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# Global towers of categorical groups as a model for exterior $\mathbb{N}$-2-types


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## 1. The category of proper and exterior $\mathbb{N}$-2-types

A continuous map $f: X \to Y$ is said to be proper if for every closed compact subset $K$ of $Y$, $f^{-1}(K)$ is a compact subset of $X$. The category of topological spaces and the subcategory of spaces and proper maps will be denoted by **Top** and **P**, respectively. This last category and its corresponding proper homotopy category are very useful for the study of non compact spaces. Nevertheless, one has the problem that **P** does not have enough limits and colimits and then we can not develop the usual homotopy constructions like loops, homotopy limits and colimits, et cetera.
An answer to this problem is given by the notion of exterior space. The new category of exterior spaces and maps is complete and cocomplete and contains as a full subcategory the category of spaces and proper maps.

**Definition 1** *Let $(X,\tau)$ be a topological space. An externology on $(X,\tau)$ is a non empty collection $\varepsilon$ of open subsets which is closed under finite intersections and such that if $E \in \varepsilon$, $U \in \tau$ and $E \subset U$ then $U \in \varepsilon$. An exterior space $(X, \varepsilon \subset \tau)$ consists of a space $(X,\tau)$ together with an externology $\varepsilon$. A map $f: (X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')$ is said to be exterior if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.*

The category of exterior spaces and maps will be denoted by **E**.

We will consider the exterior space $\mathbb{N}$ of non negative integers with the discrete topology and the cocompact (cofinite) externology $\varepsilon_{cc}^{\mathbb{N}}$ and the semi-open interval $\mathbb{R}_+$ which is also provided with the cocompact externology.
Let $\mathbf{E}^{\mathbb{N}}$ be the category of exterior spaces under $\mathbb{N}$ and $\mathbf{E}^{\mathbb{R}_+}$ the category of spaces under $\mathbb{R}_+$. If $(X,\lambda)$ is an object in the category $\mathbf{E}^{\mathbb{R}_+}$, where $\lambda: \mathbb{R}_+ \to X$ is a base ray, the natural restriction $\lambda|_{\mathbb{N}}: \mathbb{N} \to X$ gives a sequence base in $X$. Then we have a canonical forgetful functor $\mathbf{E}^{\mathbb{R}_+} \to \mathbf{E}^{\mathbb{N}}$.
For given $X$, $Z$ be exterior spaces and $Y$ a topological space, one can construct exterior spaces $X \bar{\times} Y$, $Z^Y$ and a topological space $Z^X$ having nice exponential properties. We clarify that the topology taken in $Z^X$ is finer than the compact-open topology. Let $S^q$ denote the $q$-dimensional pointed sphere. Using the exterior space $\mathbb{N}\bar{\times}S^q$ and the topological space $X^{\mathbb{N}}$, one has a canonical isomorphism

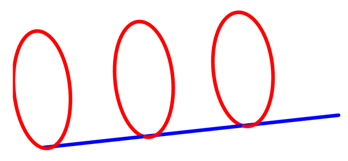
$$Hom_{\mathbf{E}}(\mathbb{N}\bar{\times}S^q, X) \cong Hom_{\mathbf{Top}}(S^q, X^{\mathbb{N}})$$

**Definition 2** *Let $(X,\lambda)$ be in $\mathbf{E}^{\mathbb{R}_+}$ and an integer $q \geq 0$. The $q$-th $\mathbb{N}$-exterior homotopy group of $(X,\lambda)$ is given by*

$$\pi_q^{\mathbb{N}}(X,\lambda) = \pi_q(X^{\mathbb{N}}, \lambda)$$

*where $\pi_q$ is the $q$-th homotopy group functor.*
*An exterior map $f: (X,\lambda) \to (X',\lambda')$ is said to be a weak $[1,2]$-$\mathbb{N}$-equivalence if $\pi_1^{\mathbb{N}}(f), \pi_2^{\mathbb{N}}(f)$ are isomorphisms. Let $\Sigma_{\mathbb{N}}$ denote the class of weak $[1,2]$-$\mathbb{N}$-equivalences*

The elements of $\pi_q^{\mathbb{N}}(X,\lambda)$ are represented by a sequence of spheres, as the given below for $q=1$, converging to an infinity point



The category of exterior $\mathbb{N}$-2-types is the category of fractions

$$\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}]^{-1}$$

and the corresponding subcategory of proper $\mathbb{N}$-2-types is

$$\mathbf{P}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}]^{-1}$$

## 2. Categorical groups

A monoidal category $\mathbb{G} = (\mathbb{G}, \otimes, a, l, r)$ consists of a category $\mathbb{G}$, a functor (tensor product) $\otimes: \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, an object $I$ (unit) and natural isomorphisms called, respectively, the associativity, left-unit and right-unit constraints

$$a = a_{\alpha,\beta,\omega} : (\alpha \otimes \beta) \otimes \omega \xrightarrow{\sim} \alpha \otimes (\beta \otimes \omega)$$

$$l = l_\alpha : I \otimes \alpha \xrightarrow{\sim} \alpha$$

$$r = r_\alpha : \alpha \otimes I \xrightarrow{\sim} \alpha$$

which satisfy that the following diagrams are commutative

$$\begin{array}{ccc} ((\alpha\otimes\beta)\otimes\omega)\otimes\tau & \xrightarrow{a\otimes 1} & (\alpha\otimes(\beta\otimes\omega))\otimes\tau \\ \downarrow a & & \downarrow a \\ (\alpha\otimes\beta)\otimes(\omega\otimes\tau) & & \alpha\otimes((\beta\otimes\omega)\otimes\tau) \\ & \searrow a \quad \swarrow 1\otimes a & \\ & \alpha\otimes(\beta\otimes(\omega\otimes\tau)) & \end{array}$$

$$\begin{array}{ccc} (\alpha\otimes I)\otimes\beta & \xrightarrow{a} & \alpha\otimes(I\otimes\beta) \\ & \searrow r\otimes 1 \quad \swarrow 1\otimes l & \\ & \alpha\otimes\beta & \end{array}$$

A categorical group is a monoidal groupoid, where every object has an inverse with respect to the tensor product in the following sense:
For each object $\alpha$ there is an inverse object $\alpha^*$ and canonical isomorphisms

$$(\gamma_r)_\alpha : \alpha \otimes \alpha^* \to I$$

$$(\gamma_l)_\alpha : \alpha^* \otimes \alpha \to I$$

A morphism of categorical groups $(T,\mu): \mathbb{G} \to \mathbb{H}$ is a functor $T: \mathbb{G} \to \mathbb{H}$ with a family of natural isomorphisms $\mu: T(\alpha \otimes \beta) \to T\alpha \otimes T\beta$ which is coherent with associativity, left-unit and right-unit constraints.
The category of categorical groups will be denoted by **CG**.

## 3. The fundamental categorical group and the classifying space functor via the small category $E(E(\hat{4}) \times EC(\Delta/2))$

We introduce the following small category with the objective that the corresponding presheaves have natural realizations in **Top** and associated categorical groups.
Consider the category $\Delta/2$. Now we can construct the pushouts

$$\begin{array}{ccc} [0] & \xrightarrow{\delta_1} & [1] \\ \delta_0 \downarrow & & \downarrow \mathrm{in}_r \\ [1] & \xrightarrow{\mathrm{in}_l} & [1]+_{[0]}[1] \end{array} \qquad \begin{array}{ccc} [1] & \xrightarrow{\mathrm{in}_l} & [1]+_{[0]}[1] \\ \mathrm{in}_r \downarrow & & \downarrow \\ [1]+_{[0]}[1] & \longrightarrow & [1]+_{[0]}[1]+_{[0]}[1] \end{array}$$

and denote by $C(\Delta/2)$ the extension of the category $\Delta/2$ given by the objects $[1]+_{[0]}[1]$, $[1]+_{[0]}[1]+_{[0]}[1]$ and all the natural maps induced by these pushouts.
In order to have vertical composition and inverses up to homotopy we extend this category with some additional maps and relations:
$V: [2] \to [1]$, $V\delta_2 = \mathrm{id}$, $V\delta_1 = \delta_1\epsilon_0$, $(V\delta_0)^2 = \mathrm{id}$,
$K: [2] \to [1]+_{[0]}[1]$, $K\delta_2 = \mathrm{in}_l$, $K\delta_0 = \mathrm{in}_r$,
$A: [2] \to [1]+_{[0]}[1]+_{[0]}[1]$, $A\delta_2 = (K\delta_1 + \mathrm{id})K\delta_1$, $A\delta_1 = (\mathrm{id} + K\delta_1)K\delta_1$, $A\delta_0 = A\delta_1\delta_0\epsilon_0$.
The new extended category will be denoted by $EC(\Delta/2)$.

With the objective of obtaining a tensor product with a unit object and inverses, we take the small category $\hat{4}$ generated by the object 1 and the induced coproducts $0,1,2,3,4$, all the natural maps induced by coproducts and three additonal maps:
$e_0: 1 \to 0$, $\nu: 1 \to 1$ and $\mu: 1 \to 2$.
This gives a category $E(\hat{4})$.

Consider the product category $E(\hat{4}) \times EC(\Delta/2)$. The object $(i,[j])$, and morphisms $\mathrm{id}_i \times g$, $f \times \mathrm{id}_{[j]}$ will be denoted by $i[j]$ and $g$, $f$, respectively.
We extend again this category by adding new maps:

$a: 1[1] \to 3[0]$, $r: 1[1] \to 1[0]$, $l: 1[1] \to 1[0]$, $\gamma_r: 1[1] \to 1[0]$, $\gamma_l: 1[1] \to 1[0]$, $t: 1[2] \to 2[0]$, $p: 1[2] \to 4[0]$,
satisfying adequate relations to induce asociativity, identity and inverse isomorphisms for the associated categorical group structure.
The new extended category will be denoted by

$$\mathbf{E(E(\hat{4}) \times EC(\Delta/2))}$$

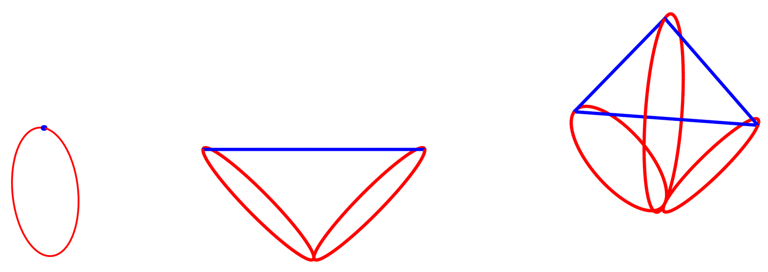
Now we take the covariant functors:
$S: E(\hat{4}) \to \mathbf{Top}^*$, preserving coproducts and such that $S(1) = S^1$, $S(\mu): S^1 \to S^1 \vee S^1$ is the co-multiplication and $S(\nu): S^1 \to S^1$ gives the inverse loop.
The standard functor $\Delta: \Delta/2 \to \mathbf{Top}$, given by $\Delta[p] = \Delta_p$, extends to a functor $\Delta: EC(\Delta/2) \to \mathbf{Top}$.
Taking the functors $()^+: \mathbf{Top} \to \mathbf{Top}^*$, $X^+ = X \sqcup \{*\}$, and the smash $\wedge: \mathbf{Top}^* \times \mathbf{Top}^* \to \mathbf{Top}^*$, we construct an induced functor

$$S \wedge \Delta^+: E(\hat{4}) \times EC(\Delta/2)) \to \mathbf{Top}^*$$

which extends to the desired functor

$$S \wedge \Delta^+: \mathbf{E(E(\hat{4}) \times EC(\Delta/2))} \to \mathbf{Top}^*$$

The images of the main objects $1[0]$, $1[1]$, $1[2]$ can be seen in the following pictures



This functor induces a pair of adjoint functors

$$\mathrm{Sing}: \mathbf{Top}^* \to \mathbf{Set}^{E(E(\hat{4}) \times EC(\Delta/2)))^{op}}$$

$$|\cdot|: \mathbf{Set}^{E(E(\hat{4}) \times EC(\Delta/2)))^{op}} \to \mathbf{Top}^*$$

We will denote by

$$\mathbf{Set}_{pp}^{E(E(\hat{4}) \times EC(\Delta/2)))^{op}}$$

the category of presheaves $X: (E(E(\hat{4}) \times EC(\Delta/2)))^{op} \to \mathbf{Set}$ such that $X(i,-)$ transforms the pushouts of $C(\Delta/2)$ in pullbacks and $X(-,[j])$ transforms the coproducts of $\hat{4}$ in products.

Given a presheaf $X$ in $\mathbf{Set}_{pp}^{E(E(\hat{4}) \times EC(\Delta/2)))^{op}}$ one can define its fundamental categorical group $G(X)$ as a quotient object. This gives a functor

$$G: \mathbf{Set}_{pp}^{E(E(\hat{4}) \times EC(\Delta/2)))^{op}} \to \mathbf{CG}$$

**Proposition 1** *The functor $G: \mathbf{Set}_{pp}^{E(E(\hat{4}) \times EC(\Delta/2)))^{op}} \to \mathbf{CG}$ is left adjoint to the forgetful functor $U: \mathbf{CG} \to \mathbf{Set}_{pp}^{E(E(\hat{4}) \times EC(\Delta/2)))^{op}}$.*

If we take the singular functor Sing and afterwards the categorical group functor $G$ of a presheaf, then the composite $\rho_2 = G\,\mathrm{Sing}: \mathbf{Top}^* \to \mathbf{CG}$ gives the fundamental categorical group of a pointed space. We also have a forgetful functor $U$ from the category of categorical groups to the category of presheaves and a realization functor, then the composite $B = |\cdot|\,U: \mathbf{CG} \to \mathbf{Top}^*$ constructs a topological pointed 0-connected space. This pair of functors gives a new version of the well known equivalence of the category of 2-types of 0-connected pointed spaces and the category of categorical groups up to weak equivalences.

## 4. Exterior $\mathbb{N}$-2-types and global towers of categorical groups

Given a category $\mathcal{C}$ we can consider the induced categories

pro$\mathcal{C}$: pro-objects $X$ in $\mathcal{C}$ ( $X: J \to \mathcal{C}$ is a functor, where $J$ is a left-filtering small category).

pro$^+\mathcal{C}$: global pro-objects $Y$ in $\mathcal{C}$ ($Y: K \to \mathcal{C}$ is a functor, where $K$ is a left-filtering small category with final object, and pro-morphisms are compatible with the final object)

tow$\mathcal{C}$: towers $X$ in $\mathcal{C}$ ($X: \mathbf{N} \to \mathcal{C}$ is a functor, where $\mathbf{N}$ is the category of ordered natural numbers )

tow$^+\mathcal{C}$ global towers $Y$ in $\mathcal{C}$ ($Y: \mathbf{N} \to \mathcal{C}$ functor, where $\mathbf{N}$ is the category of ordered natural numbers with the final object 0)

Let **Gr** be the category of groups. For $\mathbf{Top}^*$, **Gr** and **CG**, we have

$$\mathrm{pro}^+\mathbf{Top}^*, \quad \mathrm{pro}^+\mathbf{Gr}, \quad \mathrm{pro}^+\mathbf{CG}, \quad \mathrm{tow}^+\mathbf{Top}^*, \quad \mathrm{tow}^+\mathbf{Gr}, \quad \mathrm{tow}^+\mathbf{CG}$$

The fundamental categorical group $\rho_2$ and classifying functor $B$ induce

$$\mathrm{pro}^+\rho_2: \mathrm{pro}^+\mathbf{Top}^* \to \mathrm{pro}^+\mathbf{CG}, \quad \mathrm{tow}^+\rho_2: \mathrm{tow}^+\mathbf{Top}^* \to \mathrm{tow}^+\mathbf{CG}$$

$$\mathrm{pro}^+\mathrm{B}: \mathrm{pro}^+\mathbf{CG} \to \mathrm{pro}^+\mathbf{Top}^*, \quad \mathrm{tow}^+\mathrm{B}: \mathrm{tow}^+\mathbf{CG} \to \mathrm{tow}^+\mathbf{Top}^*$$

Given a exterior space $(X,\lambda) \in \mathbf{E}^{\mathbb{R}+}$ the externology $\varepsilon_X$ can be seen as a left-filtering category with a final object and we can consider the functor

$$\varepsilon(X): \varepsilon_X \to \mathbf{Top}^*, \varepsilon(X)(E) = (E \cup [0,\infty)/t \sim \lambda(t), 0),\ t \in \lambda^{-1}(E)$$

This induces a full embedding

$$\varepsilon: \mathbf{E}^{\mathbb{R}+} \to \mathrm{pro}^+\mathbf{Top}^*$$

An exterior space is said to be first countable at infinity if there is a countable base of the externology

$$X = E_0 \supset E_1 \supset E_2 \supset \cdots$$

$\mathbf{E}_{fc}^{\mathbb{R}+}$ denotes the category of rayed exterior spaces which are first countable at infinity.
We can consider the restriction functor

$$\varepsilon: \mathbf{E}_{fc}^{\mathbb{R}_+} \to \mathrm{tow}^+\mathbf{Top}^*$$

Using the functor $\mathrm{tow}^+\pi_q: \mathrm{tow}^+\mathbf{Top}^* \to \mathrm{tow}^+\mathbf{Gr}$ and the Brown´s $\mathcal{P}$ functor one has that $\mathcal{P}\mathrm{tow}^+\pi_q\varepsilon \cong \pi_q^{\mathbb{N}}$ and it is well known that if $f: X \to Y$ is a morphism in $\mathbf{E}_{fc}^{\mathbb{R}_+}$, then $\mathrm{tow}^+\pi_q\,(\varepsilon f)$ is an isomorphism if and only if $\pi_q^{\mathbb{N}}(f)$ is an isomorphism.

We also can consider the Telescopic construction $\mathrm{Tel}: \mathrm{tow}^+\mathbf{Top}^* \to \mathbf{E}_{fc}^{\mathbb{R}_+}$.

With all these functors one can prove

**Theorem 1** *The functors $\mathrm{tow}^+\rho_2\varepsilon$ and Tel $\mathrm{tow}^+\mathrm{B}$ induce an equivalence of categories*

$$\mathbf{E}_{fc}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}]^{-1} \to \mathrm{tow}^+\mathbf{CG}[\Sigma]^{-1}$$

*where $\Sigma$ is the class of maps in $\mathrm{tow}^+\mathbf{CG}$ given by the clousure of the level weak equivalences.*