. (D34).

The name "Misner-Sharp energy" seems now to be well established; however, one should say that this mass concept goes back at least to Lemaître (1933),[12] which gives a coordinate expression for it. Its geometric definition (57) was first given in Hernandez and Misner (1966) and its interpretation as the charge of a conserved current was first derived in Kodama (1980). Later, an alternative definition was given in Zannias (1990): There it is showed that the MS energy can defined in terms of the norm of the Killing fields generating the isometry group SO(3), leading directly to Eq. (58). Further relevant studies of the MS energy are Cahill and McVittie (1970a, 1970b) and, more recently, Burnett (1991) and Hayward (1996, 1998).

Some comments are needed on the above SSJC. First we note that, since n and v are spherically symmetric and hence tangent to $\mathcal{B}$ we have in view of Eqs. (D3) and (C6) that $g(n, \nabla_v v) = g_{\mathcal{B}}(n_{\mathcal{B}}, {}^{\mathcal{B}}\nabla_{v_{\mathcal{B}}} v_{\mathcal{B}}) = \varepsilon(n_{\mathcal{B}}) \mathbf{K}^{\mathcal{B}}_\gamma(v_{\mathcal{B}}, v_{\mathcal{B}})$. Hence, the quantity in (ii) is indeed (up to a possible sign) the extrinsic curvature of the curve γ in $\mathcal{B}$. Second, note that this quantity, being quadratic in v, does not depend on the sign choice of v. Third, since the matching hypersurface has the structure $\gamma \times S^2$, the words "continuous through the curve γ" can be interchanged with the words "continuous through the hypersurface Γ," depending on ones preference to think four or two dimensional.

A great advantage of the SSJC is that they are easy to verify: one simply has to impose continuity on four *scalars* along the matching curve in the two-dimensional base manifold $\mathcal{B}$. Dealing with scalars, since their value is independent on the particular coordinate choice, one does not need to worry about introducing new coordinates in both spacetimes to be glued, in order to get the different metrics in a form which is comparable. This is indeed an ingrate task: in general, these coordinates are only needed in order to check if the junction conditions are satisfied, and for nothing more. In the presence of spherical symmetry all this can be circumvented using our new junction conditions.

Furthermore, the SSJC have a good physical interpretation: The continuity condition of both, the areal radius as well as of the MS energy, can be read as equilibrium condition for the gravitational pull acting from opposite directions onto (fictitious) test masses at the location of the matching surface. Concerning the continuity of the extrinsic curvature of the matching curve, we note the following: In the case where the matching hypersurface Γ is timelike, let $\gamma = \pi(\Gamma)$ be a timelike curve in $\mathcal{B}$ and v is future-pointing tangent. One can think of γ as the "matching observer's" worldline. Hence, in the timelike case, the extrinsic curvature is nothing but the acceleration of the matching observer. In the spatial case, on the contrary, n is timelike and γ is spacelike (and v tangent to it). One can choose n to be future pointing and then think of it as an observer field defined along the spatial (one-dimensional) slice γ. In view of Eq. (C7) one then sees that the extrinsic curvature of γ is exactly[13] the shear expansion of n in direction of v "radial direction."

In Appendix B we prove the following:

Theorem 1 (Equivalence of the junction conditions). Let Γ be a smooth non-null spherically symmetric matching hypersurface between two spherically symmetric spacetimes and n be a continuous choice of unit normal vector field on Γ. Assume, moreover, that the areal radii of the two spacetimes are C^1 functions in an open neighborhood of the matching hypersurfaces. Then the DJC are equivalent to the SSJC.

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[11]The minus sign here is a relict of our signature choice.
[12]For an English translation see Lemaître (1997).

[13]Recall that, because of our signature choice, the restriction of the metric to spacelike directions is *negative* definite.

Now suppose we are faced with the following situation: We are given two spherically symmetric solutions of Einstein equation and we want to know if they can be matched together at all, and if so, how to characterize the curve γ (respectively, the hypersurface Γ) along which this is possible. Answers to these questions will be provided by the junction conditions SSJC. Note that a timelike or spacelike curve in the two-dimensional base manifold $(\mathcal{B}, \boldsymbol{g}_{\mathcal{B}})$ can be described simply by a function $R(\tau)$, where R is the areal radius and τ the curve's arc length which, in the timelike case, corresponds to the matching observer's proper time. The conditions (i) and (iii) of the SSJC are then equivalent to the condition that the functional dependence $R(\tau)$ must be the same (up to a trivial translation in τ) in both spacetimes to be matched.

B. The Eisenstaedt theorem

Perhaps the simplest attempt to model a compact body (star) in an expanding universe is trying to inglobate it in a FLRW spacetime and to assume, for simplicity, that the body is spherically symmetric. A direct consequence of the SSJC is the following intuitive appealing theorem due to Eisenstaedt (1977):

Theorem 2 (Eisenstaedt, 1977). Excise the full worldtube $\mathcal{W}_{r_0}$ of a comoving ball of comoving radius r_0 from a FLRW spacetime and insert instead a spherical symmetric inhomogeneity (hence a piece of a spherically symmetric spacetime together with a related matter model, satisfying Einstein's equations). Then a necessary condition for the resulting spacetime to satisfy Einstein's equation is that the MS energy of the inserted inhomogeneity equals that of the excised ball.

This says that the mean energy density (measured with the MS energy) of spherically symmetric inhomogeneities must be the same as the one of the FLRW spacetime. That the Eisenstaedt theorem is a consequence of the Darmois junction conditions was already pointed out in Hartl (2006).

C. The Einstein-Straus vacuole revisited

As another application of the above described matching procedure we revisit the Einstein-Straus solution (Einstein and Straus, 1945, 1946; Schücking, 1954), which originally consists on a Schwarzschild spacetime (called "vacuole") matched to a dust FLRW universe with zero cosmological constant. Later, this model was generalized also to the case of a nonvanishing cosmological constant (Balbinot *et al.*, 1988). We treat here the general case of an arbitrary cosmological constant and show that the SSJC allow substantial simplifications of the computations. This technique can also be applied to Bonnor's vacuole construction in LTB spacetimes.

Notice that the matching condition (56) implies, in particular, that the pressure must be continuous through the matching hypersurface. Since the interior is a vacuum spacetime, it follows that the pressure must van-

ish also on the exterior part of the matching hypersurface and hence everywhere on the FLRW spacetime. That is why one has to restrict to dust FLRW spacetimes.

Since we leave the cosmological constant Λ arbitrary (it may be positive, negative, or zero) the inner region is given, respectively, by a Schwarzschild–de Sitter, Schwarzschild–anti–de Sitter,[14] or Schwarzschild spacetime (all abbreviated henceforth by SdS). Recall that the SdS spacetime is given by the vacuum solution to Einstein equation with cosmological term

$$\boldsymbol{g}^{\text{SdS}} = V(R)\boldsymbol{d}T^2 - V(R)^{-1}\boldsymbol{d}R^2 - R^2\boldsymbol{g}_{S^2}, \tag{59a}$$

where

$$V(R) = 1 - 2m/R - \Lambda R^2/3. \tag{59b}$$

Above $\boldsymbol{g}_{S^2}$ denotes the metric on the unit two-sphere (31) and m is a constant which represents the central mass.

A dust FLRW spacetime is given by the metric

$$\boldsymbol{g}^{\text{FLRW}} = \boldsymbol{d}t^2 - a(t)^2\left(\frac{\boldsymbol{d}r^2}{1 - kr^2} + r^2\boldsymbol{g}_{S^2}\right) \tag{60}$$

together with the matter energy-momentum tensor[15] $\boldsymbol{T} = \varrho\boldsymbol{u} \otimes \boldsymbol{u}$, where $\boldsymbol{u} = \partial/\partial t$ is the (geodesic) velocity field of the cosmological dust and ϱ is the matter energy density, which depends on t only. Here r is the comoving radial coordinate and k is a constant which takes the values $0, -1, +1$, depending on whether the spatial slices have zero, negative, or positive curvature, respectively. The Einstein equation is then equivalent to the following set given by the Friedmann equation and a "conservation equation":

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{C}{a^3} - \frac{\Lambda}{3} = 0, \tag{61a}$$

$$\varrho a^3 = \text{const}, \tag{61b}$$

where the constant $C := 8\pi\varrho_0 a_0^3/3$ depends on the initial conditions $a_0 := a(t_0)$ and $\varrho_0 := \varrho(t_0)$ at some "initial" time t_0. Here the dot denotes differentiation with respect to t or, which is the same, along $\boldsymbol{u}$.

The central question is now the following: How shall we cut hypersurfaces $\Gamma_{\text{SdS}} = \gamma_{\text{SdS}} \times S^2$ and $\Gamma_{\text{FLRW}} = \gamma_{\text{FLRW}} \times S^2$ in the spacetimes SdS and, respectively, FLRW in order that the resulting pieces can be matched? In order to apply the SSJC we have to compute the MS energy for both spacetimes. For the FLRW spacetime one has $R_{\text{FLRW}}(t, r) = a(t)r$ and hence $\boldsymbol{d}R_{\text{FLRW}} = \dot{a}r\boldsymbol{d}t + a\boldsymbol{d}r$ and

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[14]The Schwarzschild– (anti–) de Sitter metric (59a) is often called the Kottler solution, after Friedrich Kottler, who was the first to write down this metric (Kottler, 1918). More details on its analytic and global structure may be found in Geyer (1980).

[15]Throughout we denote the metric-dual (one-form) of a vector $\boldsymbol{u}$ by underlining it, that is, $\underline{\boldsymbol{u}} := \boldsymbol{g}(\boldsymbol{u}, \cdot)$ is the one-form metric-dual to vector $\boldsymbol{u}$. In local coordinates we have $\boldsymbol{u} = u^\mu\partial_\mu$ and $\underline{\boldsymbol{u}} = u_\mu\boldsymbol{d}x^\mu$, where $u_\mu := g_{\mu\nu}u^\nu$.

$\langle dR, dR \rangle_{\text{FLRW}} = g^{\mu\nu} R_{,\mu} R_{,\nu} = r^2(k + \dot{a}^2) - 1$. From the definition (58) one then gets, using Friedmann's equation,

$$E_{\text{FLRW}} = (4\pi/3) R_{\text{FLRW}}^3 (\varrho + \varrho_\Lambda), \qquad (62)$$

where $\varrho_\Lambda := \Lambda/8\pi$ is the energy density associated with the cosmological constant. Notice that this expression, as well its derivation, is completely independent of the specific equation of state of the fluid and does not depend on the spatial curvature k. For the SdS case one has $R_{\text{SdS}} = R$ and thus $\langle dR, dR \rangle_{\text{SdS}} = -V(R)$ and

$$E_{\text{SdS}} = m + (4\pi/3) R_{\text{SdS}}^3 \varrho_\Lambda. \qquad (63)$$

Now, the last two conditions of the SSJC, that is the continuity of areal radius and MS energy across the matching hypersurface (yet to be determined), are equivalent to the continuity of the areal radius $R_{\text{FLRW}} = R_{\text{SdS}} =: R$, together with the suggestive relation

$$m = (4\pi/3) R^3 \varrho. \qquad (64)$$

These two conditions already determine the matching hypersurface. Indeed, inserting $R = R_{\text{FLRW}} = a(t) r$ into Eq. (64) and using $\varrho(t) a^3(t) = \text{const}$ valid for dust FLRW models, one obtains the matching radius in terms of the FLRW comoving radial coordinate,

$$r = r_0 := \left(\frac{m}{(4\pi/3) a_0^3 \varrho_0} \right)^{1/3} = \text{const}. \qquad (65)$$

Here $a_0 := a(t_0)$, and similarly for ϱ, where t_0 is some fixed "initial" time. This means that the matching observer moves, in the FLRW spacetime, along the integral curve of $\boldsymbol{u} = \partial/\partial t$ with initial condition (t_0, r_0) and hence is comoving with the cosmological matter.

So far we used the last two of the SSJC. As discussed above, the continuity of the areal radius and the arc length (the proper time, in the timelike case) of the matching curve are equivalent to the equality of the functional dependencies $R(\tau)$ (up to a possible trivial translation in τ) which describe the matching curves in the two spacetimes to be matched. Now, because of Eq. (65), the matching curve (worldline) in the FLRW spacetime is

$$R(\tau) = a(\tau) r_0, \qquad (66)$$

where a is the (unique) solution of the Friedmann equation (61a) with initial condition a_0 at $\tau_0 = t_0$. (Recall that in FLRW the proper time of an observer moving along an integral line of $\boldsymbol{u}$ equals the cosmological time, hence $\tau = t$.) From what we said above, the same functional relation $R(\tau)$ must hold also in the SdS—provided we identify R with the areal radius and τ with the matching observer's proper time, both referred to the SdS spacetime. This determines the matching curve in SdS.

Finally we need to show that the junction condition (ii) is satisfied, hence that the matching observer's accelerations coincide. Looking at the matching worldline from the FLRW spacetime, it is immediately clear that it is geodesic, hence its acceleration, vanishes. To conclude the matching procedure, we have to check that this is also true for the matching worldline in the SdS space-

time. For this, one has to check that the function defined in Eq. (66) satisfies the geodesic equation for a radial motion. The latter is given by

$$\dot{R}^2 + V(R) = e^2, \qquad (67)$$

where $e := \boldsymbol{g}^{\text{SdS}}(\partial/\partial T, \boldsymbol{v}) = \text{const}$ and $\boldsymbol{v} = \dot{T}\partial/\partial T + \dot{R}\partial/\partial R$ is the matching observer in the SdS spacetime. [Equation (67) can be derived from the fact that $\boldsymbol{v}(\boldsymbol{g}^{\text{SdS}}(\partial/\partial T, \boldsymbol{v})) = 0$, since $\partial/\partial T$ is Killing and $\boldsymbol{v}$ geodesic. Inserting $e := \boldsymbol{g}^{\text{SdS}}(\partial/\partial T, \boldsymbol{v}) = V(R)\dot{T}$ into the normalization condition $1 = \boldsymbol{g}^{\text{SdS}}(\boldsymbol{v}, \boldsymbol{v}) = V(R)\dot{T}^2 - \dot{R}^2/V(R)$ one arrives at Eq. (67).] Now, inserting Eq. (66) with Eq. (65) into Eq. (67) and using the Friedmann equation (61a), one gets $\dot{R}^2 + V(R) = 1 - kr_0^2$. Hence, the geodesic equation (67) is satisfied (with $e^2 = 1 - kr_0^2$) and herewith all four junction conditions.

D. The McVittie model

Among all models discussed which represent a quasi-isolated spherically symmetric gravitating system melted into a cosmological spacetime, the one that is presumably best understood as regards its analytical structure as well as its physical assumptions is that of McVittie (1933), thanks to the careful analysis of Nolan (1998, 1999a, 1999b). Here we restrict to the "flat" or $k = 0$ model, which interpolates between an exterior Schwarzschild solution, describing a local mass, and a spatially flat (i.e., $k = 0$) ambient FLRW universe. For simplicity from now on we refer to this model simply as the McVittie model. The cosmological constant is assumed to be zero, although this assumption is not essential (see the last paragraph of Sec. V.D.1).

This is not to say that this model is to be taken at face value in all its aspects. Its problems lie in the region close to the central object, where the basic assumptions on the behavior of matter definitely turn unphysical. However, at radii much larger than (in geometric units) the central mass (defined below) the $k = 0$ McVittie solution seems to provide a viable approximation for the transition between a homogeneous cosmological spacetime and a localized mass immersed in it. We now discuss this model and look at its geodesic equations, showing that they reduce to Eq. (12) in an appropriate weak-field and slow-motion limit. This provides another and more solid justification for the Newtonian approach carried out in Sec. III.

The characterization of the McVittie model is made through two sets of *a priori* specifications. The first set concerns the metric (left side of Einstein's equations) and the second set the matter (right side of Einstein's equations). The former consists in an ansatz for the metric, which can formally be described as follows: Write down the Schwarzschild metric for the mass parameter m in isotropic coordinates, add a conformal factor $a^2(t)$ to the spatial part, and allow the mass parameter m to depend on time. Hence the metric reads

$$g = \left(\frac{1 - m(t)/2r}{1 + m(t)/2r}\right)^2 dt^2 - \left(1 + \frac{m(t)}{2r}\right)^4 a^2(t)(dr^2 + r^2 \boldsymbol{g}_{S^2}),$$

$$(68)$$

where $\boldsymbol{g}_{S^2}$ is given by Eq. (31). The metric (68) is spherically symmetric with the spheres of constant radius r being the orbits of the rotation group. We discuss below what this ansatz actually entails. For later convenience we also introduce the orthonormal tetrad $\{\boldsymbol{e}_\mu\}_{\mu=0,\cdots,3}$ with respect to Eq. (68), where[16]

$$\boldsymbol{e}_\mu := \|\partial/\partial x^\mu\|^{-1}\partial/\partial x^\mu \tag{69}$$

and $\{x^\mu\}=\{t, r, \theta, \varphi\}$.

The second set of specifications, concerning the matter, is as follows: The matter is a perfect fluid with density ϱ and isotropic pressure p. Hence its energy-momentum tensor is given by

$$\boldsymbol{T} = \varrho \underline{u} \otimes \underline{u} + p(\underline{u} \otimes \underline{u} - \boldsymbol{g}). \tag{70}$$

Furthermore, and this is where the two sets of specifications make contact, the motion of the matter is given by

$$\boldsymbol{u} = \boldsymbol{e}_0. \tag{71}$$

No further assumptions are made. In particular, an equation of state, like $p = p(\varrho)$, is not assumed. The reason for this will become clear soon.

Note that the vector field (71) is not geodesic for the metric (68) (unlike for the FLRW and Gautreau metrics), which immediately implies that the pressure cannot be constant. Being spherically symmetric, $\boldsymbol{u}$ is automatically vorticity free. The last property is manifest from its hypersurface orthogonality, which is immediate from Eq. (68). Moreover, $\boldsymbol{u}$ is also shear free. This, too, can be immediately read off Eq. (68) once one takes into account the result that for spherically symmetric metrics vanishing shear for a spherically symmetric vector field is equivalent to the corresponding spatial metric being conformally related to a spherically symmetric flat metric. This is obviously the case here.

The nonvanishing components of the Einstein tensor with respect to the orthonormal basis (69) are

$$\mathbf{Ein}(\boldsymbol{e}_0, \boldsymbol{e}_0) = 3F^2, \tag{72a}$$

$$\mathbf{Ein}(\boldsymbol{e}_i, \boldsymbol{e}_j) = -\left(3F^2 + 2\frac{A}{B}\dot{F}\right)\delta_{ij}, \tag{72b}$$

$$\mathbf{Ein}(\boldsymbol{e}_0, \boldsymbol{e}_1) = \frac{2}{R^2}\left(\frac{A}{B}\right)^2 (am)^{\cdot}, \tag{72c}$$

where an overdot denotes differentiation along $\partial/\partial t$. Before explaining the functions A, B, R, and F, we make the important observation that the Einstein tensor is *spatially isotropic*, where "spatially" refers to the directions orthogonal to $\boldsymbol{e}_0$. By this we mean that $\mathbf{Ein}(\boldsymbol{e}_i, \boldsymbol{e}_j) \propto \delta_{ij}$ or, expressed more geometrically, that the spatial

restriction of the Einstein tensor is proportional to the spatial restriction of the metric.

In Eq. (72) and in the following we set

$$A(t,r) := 1 + m(t)/2r, \quad B(t,r) := 1 - m(t)/2r \tag{73}$$

and

$$R(t,r) = [1 + m(t)/2r]^2 a(t)r, \tag{74}$$

where R is the areal radius for the McVittie ansatz (68), and also

$$F := \frac{\dot{a}}{a} + \frac{1}{rB}\frac{(am)^{\cdot}}{a}. \tag{75}$$

In passing we note that F has the geometric interpretation of being one-third the expansion of the vector field $\boldsymbol{e}_0$, that is, $F = \mathbf{div}(\boldsymbol{e}_0)/3$. Hence Eq. (72) could also be written in the form $\mathbf{Ein}(\boldsymbol{e}_0, \boldsymbol{e}_0) = [\mathbf{div}(\boldsymbol{e}_0)]^2/3$. We will see later that the product *am* which appears in Eq. (72c) also has a geometric meaning: it is just the Weyl part of the MS energy; see Eq. (81).

Now, the nonvanishing components of the energy momentum tensor (70) and (71) are

$$\boldsymbol{T}(\boldsymbol{e}_0, \boldsymbol{e}_0) = \varrho, \quad \boldsymbol{T}(\boldsymbol{e}_i, \boldsymbol{e}_j) = p\,\delta_{ij}. \tag{76}$$

The $(\boldsymbol{e}_0, \boldsymbol{e}_1)$ component of Einstein's equation therefore implies $(am)^{\cdot} = 0$, which means that the Weyl part of the MS energy is constant. Physically this can be interpreted as saying that the central object does not accrete any energy from the ambient matter. Using the constancy of *am* in Eq. (75) we immediately get

$$F = \dot{a}/a =: H. \tag{77}$$

Hence Einstein's equation is equivalent to the following three relations between the four functions $m(t)$, $a(t)$, $\varrho(t,r)$, and $p(t,r)$:

$$(am)^{\cdot} = 0, \tag{78a}$$

$$8\pi\varrho = 3\left(\frac{\dot{a}}{a}\right)^2, \tag{78b}$$

$$8\pi p = -3\left(\frac{\dot{a}}{a}\right)^2 - 2\left(\frac{\dot{a}}{a}\right)^{\cdot}\left(\frac{1 + m/2r}{1 - m/2r}\right). \tag{78c}$$

Note that here Einstein's equation has only three independent components (as opposed to four for a general spherically symmetric metric), which is a consequence of the fact, already stresses above, that the Einstein tensor for the McVittie ansatz (68) is spatially isotropic.

Equation (78a) can be integrated,

$$m(t) = m_0/a(t), \tag{79}$$

where m_0 is an integration constant. Below we show that this integration constant is to be interpreted as the mass of the central body. We call the metric (68) together with condition (79) the *McVittie metric*.

Clearly the system (78) is underdetermining. This is expected since no equation of state has yet been imposed. The reason why we did not impose such a condi-

tion can now be easily inferred from Eq. (78): whereas Eq. (78b) implies that ϱ only depends on t, Eq. (78c) implies that p depends on t and r iff $(\dot{a}/a)^{\cdot} \neq 0$. Hence a nontrivial relation $p=p(\varrho)$ is incompatible with the assumptions made so far. The only possible ways to specify p are $p=0$ or $\varrho+p=0$. In the first case Eq. (78c) implies that $\dot{a}/a=0$ if $m_0 \neq 0$ (since then the second term on the right-hand side is r dependent, whereas the first is not, so that both must vanish separately), which corresponds to the exterior Schwarzschild solution, or $a(t) \propto t^{2/3}$ if $m_0=0$, which leads to the flat FLRW solution with dust. In the second case the fluid just acts like a cosmological constant $\Lambda=8\pi\varrho$ (using the equation of state $\varrho+p=0$ in **div** $T=0$ it implies $dp=0$ and this, in turn, using again the equation of state, implies $d\varrho=0$) so that this case reduces to the Schwarzschild–de Sitter solution. To see this explicitly, notice first that Eqs. (78b) and (78c) imply the constancy of $H=\dot{a}/a=\sqrt{\Lambda/3}$ and hence one has $a(t)=a_0 \exp(t\sqrt{\Lambda/3})$. With such a scale factor the McVittie metric (68) with Eq. (79) turns into the Schwarzschild–de Sitter metric (59a) and (59b) in disguise. The explicit formulas for the coordinate transformation relating the two can be found in Robertson (1928) and also in Klioner and Soffel (2005). Finally, note from Eq. (78a) that constancy of one of the functions m and a implies constancy of the other. In this case Eqs. (78b) and (78c) imply $\varrho=p=0$, so that we are dealing with the exterior Schwarzschild spacetime.

A specific McVittie solution can be obtained by *choosing* a function $a(t)$, corresponding to the scale function of the FLRW spacetime which the McVittie model is required to approach at spatial infinity, and the constant m_0, corresponding to the "central mass." Relations (78b), (78c), and (79) are then used to determine ϱ, p, and m, respectively. Clearly this "poor man's way" to solve Einstein's equation holds the danger of arriving at unrealistic spacetime-dependent relations between ϱ and p. This must be kept in mind when proceeding in this fashion. For further discussion of this point see Nolan (1998, 1999a).

1. Interpretation of the McVittie model

In this section we discuss the interpretation of the McVittie model, its singularities, trapped regions, symmetry properties, and also the motion of the matter. In doing this, we take care to isolate those properties which are intrinsic to the ansatz (68) independent of the imposition of Einstein's equation. The analysis can then also be applied to all generalizations which maintain the ansatz (68). Generalizations in this sense have been discussed in Faraoni and Jacques (2007), on which we comment at the end of this section.

According to what has just been said, we regard the McVittie solution as a candidate model for an isolated mass m_0 in an "otherwise" flat FLRW universe with scale function $a(t)$. As emphasized in the Introduction, this requires specific justification in view of the fact that simple superpositions of solutions are disallowed by the

nonlinearities. A set of criteria for when a solution represents a localized mass immersed in a flat FLRW background have been proposed and discussed in Nolan (1998). The basic idea is to employ the MS energy (in a spherically symmetric context, where it is equivalent to the Hawking mass) in order to detect localized sources of gravity. We follow this approach and for this purpose compute the Ricci and the Weyl part of the MS energy.

This we now do for the class of metrics (68), without at first making any use of Einstein's equation. The geometric definition of the MS energy in terms of the sectional curvature, together with formula (A10b) for its Ricci part specialized to metrics with spatially isotropic Einstein tensor, implies that the Ricci part of the MS energy is given by

$$E_R = \tfrac{1}{6} R^3 \mathbf{Ein}(e_0, e_0). \tag{80}$$

The Weyl part is then obtained as the difference between the full MS energy and Eq. (80). We use Eq. (58) for the former and write $\langle dR, dR \rangle = [e_0(R)]^2 - [e_1(R)]^2$. The part involving $e_0(R)$ equals Eq. (80), due to the relation $\mathbf{Ein}(e_0, e_0) = 3[dR(e_0)/R]^2$, which follows from the comment below Eqs. (75) and (D54) (for vanishing shear). The Weyl part of the MS energy is therefore given by $(R/2)(1 - [e_1(R)]^2)$. From Eq. (74) we calculate $e_1(R)$ and hence obtain for the Weyl part of the MS energy,

$$E_W = am. \tag{81}$$

Now we invoke Einstein's equation with source (70) and four-velocity (71). Then the Ricci and Weyl contributions to the MS energy can be written in the following form, also taking into account Eq. (79):

$$E_R = (4\pi/3) R^3 \varrho, \tag{82a}$$

$$E_W = m_0. \tag{82b}$$

Identifying the gravitational mass of the central object with the Weyl part of the MS energy, its constancy means that no energy is accreted from the ambient matter. As regards the Ricci part, note that the factor $(4\pi/3) R^3$ in Eq. (82a) is smaller than the proper geometric volume within the sphere of areal radius R. This can be attributed to the gravitational binding energy that diminishes the gravitational mass of a lump of matter below the value given by the proper space integral of $T(e_0, e_0)$. This is shown in Appendix D.4, in particular in the exact equation (D47) and its leading-order approximation (D49).

It is also important to note that the central gravitational mass in McVittie's spacetime may be modeled by a shear-free perfect-fluid star of positive homogeneous energy density (Nolan, 1993). The matching is performed along a world-tube comoving with the cosmological fluid, across which the energy density jumps discontinuously. This means that the star's surface is comoving with the cosmological fluid and hence, in view of Eq. (89), that it geometrically expands (or contracts). This feature, however, should be seen as an artifact of

the McVittie model [in which the relation (89) holds], rather than a general property of compact objects in any cosmological spacetimes. Positive pressure within the star seems to be only possible if $2a\ddot{a}+\dot{a}^2<0$ [see Eq. (3.27) in Nolan (1993) with $a=\exp(\beta/2)$], that is, for deceleration parameters $q>1/2$.

Next we comment on the singularity properties of the McVittie model. From Eq. (78c) it is clear that, unless $\dot{a}/a$ is constant (the Schwarzschild–de Sitter case) or $m=0$ (FLRW case), the pressure diverges at $r=m/2$ (that is at $R=2m_0=R_S$). In fact, this corresponds to a genuine curvature singularity which is built into the McVittie ansatz (68) independently of any further assumption. To see that $r=m/2$ (corresponding to $R=2am=2E_W$) is a singularity it suffices to consider the scalar curvature (i.e., the Ricci scalar) of Eq. (68),

$$\text{Scal} = -12F^2 - 6\frac{A}{B}\dot{F}, \tag{83}$$

which is readily computed from Eq. (72). Carrera and Giulini (2009) showed that this becomes singular in the limit $r\to m/2$, with the only exceptions being the following three special cases: (i) $m=0$ and a arbitrary (FLRW spacetimes), (ii) a and m constant (Schwarzschild spacetime), and (iii) $(am)^{\cdot}=0$ and $(\dot{a}/a)^{\cdot}=0$ (Schwarzschild–de Sitter spacetime). This means that, as long as we stick to the ansatz (68), at $r=m/2$ there will always (with the only exceptions listed above) be a singularity in the Ricci part of the curvature and thus, assuming Einstein's equation is satisfied, also in the energy momentum tensor, irrespectively of the details of its underlying matter model. Hence any attempt to eliminate this singularity by maintaining the ansatz (68) and merely modifying the matter model is doomed to fail. In particular, this is true for the generalizations in Faraoni and Jacques (2007), contrary to what is claimed in that work. We also remark that it makes no sense to absorb the singular factors $1/B$ in front of the time derivatives by writing $(A/B)\partial/\partial t$ as e_0 and then argue, as done in Faraoni and Jacques (2007), that this eliminates the singularity. The point is simply that then e_0 applied to any continuously differentiable function diverges as $r\to m/2$. Below we show that this singularity lies within a trapped region. Turning back to the McVittie model, recall that in this case it is assumed that the fluid moves along the integral curves of $\partial/\partial t$, which become lightlike in the limit as r tends to $m/2$. Their acceleration is given by the gradient of the pressure, which necessarily diverges in the limit $r\to m/2$, as one explicitly sees from Eq. (91). For a more detailed study of the geometric singularity at $r=m/2$, see Nolan (1999a, 1999b).

For spherically symmetric spacetimes the Weyl part of the curvature has only a single independent component [see Eq. (D8a)] which, by its very definition, is $-2/R^3$ times the Weyl part of the MS energy [see Eq. (D36)]. The square of the Weyl tensor for the ansatz (68) may then be conveniently expressed as [see Eqs. (81) and (D37)]

$$\langle\mathbf{Weyl},\mathbf{Weyl}\rangle = 48(am)^2/R^6. \tag{84}$$

This shows that $R=0$ also corresponds to a genuine curvature singularity, though this is not part of the region covered by our original coordinate system, for which $r>m/2$ (that is $R>2E_W$).

It is instructive to also determine the trapped regions of McVittie spacetime. We do this using the McVittie ansatz (68) and making no further assumptions. Recall that a spacelike two-sphere S is said to be *trapped, marginally trapped*, or *untrapped* if the product $\theta^+\theta^-$ of the expansions [defined below Eq. (D29)] for the ingoing and outgoing future-pointing null vector fields normal to S is positive, zero, or negative. Taking S to be S_R, that is, a sphere of symmetry with areal radius R, it immediately follows from Eq. (D31) that S_R is trapped, marginally trapped, or untrapped iff $\langle dR, dR\rangle$ is positive, zero, or negative, respectively. This corresponds to timelike, lightlike, or spacelike dR or, equivalently, in view of Eq. (58), to $2E-R$ being positive, zero, or negative, respectively. Using Eq. (80) together with Eq. (72a), the MS energy for the McVittie ansatz can be written as $E=E_W+R^3F^2/2$, so that

$$2E - R = F^2R^3 - R + R_S, \tag{85}$$

where $R_S:=2E_W$ denotes the "generalized" Schwarzschild radius. Note that in general F depends itself on the radial coordinate—except for the McVittie case, in which one has $F=H=:1/R_H$ (R_H denotes the Hubble radius). For computational simplicity we specialize in the following to the McVittie case, referring to Carrera and Giulini (2009) for the general case. Doing this, Eq. (85) becomes a cubic polynomial in R which is positive for $R=0$ and tends to $\pm\infty$ for $R\to\pm\infty$. Hence it always has a negative zero (which does not interest us) and two positive zeros iff

$$R_S/R_H < 2/3\sqrt{3} \approx 0.38. \tag{86}$$

This corresponds to the physical relevant case where the Schwarzschild radius is much smaller than the Hubble radius. One zero lies in the vicinity of the Schwarzschild radius and one in the vicinity of the Hubble radius, corresponding to two marginally trapped spheres. The exact expressions for the zeros can be easily written down, but are not very illuminating. In leading order in the small parameter R_S/R_H, they are given by

$$R_1 \approx R_S[1 + (R_S/R_H)^2], \tag{87a}$$

$$R_2 \approx R_H(1 - R_S/2R_H). \tag{87b}$$

From this one sees that for the McVittie model the radius of the marginally trapped sphere of Schwarzschild spacetime (R_S) increases and that of the FLRW spacetime (R_H) decreases. The first feature can be understood as an effect of the presence of cosmological matter, whereas the latter is an effect of the presence of a central mass abundance. All the spheres with $R<R_1$ or $R>R_2$ are trapped and those with $R_1<R<R_2$ are untrapped. In particular, the singularity $r=m/2$, that is R

$=2E_W=R_S$, lies within the inner trapped region.

Another aspect concerns the global behavior of the McVittie ansatz (68). We note that each hypersurface of constant time t is a complete Riemannian manifold, which, besides the rotational symmetry, admits a discrete isometry given in (r,θ,φ) coordinates by

$$\phi(r,\theta,\varphi)=[(m/2)^2 r^{-1},\theta,\varphi]. \tag{88}$$

This corresponds to an inversion at the two-sphere $r=m/2$ and shows that the hypersurfaces of constant t can be thought of as two isometric asymptotically flat pieces joined together at the totally geodesic (being a fixed-point set of an isometry and hence also minimal) two-sphere $r=m/2$. Except for the time-dependent factor $m(t)$, this is analogous to the geometry of the $t=$const slices in the Schwarzschild metric [the difference being that (88) does not extend to an isometry of the space-time metric unless $\dot{m}=0$]. This means, in particular, that the McVittie metric cannot literally be interpreted as corresponding to a point particle sitting at $r=0$ ($r=0$ is at infinite metric distance) in a flat FLRW universe, just like the Schwarzschild metric does not correspond to a point particle sitting at $r=0$ in Minkowski space. Unfortunately, McVittie seems to have interpreted his solution in this fashion (McVittie, 1933) which even until recently gave rise to some confusion in the literature [e.g., Gautreau (1984), Sussman (1988), Ferraris *et al.* (1996)]. A clarification was given by Nolan (1999a).

We now discuss the basic properties of the motion of cosmological matter for the McVittie model. We mentioned that the vorticity and shear of the four-velocity $\boldsymbol{u}$ vanish identically. On the other hand, the expansion (divergence of $\boldsymbol{u}$) is

$$\theta=3H, \tag{89}$$

just as in the FLRW case [recall that here $H:=\dot{a}/a$ is defined as in the FLRW case, see Eq. (77)]. In particular, the expansion of the cosmological fluid is homogeneous in space. The expression for the variation of the areal radius along the integral lines of $\boldsymbol{u}$ (that is, the velocity of cosmological matter measured in terms of its proper time and the areal radius) is also just as in the FLRW case,

$$\boldsymbol{u}(R)=HR, \tag{90}$$

which is nothing but Hubble's law. The acceleration of $\boldsymbol{u}$, which in contrast to the FLRW case does not vanish, is given by

$$\nabla_{\boldsymbol{u}}\boldsymbol{u}=\frac{m_0}{R^2}\left(\frac{1+m/2r}{1-m/2r}\right)\boldsymbol{e}_1. \tag{91}$$

In leading order in m_0/R this corresponds to the acceleration of the observers moving along $\partial/\partial T$ in Schwarzschild spacetime [see Eq. (132) with $\Lambda=0$].

We conclude this section by commenting on the attempts to generalize the McVittie model. The first generalization consists in allowing a nonvanishing cosmological constant. This is however trivial, since it is equivalent to the substitution $\varrho\rightarrow\varrho+\varrho_\Lambda$ and $p\rightarrow p+p_\Lambda$

into Eq. (78), where $\varrho_\Lambda:=\Lambda/8\pi$ and $p_\Lambda:=-\Lambda/8\pi$ are the energy density and pressure associated to the cosmological constant Λ, respectively. Faraoni and Jacques (2007) attempted to generalize the McVittie model by keeping the ansatz (68) and relaxing the conditions on the matter in various ways. More precisely, they showed that the McVittie case may be generalized to allow for radial fluid motions relative to the $\boldsymbol{e}_0$ observer field [that is relaxing condition (71)], provided one also allows for a nonvanishing radial heat flow. Both energy flows are necessary in order to get new solutions consistent with the ansatz (68), even though the two radial energy flows do not cancel in the energy balance. As a result, the Weyl part of the MS energy will now change in time so that the new solutions correspond to inhomogeneities of variable strength due to accretion or loss of energy. For further analysis of these solutions see Carrera and Giulini (2009). Another generalization of the McVittie model, this time away from the ansatz (68), is given in Sultana and Dyer (2005). It is constructed by applying a particular (time-dependent) conformal transformation to the Schwarzschild spacetime. As shown in Carrera and Giulini (2009), the metric so obtained cannot be written in the McVittie form (68), contrary to what is suggested in Faraoni and Jacques (2007). The corresponding energy-momentum tensor (obtained via Einstein's equation) can be interpreted as a sum of two contributions, one due to a perfect fluid and the other to a null fluid.

2. Motion of a test particle in McVittie spacetime

We are interested in the motion of a test particle (idealizing a planet or a spacecraft) in McVittie's spacetime. McVittie (1933) concluded within a slow-motion and weak-field approximation that Keplerian orbits do not expand as measured with the cosmological geodesic radius $r_*=a(t)r$. Later Pachner (1963) and Noerdlinger and Petrosian (1971) argued for the presence of the acceleration term (7) proportional to $\ddot{a}/a$ within this approximation scheme, hence arriving at Eq. (12a). In the following we show how to arrive at Eq. (12a) from the exact geodesic equation of the McVittie metric by making clear the approximations involved. Related recent discussions were given in Bolen *et al.* (2001), where the effects of cosmological expansion on the periastron precession and eccentricity are discussed for constant Hubble parameter $H:=\dot{a}/a$.

We again work with the areal radius R. Note that for fixed t the map $r\mapsto R(t,r)$ is 2 to 1 and that $R\geq 2m_0$, where $R=2m_0$ corresponds to $r=m_0/2a$. Hence we restrict the coordinate transformation (74) to the region $r>m_0/2a$ where it becomes a diffeomorphism onto the region $R>2m_0$. [The region $R<2m_0$ was investigated in Nolan (1999b).] Reintroducing factors of c, McVittie's metric assumes the (nondiagonal) form in the region $R>2m_0$ [i.e., $r>m_0/2a(t)$]

$$\boldsymbol{g} = [1 - 2\mu(R) - h(t,R)^2]c^2\boldsymbol{dt}^2 + \frac{2h(t,R)}{\sqrt{1 - 2\mu(R)}}c\boldsymbol{dt}\boldsymbol{dR}$$

$$- \frac{dR^2}{1 - 2\mu(R)} - R^2\boldsymbol{g}_{S^2}, \tag{92}$$

where

$$\mu(R) := m_0/R, \quad h(t,R) := H(t)R/c \tag{93}$$

with $H(t) := (\dot{a}/a)(t)$, as usual.

The equations for a timelike geodesic (i.e., parametrized with respect to proper time) $\tau \mapsto z^\mu(\tau)$ with $g(\dot{z}, \dot{z}) = c^2$ follows via variational principle from the Lagrangian $\mathcal{L}(z, \dot{z}) = (1/2)g_{\mu\nu}(z)\dot{z}^\mu\dot{z}^\nu$. Spherical symmetry implies conservation of angular momentum. Hence we may choose the particle's orbit to lie in the equatorial plane $\theta = \pi/2$. The constant modulus of angular momentum is

$$R^2\dot{\varphi} = L. \tag{94}$$

The remaining two equations are then coupled second-order ODEs for $t(\tau)$ and $R(\tau)$. However, we may replace the first one by its first integral that results from $\boldsymbol{g}(\dot{z}, \dot{z}) = c^2$:

$$[1 - 2\mu(R) - h^2(t,R)]c^2\dot{t}^2 + \frac{2h(t,R)}{\sqrt{1 - 2\mu(R)}}c\dot{t}\dot{R}$$

$$- \frac{\dot{R}^2}{1 - 2\mu(R)} - (L/R)^2 = c^2. \tag{95}$$

The remaining radial equation is given by

$$\ddot{R} - [1 - 2\mu(R) - h^2(t,R)]\frac{L^2}{R^3} + \frac{m_0 c^2}{R^2}[1 - 2\mu(R)]\dot{t}^2$$

$$- R\{\dot{H}(t)[1 - 2\mu(R)]^{1/2} + H(t)^2[1 - \mu(R)$$

$$- h^2(t,R)]\}\dot{t}^2 - \frac{\mu(R) - h^2(t,R)}{1 - 2\mu(R)}\frac{\dot{R}^2}{R}$$

$$+ \frac{2[\mu(R) - h^2(t,R)]}{\sqrt{1 - 2\mu(R)}}cH(t)(\dot{R}/c)\dot{t} = 0. \tag{96}$$

Recall that $m_0 = GM/c^2$, where M is the mass of the central star in standard units.

Equations (95) and (96) are exact. We are interested in orbits of slow motion (compared with the speed of light) in the region where

$$R_S \ll R \ll R_H. \tag{97}$$

Recall that R_S and R_H are the Schwarzschild and Hubble radius, respectively [see Eqs. (5) and (6)]. The latter condition covers all situations of practical applicability in the Solar System, since the Schwarzschild radius R_S of the Sun is about 3 km $= 2 \times 10^{-8}$ a.u. and the Hubble radius R_H is about 13.7×10^9 ly $= 8.7 \times 10^{14}$ a.u.

The approximation now consists in considering small perturbations of Keplerian orbits. Let T be a typical time scale of the problem, like the period for closed orbits or else R/v with v a typical velocity. The expansion is then with respect to the following two parameters:

$$\varepsilon_1 \approx v/c \approx (m_0/R)^{1/2}, \tag{98a}$$

corresponding to a slow-motion and weak-field approximation, and

$$\varepsilon_2 \approx HT, \tag{98b}$$

corresponding to the approximation for small ratios of characteristic times to the age of the universe. In order to make the expression to be approximated dimensionless, we multiply Eq. (95) by $1/c^2$ and Eq. (96) by T^2/R. Then we expand the right-hand sides in powers of the parameters (98a) and (98b), using the fact that $h := (HR/c) \approx \varepsilon_1\varepsilon_2$. From this and Eq. (94) we obtain Eq. (12) if we keep only terms to zero order in ε_1 and leading (i.e., quadratic) order in ε_2, where we also reexpress R as function of t. Note that in this approximation the areal radius R is equal to the spatial geodesic distance on the $t = \text{const}$ hypersurfaces.

We mentioned that in the special case of constant $H = \dot{a}/a$ the McVittie solution turns into the Schwarzschild–de Sitter metric (59a) and (59b), which also describes the spacetime inside the Einstein-Straus vacuoles in case of nonvanishing cosmological constant. Recently, exact expressions in terms of hyperelliptic integrals for the integrated geodesic equation in Schwarzschild–de Sitter spacetimes were derived in Hackmann and Lämmerzahl (2008a, 2008b). Moreover, a general discussion of Solar System effects in Schwarzschild–de Sitter, like gravitational redshift, light deflection, time delay, perihelion precession, geodetic precession, and effects on Doppler tracking, has been given in Kagramanova et al. (2006). For example, it was found that a nonvanishing Λ could account for the anomalous Pioneer acceleration if its value was -10^{-37} m^{-2}, which is *minus* 10^{15} times the current most probable value. That value would also give rise to a perihelion precession four orders of magnitude larger than the accuracy to which this effect has been measured today.

We conclude by commenting on the geodesic equation in the generalizations of McVittie's model given in Sultana and Dyer (2005) and Faraoni and Jacques (2007). An essential feature, which distinguish these solutions from the McVittie one, is that in the former the Weyl part of the MS energy $E_W = am$ is not a constant as for McVittie but varies in time, meaning that there is an accretion of cosmological matter by the inhomogeneity [see Carrera and Giulini (2009)]. In view of the fact that the combination

$$m/r = A^2 E_W/R \approx E_W/R \tag{99}$$

contained in the McVittie ansatz gives (minus) the "Newtonian" part of the potential in the slow-motion and weak-field approximation (see Sec. V.D.2), we deduce that in order to get the geodesic equation for the generalized McVittie models it suffices to substitute m_0 with E_W in the equation of motion derived in Sec. V.D.2.

This means that the strength of the central attraction varies in time, leading to an in or out spiraling of the orbits if E_W is increasing or decreasing, respectively.

3. Exact condition for nonexpanding circular orbits in McVittie spacetime

Analogously to Sec. IV.B, where we ask whether there exist nonexpanding circular orbits (i.e., of constant areal radius) of the electron-proton system in a FLRW spacetime, we now ask whether there exist nonexpanding circular orbits of an (uncharged) test particle around the central mass. The necessary and sufficient condition for this to happen follows from inserting $R=$const in the radial part of the geodesic equation (96) and using the normalization condition (95) of the four-velocity in order to eliminate $\dot{t}$. In terms of the dimensionless quantities $h(t) := RH(t)/c$, $l := L/Rc$, and $\mu := m_0/R$, the condition for the existence of nonexpanding circular orbits can be given in the following form:

$$\frac{R}{c}\dot{h} = \frac{(1 - 2\mu - h^2)[\mu(1 + 3l^2) - l^2 - h^2]}{(1 + l^2)\sqrt{1 - 2\mu}}. \quad (100)$$

As for the electron-proton system in an FLRW spacetime [see Eq. (51)] this is a first-order autonomous ODE for $h(t)$ and therefore the Hubble function. In the present case the ODE is even simpler since it has the elementary form $\dot{h}=p(h^2)$, where p is a polynomial of degree two with constant coefficients. From Eq. (100), to leading order in the small quantities μ, l^2, and h^2, we get the same approximate ODE (54) and hence the same approximate solutions (55a)–(55c). Hence, the same conclusions as drawn for the electron-proton system in FLRW apply here.

From Eq. (100) it follows that stationary solutions $h(t)=$const$=:h_0$, corresponding to an exponentially growing scale factor (52) (and hence leading to a Schwarzschild–de Sitter spacetime), are those where h_0 satisfies

$$(l^2 + h_0^2)/(1 + 3l^2) = \mu, \quad (101)$$

where we used that the first factor on the numerator of the right-hand side of Eq. (100) is nonzero, as can be immediately inferred from the normalization condition (95). Notice that for a vanishing Hubble parameter (that is, for $h_0=0$) the above condition reduces to the third Kepler law in Schwarzschild spacetime, as expected. The effect of a nonvanishing Hubble parameter is again that we must provide the orbiting particle with a *smaller* angular velocity (smaller l) in order to keep it on a nonexpanding circular orbit with the *same* radius. The largest radius at which in a McVittie spacetime with exponentially expanding scale factor (that is a Schwarzschild–de Sitter spacetime) there is a nonexpanding circular orbit follows from Eq. (101) in the limit $l \to 0$. Then the condition reduces to $h_0^2=\mu$ which, solving for R, gives $(R_S R_H^2/2)^{1/3}$. This, exactly corresponds to the critical radius (16), taking into account that $q_0=-1$ for an exponentially growing scale factor.

VI. KINEMATICAL EFFECTS

In this section we discuss the influence of cosmic expansion upon measurements of relative distances, velocities, and accelerations. These kinematical notions loose their *a priori* meaning in general spacetimes, in particular in time-dependent ones. Hence it is of utmost importance to carefully reconsider statements concerning such notions and their precise relations to locally observable quantities.

A. Einstein versus cosmological simultaneity

Misidentifications in the notion of simultaneity can give rise to apparent anomalies in velocities and acceleration. Such an effect has been suggested in Rosales and Sanchez-Gomez (1998) and again in Rosales (2002) to be able to account for the PA. Their argument says that in a spatially flat FLRW universe the mismatch between adapted cosmological coordinates, on the one hand, and radar coordinates, on the other hand, amount to an apparent difference in radial acceleration of magnitude (1). We agree on the existence and conceptual importance of such an effect but we disagree on the magnitude, which seems to have been grossly overestimated as shown below.

The cause of such effects lies in the way one actually measures spatial distances and determines the clock readings they are functions of (a trajectory is a "distance" for each given "time"). The point is this: equations of motions give us simultaneous (with respect to cosmological time) spatial geodesic distances as functions of cosmological time. This is what we implicitly did in the Newtonian analysis. But, in fact, spacecraft ranging is done by exchanging electromagnetic signals. The notions of spatial distance and simultaneity thereby implicitly used are not the same as those we referred to above. Hence the analytical expression of the "trajectory" so measured will be different.

We first recall the local version of Einstein simultaneity in general spacetimes $(\mathcal{M}, \boldsymbol{g})$. We take $ds = g_{\mu\nu}dx^\mu dx^\nu$ to carry the unit of length so that $d\tau = ds/c$ carries the unit of time. In general coordinates $\{x^\mu\} = \{t, x^i\}$, where $x^0=t$ denotes the timelike coordinate, the metric reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{tt}dt^2 + 2g_{ti}dtdx^i + g_{ij}dx^i dx^j. \quad (102)$$

The observer at fixed spatial coordinates is given by the vector field [normalized to $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{u})=c^2$]

$$\boldsymbol{u} = c\|\partial/\partial t\|^{-1}\partial/\partial t = \frac{c}{\sqrt{g_{tt}}}\partial/\partial t. \quad (103)$$

Consider the light cone with vertex $p \in \mathcal{M}$; one has $ds^2=0$, which allows to solve for dt in terms of the dx^i (all functions $g_{\mu\nu}$ are evaluated at p, unless noted otherwise),

$$dt_{1,2} = -\frac{g_{ti}}{g_{tt}}dx^i \pm \sqrt{\left(\frac{g_{ti}g_{tj}}{g_{tt}^2} - \frac{g_{ij}}{g_{tt}}\right)dx^i dx^j}. \qquad (104)$$

The plus sign corresponds to the future light-cone at p, the negative sign to the past light-cone. An integral line of $\boldsymbol{u}$ in a neighborhood of p cuts the light-cone in two points, q_+ and q_-. If t_p is the time assigned to p, then $t_{q_+}=t_p+dt_1$ and $t_{q_-}=t_p+dt_2$. The coordinate-time separation between these two cuts is $t_{q_+}-t_{q_-}=dt_1-dt_2$, corresponding to a proper time $\sqrt{g_{tt}}(dt_1-dt_2)/ca$ for the observer $\boldsymbol{u}$. This observer will associate a *radar distance dl_** to the event p of $c/2$ times that proper time interval, that is,

$$dl_*^2 = h = (g_{ti}g_{tj}/g_{tt} - g_{ij})dx^i dx^j. \qquad (105)$$

The event on the integral line of $\boldsymbol{u}$ that the observer will call Einstein synchronous with p lies in the middle between q_+ and q_-. Its time coordinate is in first-order approximation given by $\frac{1}{2}(t_{q_+}+t_{q_-})=t_p+\frac{1}{2}(dt_1+dt_2)=t_p+dt$, where

$$dt := \tfrac{1}{2}(dt_1 + dt_2) = -(g_{ti}/g_{tt})dx^i. \qquad (106)$$

This means the following: The integral lines of $\boldsymbol{u}$ are parametrized by the spatial coordinates $\{x^i\}_{i=1,2,3}$. Given a point p, specified by the orbit coordinates x_p^i and the time coordinate t_p, we consider a neighboring orbit of $\boldsymbol{u}$ with orbit coordinates $x_p^i+dx^i$. The event on the latter which is Einstein synchronous with p has a time coordinate t_p+dt, where dt is given by Eq. (106), or equivalently

$$\theta := dt + (g_{ti}/g_{tt})dx^i = 0. \qquad (107)$$

Using a differential geometric language we may say that Einstein simultaneity defines a *distribution* $\theta=0$.

The metric (102) can be written in terms of the radar-distance metric h (105) and the simultaneity one-form θ as follows:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{tt}\theta^2 - h, \qquad (108)$$

showing that the radar distance is just the same as the Einstein-simultaneous distance. A curve γ in $\mathcal{M}$ intersects the flow lines of $\boldsymbol{u}$ perpendicularly iff $\theta(\dot{\gamma})=0$, which is the condition that neighboring clocks along γ are Einstein synchronized.

We now apply the foregoing to isotropic cosmological metrics. In what follows we drop for simplicity the angular dimensions. Hence we consider metrics of the form

$$ds^2 = c^2 dt^2 - a(t)^2 dr^2. \qquad (109)$$

The comoving observer field

$$\boldsymbol{u} = c\boldsymbol{\partial}/\boldsymbol{\partial}t \qquad (110)$$

is geodesic and of expansion $3H$. On a hypersurface of constant t the radial geodesic distance is given by $a(t)r$.

Making this distance into a spatial coordinate r_* we consider the coordinate transformation

$$t \mapsto t_* := t, \quad r \mapsto r_* := a(t)r. \qquad (111)$$

The field $\boldsymbol{\partial}/\boldsymbol{\partial}t_*$ is given by

$$\boldsymbol{\partial}/\boldsymbol{\partial}t_* = \boldsymbol{\partial}/\boldsymbol{\partial}t - Hr\boldsymbol{\partial}/\boldsymbol{\partial}r, \qquad (112)$$

to which the observer field

$$\boldsymbol{u}_* := c\|\boldsymbol{\partial}/\boldsymbol{\partial}t_*\|^{-1}\boldsymbol{\partial}/\boldsymbol{\partial}t_* \qquad (113)$$

corresponds. In contrast to Eq. (110), whose flow connects comoving points of constant coordinate r, the flow of Eq. (112) connects points of constant geodesic distances, as measured in the surfaces of constant cosmological time. This could be called *cosmologically instantaneous geodesic distance*. It is now important to realize that this notion of distance is not the same as the radar distance that one determines by exchanging light signals in the usual (Einsteinian) way. We now explain this in detail.

From Eq. (111) we have $adr=dr_*-r_*Hdt$, where $H := \dot{a}/a$ (Hubble parameter). Rewriting the metric (109) in terms of t_* and r_* yields

$$ds^2 = c^2(1 - (Hr_*/c)^2)dt_*^2 - dr_*^2 + 2Hr_*dtdr_*$$

$$= c^2\underbrace{\{1 - (Hr_*/c)^2\}}_{g_{t_*t_*}}\underbrace{\left\{dt_* + \frac{Hr_*/c^2}{1 - (Hr_*/c)^2}dr_*\right\}^2}_{\theta}$$

$$-\underbrace{\frac{dr_*^2}{1 - (Hr_*/c)^2}}_{h}. \qquad (114)$$

Hence the differentials of radar distance and time lapse for Einstein simultaneity are given by

$$dl_* = \frac{dr_*}{\sqrt{1 - (Hr_*/c)^2}}, \qquad (115a)$$

$$dt_* = -\frac{Hr_*/c^2}{1 - (Hr_*/c)^2}dr_*. \qquad (115b)$$

Let the distinguished observer (us on Earth) now move along the geodesic $r_*=0$. Integration of Eq. (115) from $r_*=0$ to some value r_* then gives the radar distance l_* as well as the time lapse Δt_* as functions of the cosmologically simultaneous geodesic distance r_*:

$$l_* = (c/H)\arcsin(Hr_*/c)$$

$$= r_*\left\{1 + \frac{1}{6}(Hr_*/c)^2 + \mathcal{O}((Hr_*/c)^3)\right\}, \tag{116a}$$

$$\Delta t_* = (1/2H)\ln[1 - (Hr_*/c)^2]$$

$$= (r_*/c)\left\{-\frac{1}{2}(Hr_*/c) + \mathcal{O}((Hr_*/c)^2)\right\}. \tag{116b}$$

Combining both equations in Eq. (116) allows us to express the time lapse in terms of the radar distance

$$\Delta t_* = H^{-1}\ln[\cos(Hl_*/c)]$$

$$= (l_*/c)\left\{-\frac{1}{2}(Hl_*/c) + \mathcal{O}((Hl_*/c)^2)\right\}. \tag{117}$$

Now, suppose a satellite S moves on a worldline $r_*(t_*)$ in the neighborhood of our worldline $r_*=0$. Assume that we measure the distance to the satellite by radar coordinates. Then instead of the value r_* we would use l_* and instead of the argument t_* we would assign the time $t_* - \Delta t_*$ which corresponds to the value of cosmological time at that event on our worldline that is Einstein synchronous to the event (t_*, r_*); see Fig. 2.

Hence we have

$$l_*(t_*) = (c/H)\sin^{-1}\{r_*(t_* + \Delta t_*)H/c\} \tag{118a}$$

$$\approx r_* - \frac{1}{2}(v/c)(Hc)(r_*/c)^2, \tag{118b}$$

where Eq. (118b) is Eq. (118a) to leading order and all quantities are evaluated at t_*. We set $v = \dot{r}_*$.

To see what this entails we Taylor expand in t_* around $t_*=0$ (just a convenient choice),

$$r_*(t_*) = r_0 + v_0 t_* + \frac{1}{2}a_0 t_*^2 + \cdots \tag{119}$$

and insert in Eq. (118b). This leads to

$$l_*(t_*) = \tilde{r}_0 + \tilde{v}_0 t_* + \frac{1}{2}\tilde{a}_0 t_*^2 + \cdots, \tag{120}$$

where

$$\tilde{r}_0 = r_0 - (Hc)\frac{1}{2}(v_0/c)(r_0/c)^2, \tag{121a}$$

$$\tilde{v}_0 = v_0 - (Hc)(v_0/c)^2(r_0/c), \tag{121b}$$

$$\tilde{a}_0 = a_0 - (Hc)\{(v_0/c)^3 + (r_0/c)(v_0/c)(a_0/c)\}. \tag{121c}$$

These are, in quadratic approximation, the sought-after relations between the quantities measured via radar tracking (tilded) and the quantities which arise in the (improved) Newtonian equations of motion (not tilded).

In particular, the last equation (121c) shows that there is an apparent inward pointing acceleration, given by Hc times the $(v/c)^3 + \cdots$ term in curly brackets. As discussed
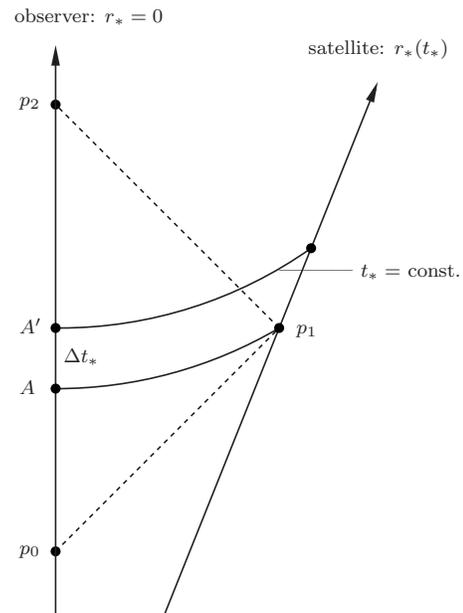

FIG. 2. An observer moves on the geodesic worldline $r_*=0$ and observes a satellite by exchanging electromagnetic signals (dashed line). Event p_0 corresponds to the signal emission by the observer, p_1 to its reflection by the satellite, and p_2 to its reabsorption by the observer. t_* denotes the cosmological time and the two curved lines correspond to hypersurfaces of constant t_*. On the observer's worldline event A is defined to be simultaneous to p_1 with respect to t_* and A' is defined to lie half-way in proper time between p_0 and p_2 on the observer's worldline. In cosmological time A' is Δt_* ahead of A. In cosmological coordinates, the satellite's trajectory is represented by the a function $t_* \mapsto r_*(t_*)$, where $r_*(t_*)$ denotes the proper geodesic distance in the hypersurface $t_*=$const between its intersection points with both worldlines. However, using radar coordinates, the observer takes A' to be simultaneous with p_1 and uses l_* as measure for the satellite's simultaneous distance. Since r_* and l_* are related by Eq. (116a), it follows that the observer uses the function $t_* \mapsto l_*(r_*(t_* - \Delta t_*))$ to characterize the satellite's trajectory, which leads to Eq. (118).

in the Introduction, Hc is indeed of the same order of magnitude as the PA, as emphasized in Rosales and Sanchez-Gomez (1998) and Rosales (2002). However, in contrast we also get the additional term in curly brackets, which in case of the Pioneer spacecraft suppresses the Hc term by 13 orders of magnitude![17] Hence we conclude that, with respect to the PA, there is no significant kinematical effect resulting from the distinct simultaneity structures inherent in radar and cosmological coordinates.

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[17]Our Eq. (117) corresponds to Eq. (10) of Rosales and Sanchez-Gomez (1998). From it, Rosales and Sanchez-Gomez (1998) and Rosales (2002) immediately jumped to the conclusion that there is "an effective residual acceleration directed toward the center of coordinates; its constant value is Hc." We were unable to follow this conclusion. Likewise, we are unable to follow the conclusion in Fahr and Siewert (2008).
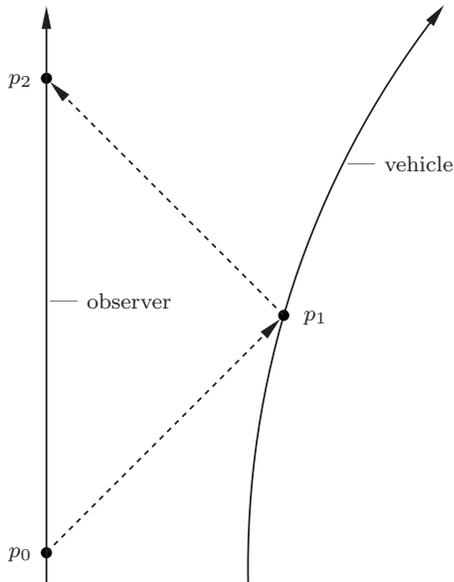
FIG. 3. Exchange of electromagnetic signals (represented by their rays at a slope of 45 degrees) between us and the space vehicle. Time runs vertically.

B. Doppler tracking in cosmological spacetimes

Doppler tracking is a common method of tracking the position of vehicles in space. It involves measuring the Doppler shift of an electromagnetic signal sent from a spacecraft to a tracking station on Earth. This signal either is coming from an on-board oscillator or is coherently transponded by the vehicle in response to a signal received from the ground station. Here we focus on the second of these modes, which is more useful for navigation, partly because the returning signal is measured against the same frequency reference as that of the originally transmitted signal and partly because the Earth-based frequency reference is also more stable than the oscillator on-board the spacecraft.

1. Minkowski spacetime

It is clear that this method will be fundamentally influenced if performed within a time varying background geometry. Before elaborating on this, we consider the simple case of static Minkowski space.

In Fig. 3 we depicted two worldlines, one of the observer (straight vertically) and one for the space vehicle. A light signal is emitted by the observer at the event p_0, reflected by the vehicle at event p_1, and finally received back by the observer at event p_2. We choose a global Minkowski frame, that is global coordinates $\{x^\mu\} = \{t, x^i\}$ with $\mathbf{g}(\partial/\partial x^\mu, \partial/\partial x^\nu) = \mathrm{diag}(1, -1, -1, -1)$, in which the observer (for simplicity assumed to be inertial) is at rest at the origin of the spatial coordinates. If $\beta := v/c$ denotes the radial velocity of the vehicle in units of c, the well-known special-relativistic Doppler formula (applied

twice) says that the ratio between the received and the emitted frequencies is[18]

$$\frac{\omega_2(t_2)}{\omega_0(t_0)} = \frac{1 - \beta(t_1)}{1 + \beta(t_1)}. \tag{122}$$

Here t_0, t_1, and t_2 refer to the global Minkowski time attributed to p_0, p_1, and p_2, respectively. In Doppler tracking one is interested in the derivative of this ratio with respect to t_2, which yields a measure for the velocity of the spacecraft. We assume ω_0 to be constant in time and note that, given the worldlines of the observer and the vehicle, t_1 and t_0 are uniquely determined by t_2 (since the events p_1 and p_0 are determined by p_2). If r denotes the spatial radius coordinate, we have $t_2 - t_1 = r(t_1)/c$. Differentiation with respect to t_1 leads to

$$\frac{dt_1}{dt_2} = \frac{1}{1 + \beta(t_1)} \tag{123}$$

and hence

$$-\frac{1}{2}\frac{\dot{\omega}_2(t_2)}{\omega_0(t_0)} = \dot{\beta}(t_1)[1 + \beta(t_1)]^{-3}$$

$$= \dot{\beta}(t_1)[1 - 3\beta(t_1) + \mathcal{O}(\beta^2)]. \tag{124}$$

This shows that $-\dot{\omega}_2/2\omega_0$, namely, (minus one-half) the derivative of the received to emitted frequency ratio with respect to the proper time of the receiving observer, gives the spacecraft's spatial acceleration up to corrections of order β. Note that in view of note 3 it would be inappropriate to call these corrections "special relativistic."

The final goal of this section is to derive the generalization of (124) for a cosmological spacetime. For this we need two things: First, we need to know what is the generalization of the concepts of spatial velocity and spatial acceleration in an arbitrary spacetime and, second, we need to know how electromagnetic signals propagate in an arbitrary spacetime. This is taken care of next.

2. General setting

In order to generalize the notions of spatial velocity and spatial acceleration to arbitrary spacetime one needs to introduce a fiducial reference "observer-field";

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[18]Note that in special relativity the Doppler formula does, of course, not distinguish between moving emitter and moving receiver. So Eq. (122) is obtained by squaring the frequency shift $\sqrt{(1-\beta)/(1+\beta)}$, which is picked up once for the ratio $\omega_R(t_R)/\omega_1(t_1)$ (receiver moving relative to the Minkowski frame) and once for $\omega_2(t_2)/\omega_1(t_1)$ (emitter moving relative to Minkowski frame). Incidentally, exactly the same formula would result in nonrelativistic physics if the observer is taken to be at rest with respect to the wave-guiding medium (e.g., the ether), which distinguishes the two states of relative motion. Indeed, in this case we have $\omega_1(t_1)/\omega_0(t_0) = [1 - \beta(t_1)]$ and $\omega_2(t_2)/\omega_1(t_1) = 1/[1 - \beta(t_1)]$, whose product is again just Eq. (122).

compare Bini *et al.* (1995) and also Carrera (2010). An *observer* at the event p is a future pointing unit timelike vector in the tangent space $T_p(\mathcal{M})$ of $\mathcal{M}$ at p. An *observer field* is a field of future pointing unit timelike vectors. Any observer $\boldsymbol{u}$ at p gives rise to an orthogonal split of the tangent space $T_p(\mathcal{M})$ at p in a part parallel to $\boldsymbol{u}$ (the local time axis) and a part orthogonal to it (the local rest space). Since $\boldsymbol{u}$ is not lightlike the two orthogonal subspaces are complementary, that is, together they span the whole tangent space and intersect only in the zero vector. The orthogonal projections of an arbitrary vector $X \in T_p(\mathcal{M})$ onto these subspaces are, respectively, given by

$$Q_u(X) := g(X,u)u, \tag{125a}$$

$$P_u(X) := X - g(X,u)u, \tag{125b}$$

which imply the decomposition identity $X = Q_u(X) + P_u(X)$.

If two observers $\boldsymbol{u}$ and $\boldsymbol{v}$ are defined at the same point, the *spatial velocity* (over c) of $\boldsymbol{v}$ with respect to $\boldsymbol{u}$ is given by

$$\boldsymbol{\beta}_u(v) := \frac{P_u(v)}{\|Q_u(v)\|} = \frac{v - g(v,u)u}{g(v,u)}, \tag{126}$$

which is an element of the local rest space $P_u T(\mathcal{M})$. Its modulus is given by

$$\beta_u(v) := \|\boldsymbol{\beta}_u(v)\| = \sqrt{1 - 1/g(u,v)^2}. \tag{127}$$

Note that for the modulus we have $\beta_u(v) = \beta_v(u)$, though the vectors $\boldsymbol{\beta}_u(v)$ and $\boldsymbol{\beta}_v(u)$ are linearly independent as they lie in $P_u T(\mathcal{M})$ and $P_v T(\mathcal{M})$, respectively. Note also that $g(u,v) = 1/\sqrt{1 - \beta_u^2(v)}$ is the ordinary "gamma factor." Finally, if $\boldsymbol{e} \in P_u T(\mathcal{M})$ is a unit vector, we define the spatial velocity of $\boldsymbol{v}$ in direction $\boldsymbol{e}$ with respect to $\boldsymbol{u}$ by

$$\beta_u^e(v) = -g(e, \boldsymbol{\beta}_u(v)) = -g(e,v)/g(u,v). \tag{128}$$

The *spatial acceleration* of a worldline γ with respect to a given observer field $\boldsymbol{u}$ is defined as the rate of change of the spatial velocity $\boldsymbol{\beta}_u(\dot{\gamma})$ within the local rest spaces $P_u T(\mathcal{M})$ of $\boldsymbol{u}$ and with respect to the clocks moving along $\boldsymbol{u}$. Denoting this acceleration (divided by c) with $\boldsymbol{\alpha}$, we have

$$\boldsymbol{\alpha}_u(\gamma) := \nabla_{\dot{\gamma}}^u \boldsymbol{\beta}_u(\dot{\gamma}), \tag{129}$$

where we used the following covariant derivative for $P_u T(\mathcal{M})$-valued vector fields along γ:

$$\nabla_{\dot{\gamma}}^u := \|Q_u \dot{\gamma}\|^{-1} P_u \circ \nabla_{\dot{\gamma}} \circ P_u. \tag{130}$$

Here $\nabla_{\dot{\gamma}}$ denotes the ordinary (Levi-Civita) covariant derivative along γ. As an application one can, for example, rewrite the geodesic equation, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, for a worldline γ in terms of the spatial quantities just introduced. One gets [see Carrera (2010) or, in a slightly different notation, Bini *et al.* (1995)]

$$\boldsymbol{\alpha}_u = -S_{\beta_u}[a_u + \theta_u(\beta_u) + \omega_u(\beta_u)], \tag{131}$$

where for better readability we omitted the arguments γ and $\dot{\gamma}$ in the spatial acceleration and spatial velocity. Here $a_u := \nabla_u u$ is the four-acceleration of the observer field $\boldsymbol{u}$, θ_u and ω_u are, respectively, its shear-expansion and rotation tensors of rank (1,1) (endomorphism), and $S_{\beta_u} := P_u + \beta_u \otimes \beta_u$ is a rank (1,1) tensor which, in a slow-motion approximation (neglecting quadratic and higher terms in β), reduces to the identity on the local rest space of $\boldsymbol{u}$. Equation (131) should be seen as a local version of Newton's equation. For example, in Schwarzschild–de Sitter spacetime (59a) and (59b), taking $\boldsymbol{u}$ to be proportional to the timelike Killing field $\partial/\partial T$, one has $\theta_u = \omega_u = 0$ (because of the Killing equation and spherical symmetry, respectively) and

$$a_u = \nabla_u u = \frac{1}{\sqrt{V}}\left(\frac{m}{R^2} - \frac{\Lambda}{3}R\right)e_R, \tag{132}$$

where e_R denotes the the normalized radial vector field $\partial/\partial R$ [we use here the coordinates and the notation of Eq. (59a) and (59b)]. Hence, in slow-motion and weak-field approximation (that is keeping only linear terms in β, m/R, and ΛR^2), the geodesic equation of motion in the form (131) reduces to

$$\boldsymbol{\alpha}_u \approx \left(-\frac{m}{R^2} + \frac{\Lambda}{3}R\right)e_R, \tag{133}$$

which gives the "improved" Newtonian equation for geodesic motions in Schwarzschild–de Sitter spacetime. It has the same form as the improved Newtonian equation studied in Sec. III.

We turn now to electromagnetic signals and restrict our attention to monochromatic waves in the geometric-optics approximation (i.e., for wavelengths negligibly small with respect to a typical radius of curvature of the spacetime and with respect to a typical length over which amplitude, polarization, and frequency vary). In this approximation an electromagnetic signal propagates on a lightlike geodesic along which the wave vector $\boldsymbol{k}$ is tangent, future pointing, and parallelly transported. Recall that $\boldsymbol{k}$ is so normalized that the frequency measured by an observer, say $\boldsymbol{u}$, is

$$\omega_u(k) := g(u,k). \tag{134}$$

Given a wave vector $\boldsymbol{k}$ and two observers $\boldsymbol{u}, \boldsymbol{v}$ at the same spacetime point, their observed frequencies are thus $\omega_v(k) = g(v,k)$ and $\omega_u(k) = g(u,k)$, and their ratio is given by

$$\frac{\omega_v(k)}{\omega_u(k)} = \frac{g(Q_u v + P_u v, k)}{g(u,k)} = g(u,v)[1 - \beta_u^{\hat{k}}(v)]. \tag{135}$$

Here the spacelike unit vector $\hat{k} := \|P_u(k)\|^{-1} P_u(k)$ defines the direction of $\boldsymbol{k}$ in the local rest space of $\boldsymbol{u}$. In deriving Eq. (135) we used Eq. (128) and $\|P_u(k)\| = g(u,k)$ to write $g(v, P_u(k)) = -g(v,u)g(u,k)\beta_u^{\hat{k}}(v)$. Equation (135) is the general form of the Doppler formula.

Now let u be an observer field along one integral line of which the distinguished observer is moving. The worldline of the vehicle is denoted by γ. The domain of the field u is assumed to include a neighborhood of γ. The wave vector k_0 emitted at p_0 suffers the following three changes: (1) propagation from p_0 to p_1: $k_0 \rightarrow k_1$; (2) reflection at p_1: $k_1 \rightarrow k_1'$; (3) propagation from p_1 to p_2: $k_1' \rightarrow k_2$.

We are interested in the ratio of the received to the emitted frequency,

$$\frac{\omega_2}{\omega_0} = \frac{g(u_2, k_2)}{g(u_0, k_0)} = \left(\frac{\omega_2}{\omega_1'}\right)\left(\frac{\omega_1'}{\omega_1}\right)\left(\frac{\omega_1}{\omega_0}\right). \tag{136}$$

What happens at reflection (the second process $k_1 \rightarrow k_1'$)? With respect to the spacecraft moving along γ with four-velocity $v = \dot{\gamma}$, the wave vector k_1 at p_1 splits according to

$$k_1 = Q_{\dot{\gamma}}(k_1) + P_{\dot{\gamma}}(k_1). \tag{137}$$

A corner-cube reflector transported along γ will reverse $P_{\dot{\gamma}}(k_1)$ while keeping $Q_{\dot{\gamma}}(k_1)$ intact (here we neglect a possible transponder shift which is irrelevant for our discussion),

$$k_1 \mapsto k_1' = Q_{\dot{\gamma}}(k_1) - P_{\dot{\gamma}}(k_1) = 2Q_{\dot{\gamma}}(k_1) - k_1. \tag{138}$$

Hence $\omega_1 := \omega_u(k_1) = g(u_1, k_1)$ and $\omega_1' := \omega_u(k_1') = g(u_1, k_1')$, the in- and out-going frequencies measured by the observer u at p_1, are related by

$$\frac{\omega_1'}{\omega_1} = 2\frac{g(u, \dot{\gamma})g(\dot{\gamma}, k)|_{p_1}}{g(u, k)|_{p_1}} - 1 = 2\frac{1 - \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}}{1 - \beta_u^2(\dot{\gamma})|_{p_1}} - 1, \tag{139}$$

where in the last step we used Eq. (135) to rewrite the ratio $g(\dot{\gamma}, k)/g(u, k)$. This accounts for the middle ratio on the right-hand side of Eq. (136).

To account for the other two ratios in Eq. (136), one uses the laws of geometric optics in (curved) spacetime to relate $\omega_0 = g(u_0, k_0)$ (at p_0) and $\omega_2 = g(u_2, k_2)$ (at p_2) to kinematical quantities of γ at p_1.[19] For example, if u is a Killing field (like $u = \partial/\partial t$ in special relativity), we have $g(u_0, k_0) = g(u_1, k_1)$ and $g(u_2, k_2) = g(u_1, k_1')$, so that

$$\frac{\omega_2}{\omega_0} = 2\frac{1 - \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}}{1 - \beta_u^2(\dot{\gamma})|_{p_1}} - 1. \tag{140}$$

As a trivial application, this includes the generalized form of Eq. (122), the latter corresponding to purely radial motion.

3. FLRW spacetimes

In standard cosmological spacetimes (FLRW), $u = \partial/\partial t$ is not Killing, though $X = a(t)\partial/\partial t$ is conformally

Killing ($L_X g = 2\dot{a}g$). One now has $a_0 g(u_0, k_0) = a_1 g(u_1, k_1)$ and $a_2 g(u_2, k_2) = a_1 g(u_1, k_1')$, so that instead of Eq. (140) one gets

$$\frac{\omega_2}{\omega_0} = \frac{a_0}{a_2}\left\{2\frac{1 - \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}}{1 - \beta_u^2(\dot{\gamma})|_{p_1}} - 1\right\}. \tag{141}$$

We now want to relate the t_2 derivative of Eq. (141) to the acceleration of γ. In order to calculate the derivative $\dot{\omega}_2(t_2)/\omega_0(t_0)$ we need to know the derivatives dt_1/dt_2 and dt_0/dt_2. Restricting to the flat FLRW case for simplicity, they follow from the law of null propagation:

$$\int_{t_1(t_2)}^{t_2} \frac{dt}{a(t)} = -\frac{1}{c}\int_{r_1(t_1(t_2))}^{r_2} dr, \tag{142a}$$

$$\int_{t_0(t_2)}^{t_2} \frac{dt}{a(t)} = \frac{1}{c}\left\{\int_{r_0}^{r_1(t_1(t_2))} dr - \int_{r_1(t_1(t_2))}^{r_2} dr\right\}. \tag{142b}$$

Differentiation with respect to t_2 yields, respectively,

$$\frac{dt_1}{dt_2} = \frac{a(t_1)}{a(t_2)}(1 + \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1})^{-1}, \tag{143a}$$

$$\frac{dt_0}{dt_2} = \frac{a(t_0)}{a(t_2)}\frac{1 - \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}}{1 + \beta_u^{\hat{k}}(\dot{\gamma})|_{p_1}}. \tag{143b}$$

The exact formula for the t_2 derivative of the frequency-shift rate can now be computed. One obtains

$$-\frac{\dot{\omega}_2(t_2)}{\omega_0(t_0)} = \frac{a_0}{a_2}\left\{2[\alpha^{\hat{k}} - g(\beta, \nabla_{\dot{\gamma}}^u \hat{k})]\frac{a_1}{a_2}[1 + \beta^{\hat{k}}]^{-1}[1 - \beta^2]^{-1}\right.$$
$$+ 4g(\alpha, \beta)\frac{a_1}{a_2}\left[\frac{1 - \beta^{\hat{k}}}{1 + \beta^{\hat{k}}}\right][1 - \beta^2]^{-2}$$
$$\left. + \left[\frac{\dot{a}_2}{a_2} - \frac{\dot{a}_0}{a_2}\left(\frac{1 - \beta^{\hat{k}}}{1 + \beta^{\hat{k}}}\right)\right]\left[\frac{1 - 2\beta^{\hat{k}} + \beta^2}{1 - \beta^2}\right]\right\}, \tag{144}$$

where we suppressed the argument $\dot{\gamma}$ and index u at β for better readability. This formula provides an exact relation between the time derivative of the observable frequency shift (defined "here") and the kinematical quantities of the vehicle (defined "there"), provided the scale function $a(t)$ is known. For purely radial motion $\nabla_{\dot{\gamma}}^u \hat{k} = 0$ and we obtain the simpler expression (now writing α for $\alpha^{\hat{k}}$)

$$-\frac{1}{2}\frac{\dot{\omega}_2(t_2)}{\omega_0(t_0)} = -\frac{a_0 a_1}{a_2^2}\left\{\alpha(1 + \beta)^{-3}\right.$$
$$\left. + \frac{1}{2}\left[\frac{\dot{a}_2}{a_1} - \frac{\dot{a}_0}{a_1}\left(\frac{1 - \beta}{1 + \beta}\right)\right]\left[\frac{1 - \beta}{1 + \beta}\right]\right\}. \tag{145}$$

In order to consistently approximate this expression in terms of small quantities β and $H\Delta t$, where $\Delta t := (t_2 - t_0)/2$, we think of Eq. (145) as being multiplied with Δt

and regard $\alpha \Delta t$ as being of order β. Then, keeping only quadratic terms in β, linear terms in $H\Delta t$ where $\Delta t := (t_2 - t_0)/2$, and also mixed terms $\beta H \Delta t$, we get

$$-\frac{1}{2}\frac{\dot{\omega}_2(t_2)}{\omega_0(t_0)} \approx \alpha(1 - 3\beta - 3H\Delta t) + H\beta. \qquad (146)$$

Hence we see that in this approximation there are two modifications, besides the -3β term already familiar from Eq. (124), due to cosmic expansion: First, there is an additional contribution $-3H\Delta t$ acting in the same way as the -3β term. It can also be interpreted in the same fashion, as its corresponds to the velocity (over c) of $H\Delta t$ that a comoving systems picks up during the time the signal went from the observer to the vehicle. Second, there is a constant contribution $H\beta$ to acceleration/c, i.e., $Hc\beta$ to acceleration, in a direction parallel to the radial velocity (i.e., outward pointing if the vehicle recedes from the observer). Hence it acts opposite to the PA and is smaller in modulus by a factor of β. Applied to the Pioneer spacecrafts, the $H\Delta t$ term amounts to a small "anomalous" acceleration of $\Delta a/a < 10^{-12}$, the $Hc\beta$ term to $\Delta a/a < 10^{-7}$.

A final point must be made regarding the choice of the reference observer field on which the kinematic quantities related to the spacecraft (spatial velocity and spatial acceleration) and the electromagnetic signal (frequency and spatial propagation direction) crucially depend. In the Minkowskian case the reference field was just $\boldsymbol{u} = \partial/\partial t$, which is inertial, that is, geodesic and of vanishing rotation, shear, and expansion. It is clear that in a general spacetime such observer fields do not exist and there is no natural choice to replace them. However, in the case of spherical symmetry there is, in fact, a distinguished observer field, namely, that one whose orbits lie within the timelike hypersurfaces of constant areal radius and there run perpendicular to the orbits of the rotation group. This clearly defines a nonrotating and "nonexpanding" (with respect to the areal radius) reference field. It is the normalization of the so-called Kodama vector field, discussed in Appendix D.3. In a FLRW spacetime it is given by Eq. (113). Notice that in the present case, where the hypersurfaces of constant cosmological time t are flat, the areal radius corresponds also to the proper distance. Hence the integral curves of $\boldsymbol{u}_*$ intersect the hypersurface of constant cosmological time at constant spatial geodesic distance. More precisely, the expansion and the shear scalar of $\boldsymbol{u}_*$ are given by $\theta_* = R^2 H \dot{H}/[1-(RH)^2]^{3/2}$ and $\sigma_* = -\theta_*/3$, respectively, showing that they are of order H^3 which we neglect. In passing we remark that the expansion and shear of $\boldsymbol{u}_*$ exactly vanish for the de Sitter case, whose metric in "static" coordinates is given by Eq. (59a) and (59b) for $m=0$. In this case $\partial/\partial t_* = \partial/\partial T$, that is, $\boldsymbol{u}_*$ is proportional to the timelike Killing vector field $\partial/\partial T$; see Eqs. (59a) and (59b). Coming back to the general FLRW case, the acceleration of $\boldsymbol{u}_*$ is given by $\boldsymbol{a}_{\boldsymbol{u}_*} = (-R\ddot{a}/a + R^3 H^4)/[1-(RH)^2]^{3/2}\boldsymbol{e}_*$, where $\boldsymbol{e}_*$ is the unit vector field orthogonal to $\boldsymbol{u}_*$ and to the two-sphere,

pointing in positive radial direction. Hence, in the slow-motion and weak-field approximation of Eq. (146), but keeping also quadratic terms in H, the geodesic equation in the form (131) with respect to the observer field $\boldsymbol{u}_*$ reads as

$$\boldsymbol{\alpha}_{\boldsymbol{u}_*}(\gamma) \approx (\ddot{a}r_*/a)\boldsymbol{e}_* \circ \gamma. \qquad (147)$$

This is just an alternative derivation of the acceleration term (7). We point out that had we we taken Eq. (110) as observer field we would have arrived at the equation of motion $\boldsymbol{\alpha}_{\boldsymbol{u}}(\gamma) \approx -H\boldsymbol{\beta}_{\boldsymbol{u}}(\gamma)$ instead of Eq. (147), that is, no acceleration term (7) would have resulted.

In the approximation within which Eq. (146) is derived this equation remains valid if the quantities in it are re-interpreted so as to refer to $\boldsymbol{u}_*$ instead of $\boldsymbol{u}$. Hence we may sum up the situation by saying that Eqs. (146) and (147) give, respectively, the two-way Doppler-tracking formula and the "Newtonian" equation in a FLRW spacetime within the mentioned approximation.

4. McVittie spacetime

The same analysis can be generalized from the spatially flat FLRW spacetime (30) to the spatially flat McVittie spacetime (68). Here the observer moves along $\partial/\partial t$, which is not geodesic. The coordinate t does now not measure proper time, denoted by τ, along the observer's worldline. The result corresponding to Eq. (146) can now be stated as follows:

$$-\frac{1}{2}\frac{\dot{\omega}_2(\tau_2)}{\omega_0(\tau_0)} \approx \alpha[1 - 3\beta - 3\Delta\tau(H - m_0c/R^2)] + H\beta. \qquad (148)$$

Here R denotes the areal radius of the observer during the measurement. Note that even though it changes along the observer's worldline according to Eq. (90), we do not need to account for the corresponding change in $\Delta\tau m_0 c/R^2$ of $(-2\Delta\tau m_0 c/R^2)(H\Delta\tau)$, which is of subleading order. The additional term in Eq. (148) has a straightforward interpretation in terms of the acceleration that the observer necessarily experiences while keeping a constant radius R away from the central inhomogeneity.

As for the FLRW case, we chose the observer field to which we refer the spatial quantities to be proportional to the Kodama vector field (along which the areal radius is constant). Putting $r_*(t,r) := A^2(t,r)a(t)r$ and $t_*(t,r) := t$, a short computation shows that the vector field $\partial/\partial t_*$ is again given by Eq. (112). In the slow-motion and weak-field approximation used in Sec. V.D.2, the geodesic equation in the form (131) with respect to the observer field $\boldsymbol{u}_*$ reads

$$\boldsymbol{\alpha}_{\boldsymbol{u}_*}(\gamma) \approx (\ddot{a}r_*/a - m_0/r_*^2)\boldsymbol{e}_* \circ \gamma, \qquad (149)$$

where again $\boldsymbol{e}_*$ denotes the unit outward-pointing vector field orthogonal to $\boldsymbol{u}_*$ and to the two-spheres of symmetry. This is an alternative derivation of the improved Newtonian equation for the McVittie spacetime carried

out in Sec. V.D.2. Notice that, again, within the approximations used, Eq. (148) remains valid if one refers the quantities to $\boldsymbol{u}_*$ instead of to $\boldsymbol{u}$. Therefore Eqs. (148) and (149) give the two-way Doppler-tracking formula and the improved "Newtonian" equation for the McVittie spacetime within the mentioned approximation.

In the special case of purely radial motion, insertion of Eq. (149) into Eq. (148) leads to a formula predicting the two-way Doppler-shift rate in linear order in $H\Delta\tau$ and m_0/r_*, and quadratic order in $\beta_{\boldsymbol{u}_*}(\dot{\boldsymbol{\gamma}})$,

$$-\frac{1}{2}\frac{\dot{\omega}_2(\tau_2)}{\omega_0(\tau_0)} = -\frac{m_0}{r_*^2}(1-3\beta^{\hat{\boldsymbol{k}}}) + H\beta^{\hat{\boldsymbol{k}}}. \tag{150}$$

Hence there are two corrections to the Newtonian contribution. One is proportional to H and stems from the cosmological expansion, the other, already familiar from the special-relativistic treatment (124), is independent of H and merely due to the finiteness of the propagation speed of light (recall footnote 18). Their ratio is (up to a factor $\sqrt{3}$) given by the square of the ratio of r_* to the geometric mean of the Schwarzschild radius m_0 and the Hubble radius c/H. The latter is of the order of 10^{23} km, so that its geometric mean with a Schwarzschild radius of 1 km is approximately given by 2400 a.u.. The ratio of the effects is therefore of the order 10^{-7}. Hence the cosmological contribution is negligible for any application in the Solar System as compared to the 3β correction. For the Pioneer 10 and 11 spacecrafts we have a radial velocity of about 12 km/s. This amounts to a 3β correction of magnitude 4×10^{-5} times the Newtonian gravitational acceleration, in an outward-pointing direction. This is indeed of the same order of magnitude as the PA but directed oppositely.

VII. SUMMARY AND OUTLOOK

We think it is fair to say that there are no theoretical hints that point towards a *dynamical* influence of cosmological expansion comparable in size to, say, that of the anomalous acceleration of the Pioneer spacecrafts. There seems to be no controversy over this point, though for completeness it should be mentioned that there exist speculations (Palle, 2005) according to which it might become relevant for future missions. But such speculations are often based on models which are not easily related to the intended physical situation, like that of Gautreau (1984). Rather, as the $(\ddot{a}/a)$-improved Newtonian analysis in Sec. III together with its justification given in the subsequent sections shows, there is no genuine relativistic effect coming from cosmological expansion at the levels of precision envisaged here.

On the other hand, as regards *kinematical* effects, the situation is less unanimous. It is important to unambiguously understand what is meant by "mapping out a trajectory," i.e., how to assign "times" and "distances."

Eventually we compare a functional relation between distance and time with observed data. That relation is obtained by solving some equations of motion and it has to be carefully checked whether the methods by which the tracking data are obtained match the interpretation of the coordinates in which the analytical problem is solved. In our way of speaking, dynamical effects really influence the worldline of the object in question whereas kinematical effects change the way in which one and the same worldline is mapped out from another worldline representing the observer. Here we have derived exact results concerning the influence of cosmic expansion on this mapping procedure, which allow one to reliably estimate upper bounds on their magnitude. They turn out to be too small to be of any relevance in current satellite trackings, which is in accord with naive expectation but in contrast to some statements found in the literature.

At this point it is useful to again recall the general philosophy behind such statements: From the Einstein-Straus solution it is clear that local overdensities inhibit cosmic expansion, or at least that part of it which is not due to a cosmological constant. Also, as mentioned, the effect of anisotropies is also to diminish the effect of global expansion [see, e.g., Dominguez and Gaite (2001)]. Hence calculating such an effect in simple models like the improved Newtonian equation discussed in Sec. II.A (backed up by the various justifications discussed in detail) means to overestimate the impact of cosmic expansion in a realistic situation, where the single overdensity (e.g., representing the Sun) is surrounded by more overdense structures (the Solar System environment, the Galaxy, etc.) with less symmetry. If this overestimation gives an already insignificant upper bound for the envisaged effect, we can conclude that it becomes even more insignificant in more realistic models.

Satellite navigation is clearly not the only potential source of interest in the question of how local inhomogeneities affect cosmological expansion. Many predictions concerning cosmological data rely on computations within the framework of the standard homogeneous and isotropic models, without properly estimating the possible effects of local inhomogeneities. Such an estimation would ideally be based on an exact inhomogeneous solution to Einstein's equations, or at least a fully controlled approximation to such a solution. The dynamical and kinematical impact of local inhomogeneities might essentially influence our interpretation of cosmological observations. As an example we mention recent serious efforts to interpret the same data that are usually taken to prove the existence of a positive cosmological constant Λ in a context with realistic inhomogeneities (Buchert, 2000; Räsänen, 2006; Wiltshire, 2007), i.e., taking into account that cosmological parameters are dressed (Buchert and Carfora, 2003). See also Buchert (2008) for a recent review. One might speculate that the measured Λ can eventually be *fully* reduced to the action of inhomogeneities, as suggested in Wiltshire (2007, 2008). For an earlier advance in this direction, see Célérier (2000).

ACKNOWLEDGMENTS

APPENDIX A: NOTATION, CONVENTIONS, AND GENERALITIES

A model for *spacetime* consists of a tuple $(\mathcal{M}, \mathbf{g})$, where M is a four-dimensional manifold and $\mathbf{g}$ a Lorentzian metric whose signature we take to be $(+,-,-,-)$, i.e., we use the "mostly minus" convention. Throughout we denote geometric objects, like tensor fields and covariant derivative operators, by bold-faced letters or words. The unique metric preserving and torsion-free covariant derivative associated with $\mathbf{g}$ will be denoted by ∇ and the covariant derivative in the direction of a vector X by ∇_X. For a smooth tensor field T on M its covariant derivative ∇T defines a linear map, $X \mapsto \nabla_X T$, from the tangent space to the tensor space at each point of $\mathcal{M}$ where T is defined. Since ∇T is again a tensor field [of rank $(p, q+1)$ if the rank of T was (p,q)] we can form $\nabla\nabla T := \nabla(\nabla T)$. Note that $(\nabla\nabla T)(X,Y) = \nabla_X \nabla_Y T - \nabla_{\nabla_X Y} T$. For a scalar function f on $\mathcal{M}$ we have $\nabla f = df$, the ordinary exterior differential, and $\nabla\nabla f = \mathbf{Hess}(f)$, the Hessian of f. The metric $\mathbf{g}$ allows one to uniquely associate to any vector X a linear form $\underline{X} := \mathbf{g}(X, \cdot)$, called the dual (with respect to $\mathbf{g}$) of X. The inverse of this map will be denoted by an overline. The gradient of a function f is then $\mathbf{grad}\, f = \overline{df}$. The metricity of ∇ implies that the latter commutes with the maps $\underline{\;\cdot\;}$ and $\overline{\;\cdot\;}$, e.g., it holds: $\nabla_X \underline{Y} = \underline{\nabla_X Y}$. The scalar product induced by $\mathbf{g}$ on the tensor bundle will be denoted by $\langle \cdot, \cdot \rangle$.

Associated with any two linearly independent vectors X, Y at a point $p \in \mathcal{M}$ is a curvature endomorphism, $R(X,Y)$, of the tangent space at p,

$$R(X,Y)Z = (\nabla\nabla Z)(X,Y) - (\nabla\nabla Z)(Y,X)$$
$$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \qquad (A1)$$

The *Riemann* or *curvature tensor* **Riem** is then defined by

$$\mathbf{Riem}(W,Z,X,Y) := \mathbf{g}(W, R(X,Y)Z). \qquad (A2)$$

It is antisymmetric under the exchange $X \leftrightarrow Y$ or $W \leftrightarrow Z$ and symmetric under the slotwise exchange of pairs $(W,Z) \leftrightarrow (X,Y)$. Moreover, the antisymmetrization over any three slots vanishes (first Bianchi identity). The *Ricci tensor* **Ric** is defined by the trace of the following endomorphism:

$$\mathbf{Ric}(Y,Z) := \mathrm{tr}(X \mapsto R(X,Y)Z), \qquad (A3)$$

which is symmetric under exchange $Y \leftrightarrow Z$. The *scalar curvature* is defined by taking the trace of **Ric**, also called the Ricci scalar, with respect to $\mathbf{g}$ (since **Ric** is not an endomorphism, we need the metric to define its trace)

$$\mathrm{Scal} = \mathrm{tr}_{\mathbf{g}}(\mathbf{Ric}). \qquad (A4)$$

Finally, the *Einstein tensor* is the following combination of **Ric** and Scal:

$$\mathbf{Ein} := \mathbf{Ric} - \tfrac{1}{2}\mathrm{Scal}\, \mathbf{g}. \qquad (A5)$$

Associated to any spacelike or timelike two-dimensional plane Π in the tangent space at $p \in \mathcal{M}$ is the *sectional curvature*. Its geometric interpretation is just that of the ordinary Gaussian curvature at p of the two-dimensional surface in $\mathcal{M}$ that is spanned by the geodesic curves through p tangent to Π. In terms of **Riem** it reads

$$k_{\Pi} := \mathbf{Riem}(X,Y,X,Y)/Q(X,Y), \qquad (A6)$$

where X, Y are any two linear independent vectors in Π and

$$Q(X,Y) := \mathbf{g}(X,X)\mathbf{g}(Y,Y) - \mathbf{g}(X,Y)^2$$
$$= (\mathbf{g} \odot \mathbf{g})(X,Y,X,Y). \qquad (A7)$$

Note that $|Q(X,Y)|$ gives the square of the area of the parallelogram spanned by X and Y which is nonzero iff the considered plane is spacelike or timelike (nondegenerate).

In Eq. (A7) we introduced the product $\odot$, which is called the Kulkarni-Nomizu product. It is a symmetric bilinear map from the space of symmetric (0,2) tensors to the space of (0,4) tensors with the same algebraic symmetries as **Riem**. Its general definition is as follows:

$$(a \odot b)(W,Z,X,Y) := \tfrac{1}{2}[a(W,X)b(Z,Y)$$
$$- a(W,Y)b(Z,X)$$
$$+ b(W,X)a(Z,Y)$$
$$- b(W,Y)a(Z,X)]. \qquad (A8)$$

This can be used to conveniently write down the $\mathbf{g}$-orthogonal decomposition of the curvature tensor into the Ricci and Weyl parts,

$$\mathbf{Riem} = \mathbf{Ricci} + \mathbf{Weyl}. \qquad (A9)$$

In four spacetime dimensions one has

$$\mathbf{Ricci} := \left(\mathbf{Ric} - \frac{1}{6}\mathrm{Scal}\, \mathbf{g}\right) \odot \mathbf{g} \qquad (A10a)$$

$$= \left(\mathbf{Ein} - \frac{1}{3}\mathrm{tr}_{\mathbf{g}}(\mathbf{Ein})\mathbf{g}\right) \odot \mathbf{g}. \qquad (A10b)$$

Inserting this into Eq. (A9) gives the definition of **Weyl**. The definition is such that the Ricci part is $\mathbf{g}$ orthogonal to the Weyl part and that the latter is totally trace free. Hence the Ricci and the Weyl part each contribute ten independent components to the 20 independent components of **Riem**. The Ricci part may be further decom-

posed according to the decomposition of **Ric** into its trace and a trace-free part, but this refinement will not be needed here.

Einstein's equation now express the local determination of the Ricci part of the curvature in terms of the energy-momentum distribution of matter, the latter being encoded in the energy-momentum tensor $\boldsymbol{T}$ of the matter. In units where Newton's constant G and the velocity of light c equal 1,[20] Einstein's equation reads

$$\mathbf{Ein} = 8\pi \boldsymbol{T}. \tag{A11}$$

Here we did not write down explicitly a cosmological term, which can always be thought of as extra contribution to $\boldsymbol{T}$ of the form $\boldsymbol{g}\Lambda/8\pi$. Now, assuming that $\boldsymbol{g}$ satisfies Einstein's equation, the Ricci part of the Riemann tensor is given in terms of $\boldsymbol{T}$ by

$$\mathbf{Ricci} = 8\pi \left[\boldsymbol{T} - \frac{1}{3}\mathrm{tr}_g(\boldsymbol{T})\boldsymbol{g} \right] \odot \boldsymbol{g}. \tag{A12}$$

APPENDIX B: PROOF OF THEOREM 1

In this section we prove Theorem 1, namely, the equivalence, in the spherically symmetric case, of the SSJC with the Darmois junction conditions.

Proof. The proof essentially consists in writing down the induced metric and extrinsic curvature for a (non-null) spherically symmetric hypersurface in a spherically symmetric spacetime. This is most easily done by introducing an adapted orthonormal frame.

We first consider the case where Γ is timelike, hence $\gamma = \pi(\Gamma)$ is a timelike curve in $\mathcal{B}$. The following construction shall be carried out in both spacetimes. One defines $\boldsymbol{v}$ as in the SSJC, hence as the (unique up to a sign) spherically symmetric, unit vector field on Γ orthogonal to $\boldsymbol{n}$. That is $\boldsymbol{v}$, seen as a vector field on $\mathcal{B}$, is tangent to γ. Since $\boldsymbol{n}$ is spacelike, $\boldsymbol{v}$ is timelike. The ambient metric can be then written as

$$\boldsymbol{g} = \underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{v}} - \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}} - R^2 \boldsymbol{g}_{S^2}, \tag{B1}$$

so that the induced metric [compare Appendix C and Eq. (C3)] on Γ is

$$\boldsymbol{g}_\Gamma = \underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{v}} - R^2 \boldsymbol{g}_{S^2}. \tag{B2}$$

In view of Eq. (C3) note that here $\varepsilon(\boldsymbol{n}) = -1$. For the extrinsic curvature (C6), using Eq. (D3) and the fact that $\boldsymbol{v}$ is spherically symmetric and hence tangent to $\mathcal{B}$, one has the decomposition

$$\boldsymbol{K}_\Gamma = -\boldsymbol{g}(\boldsymbol{n}, \nabla_v \boldsymbol{v})\underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{v}} - R dR(\boldsymbol{n})\boldsymbol{g}_{S^2}. \tag{B3}$$

Now, from Eqs. (B2) and (B3) it follows that the DJC, and hence the continuity of $\boldsymbol{g}_\Gamma$ and $\boldsymbol{K}_\Gamma$, are equivalent to the continuity of the following four functions: (a) the arc length of γ, (b) R, (c) $dR(\boldsymbol{n})$, and (d) $\boldsymbol{g}(\boldsymbol{n}, \nabla_v \boldsymbol{v})$.

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[20]Otherwise the factor 8π on the right-hand side of Eq. (A11) should be replaced with $8\pi G/c^4$.

The statement of the theorem will now follow from the following expression of the MS energy (58):

$$E = \frac{R}{2}\{1 + [\boldsymbol{d}R(\boldsymbol{v})]^2 - [\boldsymbol{d}R(\boldsymbol{n})]^2\}. \tag{B4}$$

Simply note that if R is continuous through Γ (recall the definition of this concept below the definition of DJC in Sec. V.A) the same holds for its derivatives tangent to Γ. In particular, $\boldsymbol{d}R(\boldsymbol{v})$ is continuous through Γ and hence we may substitute $\boldsymbol{d}R(\boldsymbol{n})$ by the MS energy in the above list (a)–(d). This completes the proof for timelike Γ.

In the case of spacelike Γ the unit normal $\boldsymbol{n}$ is timelike and $\boldsymbol{v}$ is chosen as the unique (up to a sign) spherically symmetric unit vector field on Γ orthonormal to $\boldsymbol{n}$. Then $\boldsymbol{v}$ is a spacelike "radial" unit vector field orthogonal to the SO(3) orbits. The proof now proceeds analogously to the timelike case. We list the expressions for the ambient metric

$$\boldsymbol{g} = \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}} - \underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{v}} - R^2 \boldsymbol{g}_{S^2},$$

the induced metric

$$\boldsymbol{g}_\Gamma = \underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{v}} + R^2 \boldsymbol{g}_{S^2}, \tag{B5}$$

the extrinsic curvature

$$\boldsymbol{K}_\Gamma = \boldsymbol{g}(\boldsymbol{n}, \nabla_v \boldsymbol{v})\underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{v}} + R dR(\boldsymbol{n})\boldsymbol{g}_{S^2}, \tag{B6}$$

and the MS energy

$$E = \frac{R}{2}\{1 + [\boldsymbol{d}R(\boldsymbol{n})]^2 - [\boldsymbol{d}R(\boldsymbol{v})]^2\}, \tag{B7}$$

and conclude exactly as in the timelike case. ∎

APPENDIX C: SUBMANIFOLDS

In a Lorentzian manifold $(\mathcal{M}, \boldsymbol{g})$ endowed with Levi-Civita connection ∇ consider a smooth submanifold Γ of co-dimension one and normal vector field $\boldsymbol{n}$. We assume Γ to be non-null, that is, either spacelike (then $\boldsymbol{n}$ is timelike) or timelike (then $\boldsymbol{n}$ spacelike). Then Γ inherits from the ambient manifold $\mathcal{M}$ a (nondegenerate) metric and a connection in a natural way. We introduce the orthogonal projectors

$$\boldsymbol{Q}_n := \varepsilon(\boldsymbol{n})\boldsymbol{n} \otimes \underline{\boldsymbol{n}}, \tag{C1a}$$

$$\boldsymbol{P}_n := \mathbf{id} - \boldsymbol{Q}_n, \tag{C1b}$$

where $\varepsilon(\boldsymbol{n})$ denotes the *indicator*, defined for any non-null vector by

$$\varepsilon(X) := \frac{\boldsymbol{g}(X,X)}{|\boldsymbol{g}(X,X)|} = \begin{cases} +1, & \text{if } X \text{ is timelike} \\ -1, & \text{if } X \text{ is spacelike}. \end{cases} \tag{C2}$$

The *induced metric* on Γ (also called first fundamental form) is given by

$$\boldsymbol{g}_\Gamma := -\varepsilon(\boldsymbol{n})\boldsymbol{P}_n\boldsymbol{g}, \tag{C3}$$

where the sign is just in order to get a positive definite metric in the case where Γ is spacelike. Given two vector fields X, Y tangent to Γ, so that $\boldsymbol{Q}_n X = \boldsymbol{Q}_n Y = 0$, one may

decompose the covariant derivative of Y with respect to X into its orthogonal components

$$\nabla_X Y = P_n(\nabla_X Y) + Q_n(\nabla_X Y) = {}^\Gamma\nabla_X Y + K_\Gamma(X,Y)n,$$
(C4)

where

$$ {}^\Gamma\nabla_X Y := P_n(\nabla_X Y) $$
(C5)

is the *induced connection* on Γ and

$$K_\Gamma(X,Y) := \varepsilon(n)g(\nabla_X Y, n) = -\varepsilon(n)g(\nabla_X n, Y)$$
(C6)

is the *extrinsic curvature* of Γ in $\mathcal{M}$ (also called the second fundamental form). The second equality sign in Eq. (C6) is an immediate consequence of the metricity of ∇ and the fact that X and Y are orthogonal to n. With this alternative expression for the extrinsic curvature one has $K_\Gamma = -\varepsilon(n)P_n\nabla\underline{n}$ and hence[21]

$$K_\Gamma = -\varepsilon(n)\mathcal{S}(P_n\nabla\underline{n}).$$
(C7)

Since n is hypersurface orthogonal (by definition) we have $\mathcal{A}(P_n\nabla\underline{n}) = 0$. Hence, the extrinsic curvature is a symmetric $(0,2)$-tensor field. We recall also that the induced connection is the Levi-Civita connection of (Γ, g_Γ), as one may easily check.

The full relations between the curvature of $\mathcal{M}$ and those (intrinsic and extrinsic) of Γ can be found in Giulini (1998). Here we are only interested in the "Einstein part" of the curvature. One gets

$$\mathbf{Ein}(n,n) = \tfrac{1}{2}[-\varepsilon(n)^\Gamma\mathrm{Scal} + (\mathrm{tr}K)^2 - \|K\|^2],$$
(C8a)

$$\mathbf{Ein}(n,P_n\cdot) = -\varepsilon(n)\mathbf{div}_\Gamma(K - (\mathrm{tr}K)g_\Gamma),$$
(C8b)

$$\mathbf{Ein}(P_n\cdot, P_n\cdot) = {}^\Gamma\mathbf{Ein} + \varepsilon(n)\left[\frac{1}{2}((\mathrm{tr}K)^2 + \|K\|^2)g_\Gamma\right.$$

$$\left. - (\mathrm{tr}K)K + \mathbf{L}_n(K - (\mathrm{tr}K)g_\Gamma)\right]. \quad \text{(C8c)}$$

APPENDIX D: SPHERICAL SYMMETRY

We recall that the *isometry group* $\mathrm{Isom}(\mathcal{M},g)$ of a spacetime $(\mathcal{M},g)$ is the subgroup of the diffeomorphism group of $\mathcal{M}$ $\mathrm{Diff}(\mathcal{M})$ which leaves the metric g invariant: $\mathrm{Isom}(\mathcal{M},g) := \{\phi \in \mathrm{Diff}(\mathcal{M}) \,|\, \phi^*g = g\}$.

Definition 3 (Spherical symmetry). A four-dimensional Lorentzian manifold $(\mathcal{M},g)$ is said to be *spherically symmetric* if its isometry group $\mathrm{Isom}(\mathcal{M},g)$ contains a subgroup g with the following two properties: (i) G is isomorphic to $\mathrm{SO}(3)$ and (ii) each orbit of G is spacelike and two dimensional (up to some closed proper subset of fixed points). A tensor field T on a spherically symmetric spacetime is said to be *spherically symmetric* if it is invariant under G, hence if $\phi^*T = T$ for all $\phi \in G$.

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[21]Here and below $\mathcal{S}$ and $\mathcal{A}$ denote the projection operators of full symmetrization and full antisymmetrization, respectively.

From this definition it follows (excluding the case where the orbits of G are diffeomorphic to the two-dimensional real-projective space) that a four-dimensional spherically symmetric Lorentzian manifold $(\mathcal{M},g)$ can, at least locally, be expressed as a warped product $\mathcal{M} = \mathcal{B} \times_R S^2$ between a two-dimensional Lorentzian manifold $(\mathcal{B}, g_\mathcal{B})$, called the "base," and the standard unit two-sphere (S^2, g_{S^2}), called the "fiber" [see O'Neill (1983) and Straumann (2004)]. This means that, at least locally, the manifold is a product

$$\mathcal{M} \overset{\mathrm{loc}}{=} \mathcal{B} \times S^2$$
(D1)

and the metric is given by

$$g = \pi^*(g_\mathcal{B}) - (R \circ \pi)^2\sigma^*(g_{S^2}).$$
(D2)

Here π and σ are the projections of $\mathcal{B} \times S^2$ onto $\mathcal{B}$ and S^2, respectively, and π^*, σ^* their pull-backs. The warping function R is nothing but the *areal radius*, since, for a point $p \in \mathcal{B}$, the area of the fiber $p \times S^2$ is just $4\pi R(p)^2$.

In this situation, a vector field X on $\mathcal{M}$ at some point $(p,q) \in \mathcal{B} \times S^2$ has then a unique decomposition $X = \tan_\mathcal{B} X + \tan_{S^2} X$ in a component tangent to the "leaves" $\mathcal{B} \times q = \sigma^{-1}(q)$ and a component tangent to the "fibers" $p \times S^2 = \pi^{-1}(p)$. Arbitrary tensor fields on $\mathcal{B}$ and on S^2 can be lifted to tensor fields on $\mathcal{M}$ in the standard way. For covariant tensor fields (and hence, in particular, for functions) this is achieved via the pull-pack of the respective projection: as an example, look at Eq. (D2). For contravariant tensor fields it suffices to consider the special case of vector fields. Let, for instance, X be a vector in the tangent space of $\mathcal{B}$ at p. Then the lift $\tilde{X}$ at (p,q) of X is defined as the unique vector in the tangent space of $\mathcal{M}$ at (p,q) with $\pi_*(\tilde{X}) = X$ and $\sigma_*(\tilde{X}) = 0$. Since this assignment is smooth, one gets the lifting of a vector field via the pointwise lifting just described. In this work, we mainly omit lifts and projections and not explicitly distinguish between original and lifted quantities. For example, when referring to a vector "tangent to $\mathcal{B}$" we refer to a vector in the tangent space of $\mathcal{B}$ or to the lift thereof in the tangent space of $\mathcal{M}$.

If X is spherically symmetric, then the component tangent to the fibers must vanish: $\tan_{S^2} X = 0$. Similarly, a spherically symmetric one-form θ on $\mathcal{M}$ must necessarily be tangent to B (i.e., normal to S^2) and thus it can be written as $\theta = \pi^*(\theta_\mathcal{B})$, where $\theta_\mathcal{B}$ is a one-form on $\mathcal{B}$. Finally, a spherical symmetric function is simply the lift of a function on $\mathcal{B}$.

1. Connection and curvature decomposition

In the following we discuss relations which express the curvature of $\mathcal{M}$ in terms of the warping function R and the curvatures of the base $\mathcal{B}$ and the fiber S^2. We start out from the relations between the Levi-Civita connection ∇ of $(\mathcal{M},g)$ and the Levi-Civita connections of the base and the fiber, denoted by ${}^\mathcal{B}\nabla$ and ${}^{S^2}\nabla$, respectively. These relations can be derived, by means of the Koszul

formula [see, e.g., O'Neill (1983), Proposition 7.35]. Let in the following X, Y, Z be vector fields tangent to $\mathcal{B}$ and U, V, W tangent to S^2. Suppressing lifts and projections, we have

$$\nabla_X Y = {}^{\mathcal{B}}\nabla_X Y, \tag{D3a}$$

$$\nabla_X V = \nabla_V X = R^{-1} X(R) V, \tag{D3b}$$

$$\tan_{S^2} \nabla_V W = {}^{S^2}\nabla_V W, \tag{D3c}$$

$$\tan_{\mathcal{B}} \nabla_V W = - g(V,W) R^{-1} \nabla R. \tag{D3d}$$

Note that, for a function f on $\mathcal{B}$, the lift of the gradient is equal to the gradient of the lifted function, that is (suppressing the lifts): $\mathbf{grad}\, f = {}^{\mathcal{B}}\mathbf{grad}\, f$. For brevity, we write just $\mathbf{grad}\, f$ for it. Take care that for the Hessian and the Laplacian this is in general not true [see Eqs. (D13) and (D14)]. Therefore we write explicitly the superscripts $\mathcal{B}$ in ${}^{\mathcal{B}}\mathbf{Hess}\, f$ and ${}^{\mathcal{B}}\Delta f$ to denote the Hessian and Laplacian of f on $\mathcal{B}$, respectively, or the lifts thereof.

By means of Eq. (D3) one can now compute the expressions for the Riemann tensor. As the sectional curvature of S^2 is obviously constant and equal to 1, the Riemann tensor, the Ricci tensor, and the Ricci scalar of S^2 are simply given by ${}^{S^2}\mathbf{Riem} = g_{S^2} \odot g_{S^2}$, ${}^{S^2}\mathbf{Ric} = g_{S^2}$, and ${}^{S^2}\mathrm{Scal} = 2$, respectively. Here we made again use of the Kulkarni-Nomizu product (A8). Moreover, since the basis manifold $\mathcal{B}$ is two-dimensional, one can express its curvature tensors in terms of the scalar curvature. The expression for the Riemann tensor, Ricci tensor, and Ricci scalar of a spherically symmetric Lorentzian manifold (D1) and (D2) are, respectively,

$$\mathbf{Riem} = \frac{{}^{\mathcal{B}}\mathrm{Scal}}{2} g_{\mathcal{B}} \odot g_{\mathcal{B}} - \frac{1}{R^2}(1 + \langle dR, dR \rangle) R^2 g_{S^2} \odot R^2 g_{S^2}$$
$$+ 2R g_{S^2} \odot {}^{\mathcal{B}}\mathbf{Hess}\, R, \tag{D4}$$

$$\mathbf{Ric} = \frac{{}^{\mathcal{B}}\mathrm{Scal}}{2} g_{\mathcal{B}} - \frac{2}{R} {}^{\mathcal{B}}\mathbf{Hess}\, R + (1 + \langle dR, dR \rangle$$
$$+ R {}^{\mathcal{B}}\Delta R) g_{S^2}, \tag{D5}$$

and

$$\mathrm{Scal} = {}^{\mathcal{B}}\mathrm{Scal} - \frac{2}{R^2}(1 + \langle dR, dR \rangle) - \frac{4}{R} {}^{\mathcal{B}}\Delta R. \tag{D6}$$

Hence, for the Einstein tensor we have

$$\mathbf{Ein} = \left(\frac{1}{R^2}(1 + \langle dR, dR \rangle) + \frac{2}{R} {}^{\mathcal{B}}\Delta R \right) g_{\mathcal{B}} - \frac{2}{R} {}^{\mathcal{B}}\mathbf{Hess}\, R$$
$$+ \left(\frac{{}^{\mathcal{B}}\mathrm{Scal}}{2} - \frac{1}{R} {}^{\mathcal{B}}\Delta R \right) R^2 g_{S^2} \tag{D7}$$

and for the Weyl tensor, using Eqs. (A9) and (A10a), the simple expression

$$\mathbf{Weyl} = w(g_{\mathcal{B}} \odot g_{\mathcal{B}} + g_{\mathcal{B}} \odot R^2 g_{S^2} + R^2 g_{S^2} \odot R^2 g_{S^2}). \tag{D8a}$$

Here we put

$$w := \frac{1}{6}\left({}^{\mathcal{B}}\mathrm{Scal} - \frac{2}{R^2}(1 + \langle dR, dR \rangle) + \frac{2}{R} {}^{\mathcal{B}}\Delta R \right) \tag{D8b}$$

$$= \frac{1}{6}\mathrm{Scal} + \frac{1}{R} {}^{\mathcal{B}}\Delta R. \tag{D8c}$$

In the derivation of Eq. (D8a)–(D8c) we made use of the formula $h_{\mathcal{B}} \odot g_{\mathcal{B}} = (1/2)(\mathrm{tr}_{g_{\mathcal{B}}} h_{\mathcal{B}}) g_{\mathcal{B}} \odot g_{\mathcal{B}}$, valid[22] for any symmetric bilinear form $h_{\mathcal{B}}$ on $\mathcal{B}$, in order to express the only term involving ${}^{\mathcal{B}}\mathbf{Hess}\, R$ in terms of the Laplacian of R. Note that from Eq. (D8a) it is immediate that the Weyl tensor of a spherically symmetric spacetime has only one independent component, as it must be the case due to it being of Petrov-type D.

Comparing Eq. (D4) with the definition of sectional curvature (A6) one can immediately read off that the sectional curvature K of the plane tangential to the (two dimensional) SO(3) orbits at a given point is

$$K = - \frac{1}{R^2}(1 + \langle dR, dR \rangle), \tag{D9}$$

and hence the MS energy, defined as (minus one-half) K times the third power of the areal radius, is given by Eq. (58). Using the MS energy (58), we can write the Einstein tensor (D7) as

$$\mathbf{Ein} = \frac{2}{R}\left({}^{\mathcal{B}}\Delta R + \frac{E}{R^2} \right) g_{\mathcal{B}} - \frac{2}{R} {}^{\mathcal{B}}\mathbf{Hess}\, R$$
$$+ \left(\frac{{}^{\mathcal{B}}\mathrm{Scal}}{2} - \frac{1}{R} {}^{\mathcal{B}}\Delta R \right) R^2 g_{S^2}. \tag{D10}$$

We conclude giving the decomposition for the divergence of a spherically symmetric vector field (that is a vector field X tangent to $\mathcal{B}$) and for the Hessian and Laplacian of a spherically symmetric function (that is a function f on $\mathcal{B}$). First, from Eqs. (D3a) and (D3b), we obtain the following decomposition for the covariant derivative of X [expressed as a (0,2) tensor]:

$$\nabla \underline{X} = {}^{\mathcal{B}}\nabla \underline{X} - X(R) R g_{S^2}. \tag{D11}$$

Note that the mixed term $(\mathcal{B} - S^2)$ vanishes—as it should due to spherical symmetry. Taking the trace of Eq. (D11) one obtains the following expression for the divergence:

$$\mathbf{div}\, X = \mathbf{div}_{\mathcal{B}}\, X + \frac{2}{R}X(R) = \frac{1}{R^2}\mathbf{div}_{\mathcal{B}}(R^2 X). \tag{D12}$$

The decompositions for the Hessian and Laplacian of a function f on $\mathcal{B}$ follow from inserting $X = \mathbf{grad}\, f$ in the above formulas. One gets

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[22]To prove this note that the only independent component of this formula is the (e_0, e_1, e_0, e_1) one, where $\{e_\mu\}$ is an adapted orthonormal basis of $(\mathcal{M}, g)$ such that e_0, e_1 are tangent to $\mathcal{B}$ and e_2, e_3 are tangent to S^2. Then, the equality follows immediately using the definition (A8) of the Kulkarni-Nomizu product.

$$\mathbf{Hess}\, f = {}^{B}\mathbf{Hess}\, f - \mathbf{g}_{B}(\mathbf{grad}\, f, \mathbf{grad}\, R)R\mathbf{g}_{S^2} \qquad \text{(D13)}$$

and

$$\Delta f = {}^{B}\Delta f + 2\mathbf{g}_{B}(\mathbf{grad}\, f, \mathbf{grad}\, R)/R, \qquad \text{(D14)}$$

respectively.

2. Einstein equation in case of spherical symmetry

A general spherically symmetric matter energy-momentum tensor has the form

$$\mathbf{T} = \mathbf{T}_{B} + pR^{2}\mathbf{g}_{S^2}, \qquad \text{(D15)}$$

where p is the spherical part of the pressure. Hence, using the decomposition (D10) of the Einstein's tensor found in Appendix D, the Einstein equation takes the form

$$\frac{2}{R}\left(\frac{E}{R^{2}} + {}^{B}\Delta R\right)\mathbf{g}_{B} - \frac{2}{R}\,{}^{B}\mathbf{Hess}\, R = 8\pi\mathbf{T}_{B}, \qquad \text{(D16a)}$$

$$\frac{{}^{B}\mathrm{Scal}}{2} - \frac{1}{R}\,{}^{B}\Delta R = 8\pi p. \qquad \text{(D16b)}$$

Using the trace of the first equation,

$$\frac{1}{R}\left({}^{B}\Delta R + \frac{2E}{R^{2}}\right) = 4\pi\mathrm{tr}\mathbf{T}_{B}, \qquad \text{(D17)}$$

to eliminate ${}^{B}\Delta R$, one can write Eq. (D16) in the equivalent form

$$\frac{1}{R}\left(\frac{E}{R^{2}}\mathbf{g}_{B} + {}^{B}\mathbf{Hess}\, R\right) = -4\pi\star\mathbf{T}_{B}\star, \qquad \text{(D18a)}$$

$$\frac{{}^{B}\mathrm{Scal}}{2} + \frac{2E}{R^{2}} = 4\pi(\mathrm{tr}\mathbf{T}_{B} + 2p). \qquad \text{(D18b)}$$

Here, and in the following, $\star$ denotes the Hodge-duality map for $(B, \mathbf{g}_{B})$ [for the definition, see Straumann (2004)]. In the first equation we used the identity $\star\boldsymbol{\tau}\star = \boldsymbol{\tau} - \mathrm{tr}(\boldsymbol{\tau})\mathbf{g}$, which is valid for any bilinear form $\boldsymbol{\tau}$ on B, where the first (second) star acts on the first (second) slot of $\boldsymbol{\tau}$.

Finally, the integrability condition $\mathbf{div}\mathbf{T} = 0$ for the energy-momentum tensor (15) reads

$$\mathbf{div}_{B}(R^{2}\mathbf{T}_{B}) + p\mathbf{d}(R^{2}) = 0. \qquad \text{(D19)}$$

3. Misner-Sharp energy

We now turn to the MS energy and its properties. We first show that it is the charge of a conserved current. The treatment presented here follows mainly Hayward (1996). In a spherically symmetric spacetime one defines the *Kodama vector field* (Kodama, 1980) as the (unique up to a sign) spherically symmetric vector field orthogonal to, and of the same norm as, the gradient of R; hence we put

$$\underline{\mathbf{k}} := \star dR. \qquad \text{(D20)}$$

With this sign choice $\mathbf{k}$ is future pointing if the gradient of R is spacelike. The orthogonality between $\mathbf{k}$ and the gradient of R is expressed by

$$\mathbf{k}(R) = 0, \qquad \text{(D21)}$$

which means that the integral curves of $\mathbf{k}$ stay at constant areal radius.

An immediate but important property of the Kodama vector field is that it is conserved,

$$\mathbf{div}\,\mathbf{k} = 0. \qquad \text{(D22)}$$

Indeed, using Eqs. (D12) and (D21), one has $\mathbf{div}\mathbf{k} = \mathbf{div}_{B}\mathbf{k} = \delta\underline{\mathbf{k}} = -\star d\star\star dR \equiv 0$.

Now, a key point for the study of spherically symmetric spacetimes is the following equation relating the MS energy with the matter's energy-momentum tensor:

$$dE = 4\pi R^{2}\star\underline{\mathbf{j}}, \qquad \text{(D23)}$$

where $\mathbf{j}$ is the so-called *Kodama current* (tangent to the base manifold B) defined by

$$\underline{\mathbf{j}} := \mathbf{T}(\mathbf{k}, \cdot). \qquad \text{(D24)}$$

Equation (D23) follows from Einstein's equation; more precisely, it is equivalent to its B part [that is Eq. (D18a)] fed with ∇R. To see this, just compute the differential of Eq. (58) as follows: $dE = (E/R)dR + (R/2)d\langle dR, dR\rangle = (E/R)dR + R\,{}^{B}\mathbf{Hess}R \cdot dR = R[(E/R^{2})\mathbf{g}_{B} + {}^{B}\mathbf{Hess}R] \cdot dR$, where the dot denotes the contraction of the last slot of the tensor on the left with the first slot of the tensor on the right of the dot. In the second step we use that $d(\langle dR, dR\rangle)(X) = X(\mathbf{g}(\nabla R, \nabla R)) = 2\mathbf{g}_{B}(\nabla_{X}\nabla R, \nabla R) = 2\,{}^{B}\mathbf{Hess}R(\nabla R, X)$ for any X tangent to B. Then, inserting Eq. (D18a) and using that $\star$ is skew adjoint on one-forms, one gets $dE = -4\pi R^{2}\star\mathbf{T}_{B}\star\cdot dR = 4\pi R^{2}\star\mathbf{T}_{B}\cdot\star dR$ and hence, using the definitions (20) and (24) together with the symmetry of $\mathbf{T}$, one arrives at Eq. (23).

From Eq. (23) it is clear that

$$\mathbf{j}(E) = 0, \qquad \text{(D25)}$$

which means that the vector field $\mathbf{j}$ is tangent to the curves in B (hypersurfaces in M) of constant MS energy. Moreover, Eq. (23) implies that $\mathbf{j}$ is also conserved

$$\mathbf{div}\,\mathbf{j} = 0, \qquad \text{(D26)}$$

where the divergence is here taken on the spacetime $(M, \mathbf{g})$. To see this, just compute the divergence of $\mathbf{j}$ with Eq. (D12) and using the Hodge-dual version of Eq. (D23): $\mathbf{div}\mathbf{j} = R^{-2}\mathbf{div}_{B}(R^{2}\mathbf{j}) = R^{-2}\delta(R^{2}\underline{\mathbf{j}}) = (4\pi R^{2})^{-1}\delta\star dE \equiv 0$.

Following Hayward (1996), we can now show that the charges corresponding to the conserved currents $\mathbf{j}$ and $\mathbf{k}$ are, respectively, the MS energy and the areal volume. Let Σ be some spatial three-dimensional hypersurface which, because of spherical symmetry, decomposes as $\Sigma = \sigma \times S^{2}$, where σ is some spatial curve in B. Recall that the charge related to a conserved current X is given by $Q_{X}(\Sigma) := \int_{\Sigma}i_{X}\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is the volume form on M, and,

because of spherical symmetry, the latter decomposes as $\boldsymbol{\mu} = \boldsymbol{\mu}_\mathcal{B} \wedge R^2 \boldsymbol{\mu}_{S^2}$, where $\boldsymbol{\mu}_\mathcal{B}$ and $\boldsymbol{\mu}_{S^2}$ are the volume forms on $\mathcal{B}$ and on the unit two-sphere, respectively. After integration of the spherical part and since $i_{\boldsymbol{j}} \boldsymbol{\mu}_\mathcal{B} = \star \boldsymbol{j}$, using Eq. (D23) one gets

$$Q_j(\Sigma) = \int_\sigma d\boldsymbol{E}, \tag{D27}$$

which means that the charge of $\boldsymbol{j}$ is the MS energy. This justifies the interpretation of the MS energy as a quantity associated to the interior of the considered sphere of symmetry. In fact, due to Eq. (D26), the charge does not depend how one choose the spatial slice to define the interior. Similarly, since $i_{\boldsymbol{k}} \boldsymbol{\mu}_\mathcal{B} = \star \boldsymbol{k} = d R$, the charge to $\boldsymbol{k}$ is simply $\int_\sigma 4\pi R^2 d R$ and hence

$$Q_k(\Sigma) = \int_\sigma d\left(\frac{4\pi}{3} R^3\right), \tag{D28}$$

which says that the charge of $\boldsymbol{k}$ is the flat-space volume computed with the areal radius.

Incidentally, the Kodama vector can be used to give an elegant proof of Birkhoff's theorem, which states that spherically symmetric solutions of Einstein's equations are, in fact, static. Indeed, by direct computation one shows that in vacuum $\boldsymbol{k}$ is Killing and, because of spherical symmetry, it is clearly also hypersurface orthogonal.

Next we turn to the relation between the MS energy and the Hawking quasilocal mass (Hawking, 1968). The latter is a quantity associated to a spatial two-sphere S in an arbitrary spacetime. It is defined by

$$M_H(S) := \sqrt{\frac{\text{Area}(S)}{16\pi}}\left(1 + \frac{1}{2\pi}\int_S \theta^+ \theta^- \boldsymbol{\mu}_S\right). \tag{D29}$$

Here $\theta^\pm := \text{tr}_{S^2}(\boldsymbol{\nabla} \boldsymbol{l}^\pm)/2$ are, respectively, the expansions of the outgoing and ingoing future-pointing null vector fields $\boldsymbol{l}^\pm$ normal to S, the latter being partially normalized such that $g(\boldsymbol{l}^+, \boldsymbol{l}^-) = 1$ (there remains the freedom to rescale $\boldsymbol{l}^\pm \to \alpha^{\pm 1} \boldsymbol{l}^\pm$, where α is a positive real-valued function). In the special case of spherical symmetry we take S to be an orbit of the rotation group. Then we have $\text{Area}(S) = 4\pi R^2$. It is also obvious that the metric of the base $\mathcal{B}$, evaluated on S, can be written in the form

$$g_\mathcal{B} = \underline{\boldsymbol{l}}^+ \otimes \underline{\boldsymbol{l}}^- + \underline{\boldsymbol{l}}^- \otimes \underline{\boldsymbol{l}}^+. \tag{D30}$$

Now, for $\boldsymbol{V}$ tangent to S, Eq. (D3b) gives $\boldsymbol{\nabla}_{\boldsymbol{V}} \boldsymbol{l}^\pm = R^{-1} \boldsymbol{l}^\pm(R) \boldsymbol{V}$ so that $\theta^\pm = R^{-1} \boldsymbol{l}^\pm(R)$. Hence we have

$$2\theta^+ \theta^- = 2R^{-2} d R(\boldsymbol{l}^+) d R(\boldsymbol{l}^-) = \langle d R, d R \rangle / R^2, \tag{D31}$$

where we used Eq. (D30), or rather its contravariant version, in the last step. Equation (58) now establishes the equality between the MS energy at p and the Hawking quasilocal mass of S, where p is any point on S,

$$E(p) = M_H(S). \tag{D32}$$

As is the case for the Hawking quasilocal mass, the MS energy can be naturally decomposed into a Ricci and a Weyl part,

$$E = E_R + E_W, \tag{D33}$$

where

$$E_R := -\tfrac{1}{2} R^3 K_R, \tag{D34a}$$

$$E_W := -\tfrac{1}{2} R^3 K_W. \tag{D34b}$$

Here K_R and K_W denote, respectively, the Ricci and the Weyl parts of the sectional curvature of the plane tangential to the SO(3) orbits. These are obtained inserting the decomposition of the Riemann tensor (A9) in the definition of the sectional curvature (A6). The Ricci part of the MS energy is determined by the local matter distribution via Einstein's equation: Using Eqs. (D15) and (D2) for an arbitrary spherically symmetric energy-momentum tensor and, respectively, metric in Eq. (A12) one gets

$$E_R = \frac{4\pi}{3} R^3 (\text{tr} \boldsymbol{T}_\mathcal{B} + p). \tag{D35}$$

For the Weyl part of the MS energy we have, in view of Eq. (D8a), that

$$E_W = -\frac{1}{2} R^3 w, \tag{D36}$$

where w is given by Eq. (D8b) or (D8c). Hence, in particular, the Weyl tensor vanishes iff E_W does. Since the square of the Weyl tensor is $\langle \boldsymbol{Weyl}, \boldsymbol{Weyl} \rangle \equiv W_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta} = 12w^2$, with Eq. (36) we obtain

$$\langle \boldsymbol{Weyl}, \boldsymbol{Weyl} \rangle = 48 E_W^2 / R^6. \tag{D37}$$

From this one sees that, in a spherically symmetric spacetime, the nonvanishing of E_W (that is the nonvanishing of the Weyl tensor) for $R \to 0$ implies a curvature singularity at $R = 0$.

To gain a better physical understanding of the Weyl part of the MS energy we take a look at the equation of geodesic deviation (sometime called Jacobi equation). Let $\boldsymbol{\mu}$ be a geodesic observer field and s the spatial $[g(\boldsymbol{u}, \boldsymbol{s}) = 0]$ separation vector between two nearby integral curves of $\boldsymbol{u}$. Then the equation of geodesic deviation [see, e.g., Straumann (2004)] is $\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{s} = \boldsymbol{R}(\boldsymbol{u}, \boldsymbol{s}) \boldsymbol{u} = \boldsymbol{B}_{\boldsymbol{u}}(\boldsymbol{s}) + \boldsymbol{C}_{\boldsymbol{u}}(\boldsymbol{s})$. In the last step we decomposed the endomorphism on the r.h.s. of the geodesic deviation equation in its Ricci and Weyl parts, denoted here by $\boldsymbol{B}_{\boldsymbol{u}}$ and $\boldsymbol{C}_{\boldsymbol{u}}$, respectively. For an arbitrary spherically symmetric spacetime the latter is given by [see Eqs. (8) and (36)]

$$\boldsymbol{C}_{\boldsymbol{u}} = \frac{2 E_W}{R^3} \boldsymbol{P}_{\boldsymbol{u}}^\mathcal{B} - \frac{E_W}{R^3} \boldsymbol{P}_{\boldsymbol{u}}^{S^2}, \tag{D38}$$

where $\boldsymbol{P}_{\boldsymbol{u}}^\mathcal{B}$ and $\boldsymbol{P}_{\boldsymbol{u}}^{S^2}$ are, respectively, the $\mathcal{B}$ and S^2 parts of the projector $\boldsymbol{P}_{\boldsymbol{u}}$. Recall that $\boldsymbol{P}_{\boldsymbol{u}}$ projects onto the subspace of the tangential space orthogonal to $\boldsymbol{u}$ [see Eq. (125b)]. Equation (D38) is exactly the same expression one gets in Newtonian gravity—provided one identifies E_W with the mass of the central object. The spatial endomorphism $\boldsymbol{C}_{\boldsymbol{u}}$ describes the familiar volume-preserving tidal deformation which produces an expan-

sion in radial direction and a contraction in the orthogonal directions tangential to the SO(3) orbits.

Concerning the Ricci part $\boldsymbol{B_u}$, in the case where $\boldsymbol{u}$ is the velocity field of dust,[23] making use of Einstein's equation [see Eq. (A12)] we have

$$\boldsymbol{B_u} = -(4\pi/3)\varrho\boldsymbol{P_u}. \tag{D39}$$

This says that the local effect of matter (here given by dust) is an isotropic contraction.

4. Spherically symmetric perfect fluids

We specialize now to a perfect fluid, which is described by a four-velocity vector field $\boldsymbol{u}$, density ϱ, and pressure p. In case of spherical symmetry $\boldsymbol{u}$ is tangent to the basis manifold and the matter energy-momentum tensor (79) decomposes as

$$\boldsymbol{T} = \varrho\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + p(\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} - \boldsymbol{g_B}) + pR^2\boldsymbol{g}_{S^2}, \tag{D40}$$

from which one can read off the part tangent to $\mathcal{B}$,

$$\boldsymbol{T}_\mathcal{B} = \varrho\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + p(\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} - \boldsymbol{g_B}). \tag{D41}$$

Usually, the description is to be completed with the specification of an equation of state. We will not assume any equation of state yet, since in some cases (e.g., McVittie spacetime) this happens to be determined by Einstein's equation.

Inserting Eq. (41) into Eq. (35) we get for the Ricci part of the MS energy the simple expression

$$E_R = (4\pi/3)R^3\varrho. \tag{D42}$$

Also Eq. (23) for the differential of the MS energy simplifies in case of a perfect fluid. Using Eq. (41) the Kodama current (as one-form) becomes

$$\underline{\boldsymbol{j}} = (\varrho + p)\boldsymbol{g(k,u)}\underline{\boldsymbol{u}} - p\underline{\boldsymbol{k}}. \tag{D43}$$

It is useful to introduce an adapted orthonormal basis $\{\boldsymbol{u}, \boldsymbol{e}\}$ tangent to the basis manifold, where $\boldsymbol{u}$ is the velocity vector field of the fluid and $\boldsymbol{e}$ is chosen to point in direction of increasing areal radius. Because of our choice of orientation we have $\underline{\boldsymbol{e}} = \star\underline{\boldsymbol{u}}$ (the volume form on $\mathcal{B}$ is simply $\boldsymbol{\mu}_\mathcal{B} = \underline{\boldsymbol{u}} \wedge \underline{\boldsymbol{e}}$). Using this expression for $\boldsymbol{e}$ and the definition of the Kodama vector field (20) we have

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$$\boldsymbol{g(k,u)} = \langle\underline{\boldsymbol{k}},\underline{\boldsymbol{u}}\rangle = \langle\star dR,\underline{\boldsymbol{u}}\rangle = -\langle dR,\star\underline{\boldsymbol{u}}\rangle = -\langle dR,\underline{\boldsymbol{e}}\rangle = -dR(\boldsymbol{e})$$

and, hence, the Hodge star of the Kodama current becomes

$$\star\underline{\boldsymbol{j}} = -pdR(\boldsymbol{u})\underline{\boldsymbol{u}} - \varrho dR(\boldsymbol{e})\underline{\boldsymbol{e}}, \tag{D44}$$

which, inserted into Eq. (23), gives the following expression for the differential of the MS energy for a perfect fluid:

$$dE = -4\pi R^2[pdR(\boldsymbol{u})\underline{\boldsymbol{u}} + \varrho dR(\boldsymbol{e})\underline{\boldsymbol{e}}]. \tag{D45}$$

Hence, the variation of the MS energy along $\boldsymbol{u}$ and $\boldsymbol{e}$ is, respectively,

$$dE(\boldsymbol{u}) = -4\pi R^2 pdR(\boldsymbol{u}), \tag{D46a}$$

$$dE(\boldsymbol{e}) = +4\pi R^2\varrho dR(\boldsymbol{e}). \tag{D46b}$$

These expressions have a good physical interpretation: Since the matter moves along $\boldsymbol{u}$, Eq. (D46a) expresses the fact that the energy can only increase (decrease) if the motion along $\boldsymbol{u}$ does (releases) work against (with) the action of the pressure. Equation (D46b) expresses the almost obvious increase (decrease) of gravitational mass with increase (decrease) of volume in the rest system of the matter. We said "almost" because $4\pi R^2 dR(\boldsymbol{e})$ is not quite the increment of proper volume. The difference accounts for the fact that kinetic and gravitational binding energy are themselves gravitationally active. To see that this is indeed what Eq. (D46b) implies, let p be some point in spacetime and S_p the two-sphere of spherical symmetry through p. Assume S_p to have a regular interior, that is, that S_p bounds a three-ball B_p in the hypersurface Σ orthogonal to $\boldsymbol{u}$. Except for the origin of B_p, we can write $B_p = \sigma \times S^2$, where σ is a spacelike curve in $\mathcal{B}$ orthogonal to $\boldsymbol{u}$, going from the center of symmetry to $\pi(p)$. Using $E = (R/2)\{1 + [dR(\boldsymbol{u})]^2 - [dR(\boldsymbol{e})]^2\}$ for the MS energy to eliminate $dR(\boldsymbol{e})$ in Eq. (D46b), integrating the latter over σ, and reexpressing the result as a volume integral, one gets

$$E(p) = \int_{B_p} \varrho\left(1 + [dR(\boldsymbol{u})]^2 - \frac{2E}{R}\right)^{1/2}\boldsymbol{\mu}_\Sigma. \tag{D47}$$

One sees that the MS energy contains the contribution from the proper mass contained in the ball B_p,

$$M(p) = \int_{B_p} \varrho\boldsymbol{\mu}_\Sigma, \tag{D48}$$

as well as contributions from the "kinetic" and "potential" energy (Misner and Sharp 1964; Hayward, 1996). In a Newtonian approximation, that is for small "velocity" $dR(\boldsymbol{u})$ and weak field (small E/R) one can expand the square root in Eq. (D47) and gets, in leading order,

$$E(p) \approx \int_{B_p} \left(\varrho + \frac{1}{2}\varrho[dR(\boldsymbol{u})]^2 - \frac{\varrho M}{R}\right)\boldsymbol{\mu}_\Sigma. \tag{D49}$$

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[23]Recall that dust particles moves along geodesics by the Euler equation.

In this approximation the MS energy is therefore just the sum of the proper mass and the Newtonian kinetic and potential energies contained in the ball B_p. This provides a sound justification for the interpretation of the MS energy as the active gravitational energy.

At this point we can also compute the differentials of the two parts (D34) of the MS energy separately. The differential of the Ricci part follows directly from Eq. (D42),

$$dE_R = 4\pi R^2\left(\varrho\, dR + \frac{1}{3}R\, d\varrho\right) \tag{D50}$$

and the differential of the Weyl part is the difference of this with Eq. (D45),

$$dE_W = -4\pi R^2\left[(\varrho + p)dR(\boldsymbol{u})\underline{\boldsymbol{u}} + \frac{1}{3}R\, d\varrho\right]. \tag{D51}$$

Its components in the directions $\boldsymbol{u}$ and $\boldsymbol{e}$ are then

$$dE_W(\boldsymbol{u}) = -4\pi R^2(\varrho + p)dR(\boldsymbol{u}) - \frac{4\pi}{3}R^3 d\varrho(\boldsymbol{u}), \tag{D52a}$$

$$dE_W(\boldsymbol{e}) = -\frac{4\pi}{3}R^3 d\varrho(\boldsymbol{e}). \tag{D52b}$$

It is now instructive to express the variation along $\boldsymbol{u}$ of the Ricci and Weyl parts of the MS energy in terms of the kinematical properties of the fluid velocity $\boldsymbol{u}$. Recall that, because of spherical symmetry, the rotation tensor vanishes identically and the shear tensor has only one independent component. The kinematical quantities reduces thus to two scalars: the expansion

$$\theta := \operatorname{\mathbf{div}} \boldsymbol{u} \tag{D53}$$

and the shear scalar

$$\sigma := dR(\boldsymbol{u})/R - \tfrac{1}{3}\theta. \tag{D54}$$

The shear tensor is then given by the trace-free endomorphism $\boldsymbol{\sigma} = \sigma(\boldsymbol{Q}_{S^2} - 2\boldsymbol{Q}_e)$, where $\boldsymbol{Q}_{S^2}$ and $\boldsymbol{Q}_e$ denote, respectively, the projections onto the two-dimensional subspace of $T(\mathcal{M})$ tangential to the two-sphere and onto the one-dimensional space parallel to $\boldsymbol{e}$ [for the latter see Eq. (C1a)]. We recall that the divergence-freeness of the energy-momentum tensor (D40) is equivalent to

$$(\varrho + p)\theta = -d\varrho(\boldsymbol{u}), \tag{D55a}$$

$$(\varrho + p)b = -dp(\boldsymbol{e}), \tag{D55b}$$

where $b := -\boldsymbol{g}(\nabla_{\boldsymbol{u}}\boldsymbol{u}, \boldsymbol{e})$ is the acceleration (scalar) of $\boldsymbol{u}$ in positive radial direction (the minus sign in the latter formula is because the metric is negative definite in spatial directions).

Now, using Eqs. (D55a) and (D54) we get

$$dE_R(\boldsymbol{u}) = \frac{4\pi}{3}R^3(3\varrho\sigma - p\theta), \tag{D56a}$$

$$dE_W(\boldsymbol{u}) = -\frac{4\pi}{3}R^3(\varrho + p)3\sigma. \tag{D56b}$$

With the equations just derived we can now say when the MS energy, and its Ricci and Weyl parts, are temporally or spatially constant. Here by temporally (spatially) constant we mean that the variation in direction of $\boldsymbol{u}$ ($\boldsymbol{e}$) vanishes. We collect the results in the following:

Theorem 4. Consider a spherically symmetric fluid with $\varrho + p \neq 0$ and restrict to the region where dR is spacelike. Then for the MS energy E and its Ricci and Weyl parts E_R and E_W the following statements hold true: (i) E is temporally constant iff $p = 0$ or $\boldsymbol{dR}(\boldsymbol{u}) = 0$; (ii) E is spatially constant iff $\varrho = 0$; (iii) E_W is temporally constant iff $\sigma = 0$; (iv) E_W is spatially constant iff ϱ is spatially constant; (v) E_R is temporally (spatially) constant iff $R^3\varrho$ is temporally (spatially) constant.

The proof is a straightforward application of the formulas just derived above. Note that the assumption $\varrho + p \neq 0$ is needed only for (iii). The assumption that $\boldsymbol{dR}$ is spacelike is needed only for (ii): If dR is spacelike, then for any spacelike spherically symmetric vector $\boldsymbol{e}$ (hence tangent to the basis manifold $\mathcal{B}$) it holds $\boldsymbol{dR}(\boldsymbol{e}) \neq 0$, since in a two-dimensional Lorentzian manifold any two spacelike vectors are linearly dependent.

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