Non-convolution nonlinear integral Volterra equations with monotone operators

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Abstract

In this paper existence and uniqueness results are obtained for nonlinear Volterra integral equations with non-convolution kernels. Results known for convolution equations are used, together with some comparison techniques, in order to extend such results to non-convolution equations.

1 Introduction

This paper is devoted to the study of the nonlinear Volterra integral equation

\[ u(x) = \int_0^x k(x, s)g(u(s)) \, ds. \tag{1} \]

For this equation, that we shall denote \((k, g)\), we will assume that the following conditions are held:

\[ K_1. \quad \text{The kernel } k : \mathbb{R}^2 \to \mathbb{R}^+ \text{ is a locally bounded function such that } k(x, s) = 0 \text{ whenever } s > x. \]

\[ K_2. \quad \text{For every } x \in \mathbb{R}, \text{ the map } s \to k(x, s) \text{ is locally integrable, and } K(x) = \int_0^x k(x, s) \, ds \text{ is a strictly increasing function.} \]

\[ G_1. \quad \text{The nonlinearity } g \text{ is a continuous, strictly increasing function, vanishing on } (-\infty, 0], \text{ and such that } g' > 0 \text{ almost everywhere.} \]

Solutions of equation \((k, g)\) are fixed points of the operator \(T_{kg}\) defined as

\[ T_{kg}f(x) = \int_0^x k(x, s)g(u(s)) \, ds. \tag{2} \]

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With the assumption \( G_1 \) made above, \( T_{kg} \) is a monotone increasing operator, i.e. if \( f_1 \leq f_2 \), then \( T_{kg} f_1 \leq T_{kg} f_2 \). We also have that, since \( g(0) = 0 \), the function zero is a soution of (1), known as the trivial solution.

The following results allow us to consider only solutions that are bounded on a certain interval \( [0, \delta] \), for some positive \( \delta \).

**Lemma 1.1.** Let \( k \) be a kernel satisfying the following condition

\[
k(x, s) \leq k(y, s), \quad \text{for } x \leq y.
\]

Then, the operator \( T_{kg} \) transforms positive functions into increasing functions.

**Proof.** Let \( f \) be a positive function, and let \( x \leq y \).

\[
T_{kg} f(x) = \int_0^x k(x, s) g(f(s)) \, ds = \int_0^y k(x, s) g(f(s)) \, ds \\
\leq \int_0^y k(y, s) g(f(s)) \, ds = T_{kg} f(y).
\]

**Corollary 1.1.** Let \( k \) be a kernel verifying condition \( K_1 \), and let \( f \) be a positive function. Then, for every \( x \) in its domain of definition, \( T_{kg} f \) is bounded on \( [0, x] \).

**Proof.** Defining the auxiliar kernel, \( \overline{k}(x, s) = \max\{k(t, s); 0 \leq t \leq x\} \), we have that \( \overline{k} \) verifies condition (3) and \( k \leq \overline{k} \). Then, if \( f \) is a positive function, \( T_{kg} f \leq T_{\overline{k}g} f \). From Lemma 1.1, it follows that \( T_{\overline{k}g} f \) is an increasing function; thus, for every \( x \) where \( T_{kg} f \) is defined, we have that \( T_{kg} f \) is bounded by \( T_{\overline{k}g} f(x) \) on \( [0, x] \).\[\square\]

Taking into account last corollary, positive solutions for equation (1) will be bounded near zero, i.e. bounded on an interval \( [0, \delta] \), for some \( \delta > 0 \). So, unless otherwise stated, any function considered in this paper will be bounded near zero.

A particular case of equation \((k, g)\) is the well known convolution equation

\[
u(x) = \int_0^x \phi(x - s) g(u(s)) \, ds,
\]

where the kernel \( k(x, s) = \phi(x - s) \), being \( \phi \) a locally bounded function of one real variable. For such equations, the existence of a nontrivial solution is equivalent to the existence of a nontrivial subsolutions, i.e a function \( f \) such that \( f \leq T_{\phi g} f \). Moreover, if a positive solution of (4) exists, then it is unique, strictly increasing, continuous and it is a global attractor of all positive and measurable function. That means that
for every positive and measurable function $f$, the sequence $(T^n_{\phi g} f)_{n \in \mathbb{N}}$ converges to the solution, where $T^n_{\phi g}$ denotes the composition of $T_{\phi g}$ with itself $n$ times.

In this paper we are going to study how the results known for the convolution equation (4), can be used in order to obtain properties of the solutions for the non-convolution equation (1).

## 2 Existence of solutions

First we will show that as it occurs with convolution equations, for non-convolution equations the existence of a solution is equivalent to the existence of a subsolution.

**Theorem 2.1.** Equation (1) admits a solution if and only if it admits a subsolution.

**Proof.** Since every solution of equation (1) is a particular case of subsolution, the proof is reduced to the necessary part.

Let $v$ be a positive subsolution of (1). Then there exits some positive $\delta_0$, $\delta_1$ and $M$, such that

$$v \leq T_{kg}v \text{ on } [0, \delta_0] \quad \text{and} \quad v \leq M \text{ on } [0, \delta_1].$$

Moreover, since the kernel is locally bounded, $\lim_{x \to 0^+} K(x) = 0$, and taking into account that $T_{kg}M(x) = g(M)K(x)$, it follows that there exists $\delta_2 > 0$ such that

$$T_{kg}M \leq M \text{ on } [0, \delta_2].$$

Let $\delta = \min\{\delta_0, \delta_1, \delta_2\}$. From (5) and (6), we have that

$$v \leq T_{kg}v \leq T_{kg}M \leq M \text{ on } [0, \delta].$$

Thus, $(T^n_{kg} v)_{n \in \mathbb{N}}$ is a nondecreasing sequence bounded by above by $M$. Then we can define the pointwise limit

$$u(x) = \lim_{n \to \infty} T^n_{kg} v(x), \quad \forall x \in [0, \delta].$$

Using, for each $x \in [0, \delta]$, the Monotone Convergence Theorem on the sequence $(\phi_n)_{n \in \mathbb{N}}$, where

$$\phi_n(s) = k(x, s)g(T^n_{kg} v(s)),$$

we obtain that $u$ is a solution of equation (1) on $[0, \delta]$. □
In order to obtain more results from convolution equations, next we will see that every locally bounded kernel can be bounded from above and from below by convolution kernels on any bounded region of $\mathbb{R}^2$.

We shall say that a kernel $k$ is invariant if $k(x, s) = k(x + \lambda, s + \lambda)$ for all $\lambda \in \mathbb{R}$. Obviously a convolution kernel is invariant, since for any $\lambda \in \mathbb{R}$

$$k(x + \lambda, s + \lambda) = \phi(x + \lambda - s - \lambda) = \phi(x - s) = k(x, s).$$

Moreover, every invariant kernel can be written as a convolution kernel. If a kernel $k$ is invariant, then

$$k(x, s) = \phi(x - s).$$

Thus, considering a convolution kernel as a function of two variables, defined in the $x$-$s$-plane, its level curves are straight lines parallel to the line $x = s$.

Now let us consider a kernel $k$ satisfying $K_1$, and let us restrict our equation to a bounded interval $[0, x_0]$, for some fixed arbitrary $x_0 > 0$. If we define

$$\phi_{x_0}(x) = \min\{k((1 - \lambda)x + \lambda x_0, \lambda(x_0 - x)) : \lambda \in [0, 1]\}$$

and

$$\psi_{x_0}(x) = \max\{k((1 - \lambda)x + \lambda x_0, \lambda(x_0 - x)) : \lambda \in [0, 1]\}.$$  

Let $T_{x_0}$ denote the triangle with vertex $(0, 0)$, $(x_0, 0)$ and $(x_0, x_0)$. Then, for every $x \in [0, x_0]$, $\phi$ and $\psi$ are the minimum and the maximum, respectively, of $k$ on the segment with end points $(x_0, 0)$ and $(x_0, x_0 - x)$, which is parallel to the line $x = s$. Thus, for every $(x, s) \in T_{x_0}$ we have that

$$\phi_{x_0}(x - s) \leq k(x, s) \leq \psi_{x_0}(x - s).$$

With such bounds, and taking into account Theorem 2.1, it follows that the existence of a solution for a equation $(\phi_{x_0}, g)$ implies the existence of a solution for equation $(k, g)$; which also implies the existence of a solution for $(\psi_{x_0}, g)$. The converse is not true in general, but if we can find a positive constant $c$ such that $\psi \leq c\phi$ (or equivalently such that $c\psi \leq \phi$), then we will have that

$$\phi_{x_0}(x - s) \leq k(x, s) \leq c\phi_{x_0}(x - s).$$

In this case the existence of a solution for equation $(k, g)$ is equivalent to the existence of a solution for equation $(\phi, g)$. Such bounds can be found, for example, when

$$\lim_{x \to 0^+} \frac{\psi(x)}{\phi(x)} = l \in [0, +\infty).$$

Let us see some examples:
Example. The equation
\[ u(x) = \int_0^x (a^{x+s} + 1) \sqrt{2u(s)} \, ds, \quad a > 0, x \geq 0, \quad (9) \]
has, at least, one locally bounded solution.

Proof. Let us consider an arbitrary positive constant \( x_0 > 0 \), and restrict the problem to the interval \([0, x_0]\). Consider the triangle
\[ T_{x_0} = \{(x, s) \in \mathbb{R}^2 : 0 \leq x \leq x_0, 0 \leq s \leq x\}. \]

Since the kernel is increasing with respect both variables, the functions \( \phi \) and \( \psi \), defined in (7) and (8), are
\[ \psi(x) = k(x, 0) = a^x + 1 \quad \text{and} \quad \psi(x) = k(x_0 - x) = a^{2x_0-x} + 1. \]

We also have that
\[ \lim_{x \to 0^+} \frac{\phi(x)}{\psi(x)} = \frac{2}{a^{2x_0} + 1}, \]
thus the existence of solutions for equation (9) is equivalent to the existence of a solution for the equation
\[ u(x) = \int_0^x \phi(x-s) \sqrt{2u(s)} \, ds, \quad x \in [0, x_0]. \quad (10) \]

A necessary and sufficient condition for the existence of a solution for equation (10) is (see [1]) that there exists a locally integrable function \( f \) such that \( H[f] \geq g^{-1} \), where \( H \) is the operator
\[ H[f](x) = \int_0^x \Phi \left( \int_s^x f(t) \, dt \right) \, ds, \]
being \( \Phi(x) = \int_0^x \phi(s) \, ds \). In our case
\[ \Phi(x) = \int_0^x a^s + 1 \, ds = \frac{a^x - 1}{\log a} + x. \]

Thus, considering \( f(t) \equiv 1 \), we obtain
\[ H[1](x) = \int_0^x \Phi \left( \int_s^x dt \right) \, ds = \int_0^x \Phi(x-s) \, ds = \int_0^x \frac{a^x - 1}{\log a} + (x-s) \, ds = \frac{a^x - 1}{(\log a)^2} - \frac{x^2}{\log a} + \frac{x^2}{2} \geq \frac{x^2}{2} = g^{-1}(x). \]

Hence the convolution equation has a solution, and consequently the nonconvolution equation \((k, g)\) also has a solution. \( \square \)
3 Uniqueness

For convolution equations with locally bounded (e.g. continuous) kernels, nontrivial solutions are unique. As we shall see next, adding a subinvariance condition to the nonconvolution kernel, a similar result is obtained.

**Lemma 3.1.** Let us suppose that, in addition to $K_1$ and $K_2$, the kernel is such that the function $K(x) = \int_0^x k(x, s)\,ds$ is continuous (e.g. $k$ is continuous in $\mathbb{R}^2$). Then the operator $T_{kg}$ transforms bounded functions into continuous functions.

**Proof.** Let $f$ be a function bounded by $M$. Let $x_1 \leq x_2$, then, since $k(x, s) = 0$ whenever $s > x$, we have that

\[
T_{kg}(x_2) - T_{kg}(x_1) = \int_0^{x_2} k(x_2, s)g(f(s))\,ds - \int_0^{x_1} k(x_1, s)g(f(s))\,ds
\]

\[
= \int_0^{x_2} (k(x_2, s) - k(x_1, s))g(f(s))\,ds \leq g(M)\int_0^{x_2} (k(x_2, s) - k(x_1, s))\,ds
\]

\[
= g(M)(K(x_2) - K(x_1)).
\]

(11)

From the continuity of $K$ on $x_1$, it follows the continuity of $T_{kg}f$ on $x_1$. □

As a simple corollary we obtain

**Corollary 3.1.** Every solution of equation $(k, g)$ is a continuous function.

**Lemma 3.2.** Let $k$ be a kernel under the assumptions of Lemma 3.1. Let us suppose that $k$ also verifies the following inequality

\[
k(x, s) \leq k(x+\lambda, s+\lambda), \quad \forall \lambda \geq 0.
\]

(12)

Then every subsolution of equation $(k, g)$ is bounded by any solution of equation $(k, g)$.

**Proof.** Let $v$ and $u$ be a subsolution and a solution of equation $(k, g)$, respectively. By Corollary 3.1 they are both continuous functions. We will first show that for every $c > 0$, the function

\[
v_c(x) = \begin{cases} 0, & \text{if } x \in [0, c] \\ v(x-c), & \text{if } x > c, \end{cases}
\]

(13)

is also a subsolution of equation $(k, g)$. 

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For $x \in [0, c]$ this is trivial, since $v_c(x) = T_k v_c(x) = 0$. For $x > c$ we have that

$$v_c(x) = v(x - c) \leq T_k v(x - c) = \int_0^{x-c} k(x - c, s) g(v(s)) \, ds;$$

making the change of variable $t = s + c$, it follows, since $k$ verifies (12),

$$v_c(x) \leq \int_c^x k(x - c, t - c) g(v(t - c)) \, dt \leq \int_c^x k(x, t) g(v_c(t)) \, dt = T_k v_c(x).$$

Let us compare now $v_c$ and $u$. For $0 < x < c$ it is obvious that $0 = v_c(x) < u(x)$. Since $v_c$ and $u$ are continuous, exists an interval $(0, x_0)$, with $x_0 > c$, where $v_c \leq u$. Then we have that

$$u(x_0) - v_c(x_0) \geq \int_0^{x_0} k(x_0, s) [g(u(s)) - g(v_c(s))] \, ds$$

$$> \int_0^c k(x_0, s) g(u(s)) \, ds > 0.$$

Again, from the continuity of $v_c$ and $u$, we can assure that $v_c < u$ in the whole domain of the solution. Finally, taking limits as $c \to 0^+$, we obtain that $v \leq u$. \hfill \Box

As a consequence of this Lemma, the uniqueness of positive solutions for the equation $(k, g)$ is granted.

**Theorem 3.1.** Under the hypothesis of Lemma 3.2, equation $(k, g)$ has at most one positive solution.

**Proof.** The proof is trivial since every solution can be considered as a special case of subsolution. Then if $u_1$ and $u_2$ are two solutions of $(k, g)$, considering $u_1$ as a subsolution, by Lemma 3.2 $u_1 \leq u_2$, but considering $u_2$ as a subsolution, we have that $u_2 \leq u_1$, so $u_1 \equiv u_2$. \hfill \Box

### 4 Attracting behaviour

In this section we are going to study how the attracting character of the solutions for the convolution equations $(\phi, g)$ and $(\psi, g)$, given in Section 2, let us obtain some attracting properties for the solutions of the nonconvolution equation $(k, g)$.

Let $u_\phi$ and $u_\psi$ denote the solutions for the convolution equations $(\phi, g)$ and $(\psi, g)$, respectively; and let $u$ denote a solution for the equation $(k, g)$. Since $u_\phi hi$ and $u_\psi si$ are global attractor of all positive and measurable functions, we have the following result:
Proposition 4.1. Let $u$ be any solution of the nonconvolution equation $(k, g)$. Then $u_φ ≤ u ≤ u_ψ$.

Proof. It follows from the inequalities $φ ≤ k ≤ ψ$, that $T_φ u ≤ T_k u ≤ T_ψ u$. Thus, $T_φ u ≤ u = T_k u ≤ T_ψ u$ and then, $T_n^φ u ≤ u ≤ T_n^ψ u$, for every natural $n$. Taking limits as $n$ tends to $∞$, and taking into account that, since both $u_φ$ and $u_ψ$ are global attractor, the sequences $(T_n^φ u)_{n∈N}$ and $(T_n^ψ u)_{n∈N}$ converge to $u_φ$ and $u_ψ$, respectively, we have that $u ≤ u_ψ$.

Proposition 4.2. The sequence $(T_k u_ψ)_{n∈N}$ converges to the maximum solution of the equation $(k, g)$.

Proof. By Proposition 4.1 $u ≤ u_ψ$. Thus, from the monotony of the operators $T_k$ and $T_ψ$ it follows that

$$u = T_k u ≤ T_k u_ψ ≤ T_ψ u_ψ = u_ψ.$$  

Hence, for every $x ≥ 0$, the decreasing sequence $(T_n^k u_ψ(x))_{n∈N}$ is bounded from below by $u(x)$, so it converges pointwisely to a function

$$v(x) = \lim_{n→∞} T_n^k u_ψ(x) = \inf\{T_n^k u_ψ(x) : n ∈ N\}.$$  

By the Monotone Convergence Theorem, we can assure that $v$ is a solution of the equation $(k, g)$; moreover from the way of constructing $v$, it is immediate that $v$ is the maximum solution.

With an similar proof we obtain an analogous result for the minimum solution.

Proposition 4.3. The sequence $(T_n^k u_φ)_{n∈N}$ converges to the minimum solution of the equation $(k, g)$.

Now we are in position to give a result on the attracting character of the maximum and minimum solutions of equation $(k, g)$.

Theorem 4.1. The maximum (resp. minimum) solution of the equation $(k, g)$ attracts globally any measurable function lower (resp. upper) bounded by the solution.

Proof. We shall prove the theorem for the maximum solution. The proof for the minimum solution is similar.

Let $u_{max}$ denote the maximum solution of the equation $(k, g)$, and let $f$ be a measurable function such that $u_{max} ≤ f$. We have to show that $(T_n^k f)_{n∈N}$ converges to $u_{max}$. We already know that $f$ is globally attracted by $u_ψ$, and by Proposition 4.1 we have that

$$u_{max} = T_n^k u_{max} ≤ T_n^k f ≤ T_n^ψ f, \quad ∀n ∈ N.$$
Thus, for every $x \geq 0$, the sequence $(T_{kg}^n f(x))_{n \in \mathbb{N}}$ is bounded from below by $u_{\max}$, and from above by the sequence $(T_{kg}^n \psi g)_{n \in \mathbb{N}}$, which, as said above, converges to $u_\psi$. Then it can be assured that the set of accumulation points of the sequence $(T_{kg}^n f(x))_{n \in \mathbb{N}}$, that we shall denote by $\Omega_f(x)$, verifies $u_{\max}(x) \leq \Omega_f(x) \leq u_\psi(x)$.

To finish the proof it suffices to show that $\Omega_f(x) = \{u_{\max}(x)\}$. But this is obvious since $\Omega_f(x)$ is invariant under $T_{kg}$ and, by the Proposition 4.2, the sequence $(T_{kg}^n u_\psi)_{n \in \mathbb{N}}$ converges to $u_{\max}$. Hence $u_{\max}(x) \leq \Omega_f(x) \leq u_{\max}(x)$.

**Remark 4.1.** Note that if, for instance, we can assure the uniqueness of solutions for the nonconvolution equation $(k, g)$, then the maximum and minimum solutions are the same. In that case, with a simple comparison reasoning, it can be guaranteed that the unique solution is a global attractor of any positive and measurable function.

**References**


