XIV Encuentro de Topologa

WEAK INCLUSIONS: A NEW APPROACH TO DIGITAL SPACES

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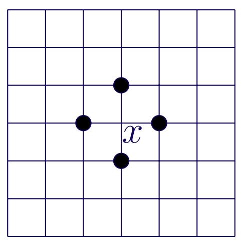
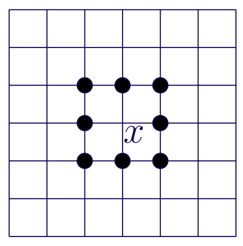
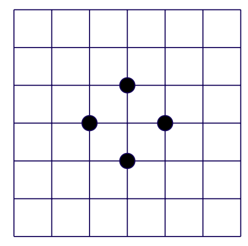
DIGITAL SPACES

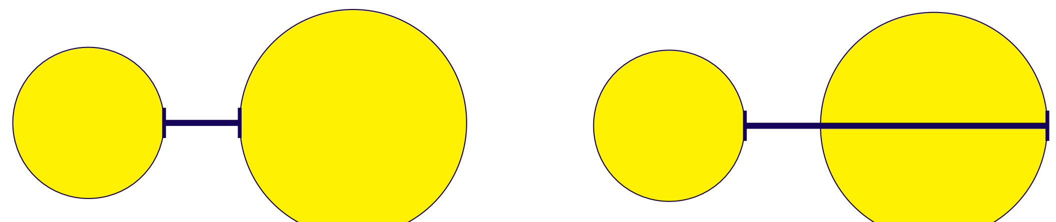
Kong and Rosenfeld [4, 7] (see also [2]) gave an important contribution to the beginning of the so-called digital topology. Their idea consists in considering a digital image as a set of points of a grille or a mesh like $\mathbb{Z}^2$ endowed with an adjacent relation. In this way, the points represent the pixels.

4-neighbours of x  8-neighbours of x 

A **digital picture** is a quadruple $(\mathbb{Z}^2, m, n, B)$ where $B \subseteq \mathbb{Z}^2$, and where $(m, n) = (4, 8)$ or $(8, 4)$.
The points in B are called the **black points** of the picture; the points in $\mathbb{Z}^2 - B$ are called the **white points** of the picture. Two black points are said to be **adjacent** if they are m-neighbours, and two white points or a white point and a black point are said to be adjacent if they are n-neighbours.
A subset S of $\mathbb{Z}^2$ is said to be **connected** if for every $p, t \in S$ we can find a finite subset $\{s_1, \dots, s_r\} \subseteq S$ such that $s_1 = p$, $s_r = t$ and s_{i+1} is adjacent to s_i for all $i \in \{1, \dots, n-1\}$. The set $\{s_1, \dots, s_r\}$ is called a path from p to t.
It is natural to wonder if there exists a topology on $\mathbb{Z}^2$ in such a way that the topological and digital concepts of connectedness coincide. However, this is not possible. Nevertheless, many things similar to topics that are normally considered in classical topology appear in digital spaces: Jordan curve theorem, simple connectedness, etc.



This digital picture is a 8-Jordan digital curve. However it does not divide the space into two 8-connected components. This is the reason for considering two different kinds of connectedness in a digital picture.
In [5], the authors used the concept of a semi-proximity to provide a mathematical context for digital spaces. This model is appropriate for the connectedness of transformation of images. However, this theory produce some incongruities because there exist examples of two different images which are considered to be near and only two points are close so just two near points suffice to declare near two completely dissimilar pictures. This is due to the fact that this model parallels the deceitful gap distance and not the truthful Hausdorff metric.



Here [1], we introduce the concept of weak inclusion which is very related with the concept of a semi-proximity [6]. This structure unifies the advantages of the proximities and the Hausdorff metric [3]. We will show that a particular weak inclusion is compatible with the usual adjacency relation of digital pictures. Furthermore, the weak inclusion allows us to obtain a relation for comparing images which parallels the behavior of the Hausdorff metric.

WEAK INCLUSIONS

Definition. *Let X be a nonempty set. A relation $\sqsubseteq$ on $\mathcal{P}(X) \times \mathcal{P}(X)$ is called a* ***weak inclusion*** *on X if it satisfies the following properties:*

1. *$A \not\sqsubseteq \varnothing$ for all nonempty $A \subseteq X$;*
2. *if $A \subseteq B$ then $A \sqsubseteq B$;*
3. *if $A \subseteq B \sqsubseteq C \subseteq D$ then $A \sqsubseteq D$;*
4. *if $C \sqsubseteq A \cup B$ then there exist C_1, C_2 such that at least one of them is nonempty, $C_1 \sqsubseteq A$ and $C_2 \sqsubseteq B$;*
5. *if $A \sqsubseteq B$ and $C \sqsubseteq D$ then $A \cup C \sqsubseteq B \cup D$.*

If $\sqsubseteq$ also satisfies

6. *if every subset C of A verifies that $C \not\sqsubseteq B$ then there exists E containing A such that $C \not\sqsubseteq X \backslash E$ for every $C \subseteq A$ and $G \not\sqsubseteq B$ for every $G \subseteq E$,*

then we say that $\sqsubseteq$ is an EF-weak inclusion.
If the weak inclusion $\sqsubseteq$ satisfies that if $C \sqsubseteq \{x\}$ then $\{x\} \sqsubseteq C$, we say that this weak inclusion is ***separated****.*
Given a weak inclusion $\sqsubseteq$ on X we say that A and B are ***similar****, denoted by $A \approx B$, if $A \sqsubseteq B$ and $B \sqsubseteq A$.*

Lemma. *Let $\sqsubseteq$ be a weak inclusion on a nonempty set X. Then the function $\mathrm{cl}(\sqsubseteq) : \mathcal{P}(X) \to \mathcal{P}(X)$ given by $\mathrm{cl}(\sqsubseteq) - A = \cup\{B : B \sqsubseteq A\} = \cup\{x : \{x\} \sqsubseteq A\}$ is a Čech closure operator.*

Example. *Let (X, d) be a metric space. Recall that the Hausdorff extended metric is defined as*

$$H(A,B) = \max\{H_d^+(A,B), H_d^-(A,B)\} \text{where} \begin{cases} H_d^+(A,B) = \sup_{b \in B} d(A,b) \\ H_d^-(A,B) = \sup_{a \in A} d(a,B) \end{cases}$$

It is obvious that

$$H_d^-(A,B) = 0 \iff A \subseteq \overline{B} \text{ and } H_d^+(A,B) = 0 \iff B \subseteq \overline{A}$$

Then it is easy to see that the relation $\sqsubseteq_d$ given by $A \sqsubseteq_d B$ if and only if $H_d^-(A,B) = 0$, i. e. $A \subseteq \overline{B}$, is an EF-weak inclusion. Furthermore, $A \approx B$ if and only if $H_d(A,B) = \max\{H_d^-(A,B), H_d^+(A,B)\} = 0$ which is equivalent to assert that $A \subseteq \overline{B}$ and $B \subseteq \overline{A}$, i. e. $\overline{A} = \overline{B}$.

Example. *Let $(\mathbb{Z}^2, m, n, B)$ be a digital picture. Then we can define the weak inclusion $\trianglelefteq$ given by $A \trianglelefteq B$ if every element of A is adjacent to an element of B.*

CONNECTEDNESS IN DIGITAL PICTURES

Definition. *Let $\sqsubseteq$ be a weak inclusion on a nonempty set X. Then X is said to be* ***$\sqsubseteq$-connected*** *if we cannot find two sets X_1 and X_2 such that $X_1 \cup X_2 = X$ and if $C \sqsubseteq X_i$ then $C \cap X_j = \varnothing$ where $i, j \in \{1, 2\}$ and $i \neq j$.*

Proposition. *The concept of connectedness for a digital picture is equivalent to that of $\trianglelefteq$-connectedness.*

Definition. *Let $(\mathbb{Z}^2, m, n, B)$ be a digital picture. We denote by $cl_b(\sqsubseteq_m) - A = \{p \in B : p$ is m-adjacent to $A\}$ and $cl_w(\sqsubseteq_n) - A = \{p \in \mathbb{Z}^2 - B :$ is n-adjacent to $A\}$.*

Definition. *Let $(\mathbb{Z}^2, m, n, B)$ be a digital picture. Given $p \in \mathbb{Z}^2 - B$ (resp. $p \in B$), the $\sqsubseteq_n^w$-component (resp. the $\sqsubseteq_m^b$-component) of p is the set $C = \cup_{k \in \mathbb{N}} cl_w^k(\sqsubseteq_n) - \{p\}$ (resp. $C = \cup_{k \in \mathbb{N}} cl_b^k(\sqsubseteq_m) - \{p\}$) where $cl_w^k(\sqsubseteq_n) - \{p\} = cl_w(\sqsubseteq_n) - (cl_w^{k-1}(\sqsubseteq_n) - \{p\})$.*

Recall that a black component (resp. white component) in a digital picture is a set of black (resp. white) points which is not adjacent to other black (resp. white) point.

Proposition. *Let $(\mathbb{Z}^2, m, n, B)$ be a digital picture. Then the black components (resp. white components) are equal to the $\sqsubseteq_m^b$-components (resp. $\sqsubseteq_n^w$-components).*

References

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