Givens rotations to annihilate the inferior triangular part of the matrix $\tilde{\mathbf{Y}}^{(t+1)}$. The update is operated on the change in the parameter (see [10]) $d\tilde{\mathbf{w}}_i(t) = \tilde{\mathbf{w}}_i(t+1) - \tilde{\mathbf{w}}_i(t)$. The algorithm consists of the following steps:

1) Compute the prediction error $u(t+1) = \tilde{z}_i(t+1) - \tilde{\mathbf{y}}(t+1)^T \tilde{\mathbf{w}}_i(t)$.
2) Form the matrix
$$\begin{bmatrix} \lambda \mathbf{V}(t) & \mathbf{0} \\ \tilde{\mathbf{y}}(t+1)^T & u(t+1) \end{bmatrix}.$$
3) Sweep the bottom part of this matrix using Givens rotations.
4) Solve the triangular system $\mathbf{V}(t+1)\, d\tilde{\mathbf{w}}_i(t) = \overline{\mathbf{Z}}_i^{(t+1)}$.
5) Obtain $\tilde{\mathbf{w}}_i(t+1) = \tilde{\mathbf{w}}_i(t) + d\tilde{\mathbf{w}}_i(t+1)$.

V. Simulation Results

To evaluate the performance of the multichannel parameter estimation method proposed, we simulated the algorithm. We adopted as a measure of performance for a generic channel of the AR model $MSE_i = [\sum_{j=1}^{\bar{r}} \sum_l |\hat{a}_{i,j}(l) - a_{i,j}(l)|^2 / \sum_l |a_{i,j}(l)|^2$, where $\hat{a}_{i,j}(l)$ are the estimated parameters of the AR model, and $a_{i,j}(l)$ are the true values. The system we used to generate data is a stable, noncausal AR model 2×2 with

$$\mathbf{A}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{A}(1) = \begin{bmatrix} -0.5 & 0 \\ 1 & -0.8 \end{bmatrix}$$
$$\mathbf{A}(2) = \begin{bmatrix} -0.9375 & 0 \\ -1 & 0.65 \end{bmatrix}$$

with poles at $(1.25, -0.75, 0.4 + j0.7, 0.4 - j0.7)$. The input data was generated by a zero mean vector process with independent and identically distributed components with the desired HOS properties and a power of one. The values of $\text{cum}_3(x)$ are -0.2474. A vector Gaussian process with statistically i.i.d. components is added at each output of the model. To start up the adaptive algorithm, 100 samples were used to initialize data matrices. In Fig. 2, the SNR is 50 dB, and the value of λ is 1. The dashed lines are the true values of the AR parameters for channel 1. In Fig. 3, the MSE convergence process is shown at SNR = 8 dB and SNR = 50 dB. In Fig. 4, the results of the batch algorithm running on a window of data samples of length 5000 are given. The point of convergence is affected by the additive noise. Fig. 5 shows the traces of two AR parameters and the sensitivity of the adaptive algorithm to the number of samples used for initialization, particularly that the traces of the AR parameters are considerably smoother with 100 samples initialization.

VI. Conclusions

We have proposed an estimation technique for multichannel causal or noncausal AR models when the input is non-Gaussian. We have shown how the super exponential algorithm presented in [1] can be generalized to vector processes and applied to the non-Gaussian multivariate AR estimation problem. Since the algorithm is iterative, further investigation is required to fully characterize the convergence behavior of the method. An adaptive implementation has been presented that is attractive from the computational point of view with respect to other identification procedures based on HOS. Some simulation results have been shown for the adaptive and the batch (block processing) implementation.

Acknowledgment

The author is grateful to Prof. J. B. Anderson and Prof. G. J. Saulnier for their help when the author was at RPI and to D. Bell for his valuable suggestions.

References

[1] O. Shalvi and E. Weinstein, "Super exponential methods for blind deconvolution," *IEEE Trans. Inform. Theory,* vol. 39, pp. 504–519, Mar. 1993.

[2] ——, "Universal methods for blind deconvolution," in *Blind Deconvolution,* S. Haykin, Ed. Englewood Cliffs, NJ: Prentice-Hall, 1994.

[3] A. Swami, G. Giannakis, and S. Shamshnder, "Multichannel ARMA processes," *IEEE Trans. Signal Processing,* vol. 42, pp. 898–913, Apr. 1994.

[4] J. H. Winters, J. Salz, and R. D. Gitlin, "The impact of antenna diversity on the capacity of wireless communication systems," *IEEE Trans. Commun.,* vol. COM-42, pp. 1740–1751, Feb–Apr. 1994.

[5] H. Sakai, "Circular lattice filtering using Pagano's method," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. ASSP-30, pp. 279–287, Apr. 1982.

[6] P. Lewis, "QR based algorithms for multichannel adaptive least squares lattice filters," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. ASSP-30, pp. 421–432, Mar. 1990.

[7] B. Porat, *Digital Processing of Random Signals.* Englewood Cliffs, NJ: Prentice-Hall, 1994.

[8] S. Haykin, *Adaptive Filter Theory.* Englewood Cliffs, NJ: Prentice-Hall, 1986.

[9] G. B. Giannakis, J. M. Mendel, and Y. Inouye, "Cumulant based identification of multichannel moving average models," *IEEE Trans. Automat. Contr.,* vol. 34, pp. 783–787, July 1989.

[10] J. F. Bobrow and W. Murray, "An algorithm for RLS identification of parameters that vary quickly with time," *IEEE Trans. Automat. Contr.,* vol. 38, pp. 351–354, Feb. 1993.

Generating Matrices for the Discrete Sine Transforms

Victoria Sánchez, Antonio M. Peinado, José C. Segura, Pedro García, and Antonio J. Rubio

***Abstract*— In this correspondence, we obtain a general form for the generating matrices of the eight types of discrete sine transforms. These matrix forms can be used to study the performance of the different discrete sine transforms as substitutes of the Karhunen–Loeve transform and will allow us to show in a very straightforward way that the discrete sine transforms, as the discrete cosine transforms, are asymptotically optimal for any finite order Markov process.**

I. Introduction

The discrete sine transform (DST) was first introduced by Jain [1] in 1976, and several versions of this original DST were later developed by Kekre *et al.* [2], Jain [3], and Wang *et al.* [4], who finally established that there exist four even DST's and four odd DST's, which he numbered from *I* to *IV* with letter *E* or *O*, indicating

Manuscript received November 12, 1995; revised March 10, 1996. The associate editor coordinating the review of this paper and approving it for publication was Dr. Ananthram Swami.

The authors are with the Departamento de Electrónica y Tecnología de Computadores, Facultad de Ciencias, Universidad de Granada, 18071-Granada, Spain.

Publisher Item Identifier S 1053-587X(96)07124-3.

whether they are an even or an odd transform. Ever since the introduction of the first version of the DST, the different DST's have found wide application in several areas of digital signal processing such as image processing [1], [5], [6], adaptive digital filtering [7], and interpolation [8].

We have recently obtained [9] a general form of matrices whose eigenvectors constitute the different discrete cosine transforms (DCT's). In this correspondence, we now propose a general form for the generating matrices of the different types of DST's and use these matrices to study the asymptotic performance of the different DST's with block length N when applied to a stationary process.

II. GENERATING MATRICES

In [9], we obtained the eight types of discrete cosine transforms as the complete orthonormal set of eigenvectors generated by a general form of matrices that can be decomposed as the sum of a symmetric Toeplitz matrix plus a Hankel or close to Hankel matrix scaled by some constant factors. Our development was based on a recent work by Martucci [10], where the relation between symmetric convolution and the discrete sine and cosine transforms was established. Following a similar procedure to that developed in [9], we are going now to obtain a general form for the generating matrices of the eight types of discrete sine transforms. In this case, if we analyze the list of the 40 different types of symmetric convolution [10], we can observe that for each DST in convolution form $\mathcal{S}_a$, we now have a convolution-multiplication expression of the form

$$w_n = \varepsilon_a\{x_n\} \circledast \varepsilon_b\{y_n\} = \mathcal{S}_a^{-1}\{\mathcal{S}_a\{x_n\} \times \mathcal{C}_b\{y_n\}\} \tag{1}$$

where the inverse discrete sine transform applied $\mathcal{S}_a^{-1}$ is of the same type as one of the direct transforms used $\mathcal{S}_a$ and where transform $\mathcal{C}_b$ is one of the discrete cosine transforms in convolution form. w_n is the symmetric convolution of sequences $\boldsymbol{x}$ and $\boldsymbol{y}$, ε_a and ε_b are two symmetric extension operators, and the symbol $\circledast$ represents the convolution operation that can be either circular or skew circular. We will use $[\mathcal{S}^a]$ to indicate the matrix form of transform $\mathcal{S}_a$, and the relationship between the orthogonal and the convolution forms of each discrete sine transform will be expressed as

$$[S^A] = [D_l^a][\mathcal{S}^a][D_r^a] \tag{2}$$

where $[S^A]$ denotes the orthogonal form of transform $[\mathcal{S}^a]$, and $[D_l^a]$ and $[D_r^a]$ are two nonsingular diagonal matrices that depend on the type of DST being considered.

Expressing (1) in matrix form and following a similar development to that presented in [9] for the discrete cosine transforms, we obtain

$$[S^A][D_r^a]^{-1}[\mathcal{Y}^a][D_r^a][S^A]^{-1} = [D([\mathcal{C}^b]\boldsymbol{y})] \tag{3}$$

where $[\mathcal{Y}^a]$ is a matrix whose elements are built from the elements of sequence $\boldsymbol{y}$, and $[D([\mathcal{C}^b]\boldsymbol{y})]$ represents a diagonal matrix with diagonal elements given by $[\mathcal{C}^b]\boldsymbol{y}$. From (3), we obtain that matrix $[Y^A] = [D_r^a]^{-1}[\mathcal{Y}^a][D_r^a]$ is diagonalized by the DST given by $[S^A]$ and therefore, we can conclude that the DST's can be obtained as the eigenvectors of such matrices with eigenvalues λ^{S_A} given by $[\mathcal{C}^b]\boldsymbol{y}$. We have built matrix $[Y^A] = [D_r^a]^{-1}[[\mathcal{Y}_t^a] + [\mathcal{Y}_h^a]][D_r^a]$ for each type of discrete sine transform, and we have obtained that it can also be decomposed as the sum of a Toeplitz symmetric matrix $[\mathcal{Y}_t^a]$ plus a Hankel or close to Hankel matrix $[\mathcal{Y}_h^a]$ scaled by some constant factors.

We give below the listing of the generating matrices for the eight types of discrete sine transforms where we have indicated by a subindex included in parenthesis ( ) the dimension of the corresponding matrix. We only give explicitly the form of the Hankel or close-to-Hankel matrix as the Toeplitz symmetric matrix is the same one in all cases (except for the dimension in some cases) and is given by

$$[\mathcal{Y}_t^a] = \begin{pmatrix} y_0 & y_1 & \cdots & y_{N-2} & y_{N-1} \\ y_1 & \ddots & \ddots & & y_{N-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ y_{N-2} & & \ddots & \ddots & y_1 \\ y_{N-1} & y_{N-2} & \cdots & y_1 & y_0 \end{pmatrix}. \tag{4}$$

A. Matrix Forms

DST-*IE*:

$$y_n, \qquad 0 \le n \le N$$

$$[D_{(N-1),r}^{1e}] = [I]$$

$$[\boldsymbol{\mathcal{Y}}_{(N-1),h}^{1e}] = \begin{pmatrix} -y_2 & -y_3 & \cdots & -y_{N-1} & -y_N \\ -y_3 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & -y_{N-1} \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ -y_{N-1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & -y_3 \\ -y_N & -y_{N-1} & \cdots & -y_3 & -y_2 \end{pmatrix}$$

$$[Y_{(N-1)}^{IE}] = [\mathcal{Y}_{(N-1),t}^{1e}] + [\mathcal{Y}_{(N-1),h}^{1e}]$$

$$\lambda_{(N-1)}^{IE} = \{[\mathcal{C}_{(N+1)}^{1e}]\boldsymbol{y}\}_{1,\cdots,N-1}.$$

DST-*IIE*:

$$y(n), \qquad 0 \le n \le N$$

$$[D_{(N),r}^{2e}] = [I]$$

$$[\mathcal{Y}_{(N),h}^{2e}] = \begin{pmatrix} -y_1 & -y_2 & \cdots & -y_{N-1} & -y_N \\ -y_2 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & -y_{N-1} \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ -y_{N-1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & -y_2 \\ -y_N & -y_{N-1} & \cdots & -y_2 & -y_1 \end{pmatrix}$$

$$[Y_{(N)}^{IIE}] = [\mathcal{Y}_{(N),t}^{2e}] + [\mathcal{Y}_{(N),h}^{2e}]$$

$$\lambda_{(N)}^{IIE} = \{[\mathcal{C}_{(N+1)}^{1e}]\boldsymbol{y}\}_{0,\cdots,N-1}.$$

DST-*IIIE*:

$$y_n, \qquad 0 \le n \le N-1$$

$$[D_{(N),r}^{3e}] = \mathrm{diag}(1, \cdots, 1, \sqrt{2})$$

$$[\mathcal{Y}_{(N),h}^{3e}] = \begin{pmatrix} -y_2 & \cdots & -y_{N-1} & 0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & y_{N-1} & 0 \\ -y_{N-1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & y_{N-2} & 0 \\ 0 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots & 0 \\ y_{N-1} & y_{N-2} & \cdots & y_1 & 0 \end{pmatrix}$$

$$[Y_{(N)}^{IIIE}] = [D_{(N),r}^{3e}]^{-1}[[\mathcal{Y}_{(N),t}^{3e}] + [\mathcal{Y}_{(N),h}^{3e}]][D_{(N),r}^{3e}]$$

$$\lambda_{(N)}^{IIIE} = [\mathcal{C}_{(N)}^{3e}]\boldsymbol{y}.$$

DST-*IVE*:

$$y_n, \qquad 0 \le n \le N-1$$

$$[D_{(N),r}^{4e}] = [I]$$

$$[\mathcal{Y}_{(N),h}^{4e}] = \begin{pmatrix} -y_1 & -y_2 & \cdots & -y_{N-1} & 0 \\ -y_2 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & y_{N-1} \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ -y_{N-1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & y_2 \\ 0 & y_{N-1} & \cdots & y_2 & y_1 \end{pmatrix}$$

$$[Y_{(N)}^{IVE}] = [\mathcal{Y}_{(N),t}^{4e}] + [\mathcal{Y}_{(N),h}^{4e}]$$

$$\lambda_{(N)}^{IVE} = [\mathcal{C}_{(N)}^{3e}]\boldsymbol{y}.$$

DST-*IO*:

$$y_n, \qquad 0 \le n \le N-1$$

$$[D^{1o}_{(N-1),r}] = [I]$$

$$[\mathcal{Y}^{1o}_{(N-1),h}] = \begin{pmatrix} -y_2 & -y_3 & \cdots & -y_{N-1} & -y_{N-1} \\ -y_3 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & -y_{N-2} \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ -y_{N-1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & -y_2 \\ -y_{N-1} & -y_{N-2} & \cdots & -y_2 & -y_1 \end{pmatrix}$$

$$[Y^{IO}_{(N-1)}] = [\mathcal{Y}^{1o}_{(N-1),t}] + [\mathcal{Y}^{1o}_{(N-1),h}]$$

$$\lambda^{IO}_{(N-1)} = \{[\mathcal{C}^{1o}_{(N)}]\boldsymbol{y}\}_{1,\cdots,N-1}.$$

DST-*IIO*:

$$y_n, \qquad 0 \le n \le N-1$$

$$[D^{2o}_{(N-1),r}] = [I]$$

$$[\mathcal{Y}^{2o}_{(N-1),h}] = \begin{pmatrix} -y_1 & -y_2 & \cdots & -y_{N-2} & -y_{N-1} \\ -y_2 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & -y_{N-1} \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ -y_{N-2} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & -y_3 \\ -y_{N-1} & -y_{N-1} & \cdots & -y_3 & -y_2 \end{pmatrix}$$

$$[Y^{IIO}_{(N-1)}] = [\mathcal{Y}^{2o}_{(N-1),t}] + [\mathcal{Y}^{2o}_{(N-1),h}]$$

$$\lambda^{IIO}_{(N-1)} = \{[\mathcal{C}^{1o}_{(N)}]\boldsymbol{y}\}_{0,\cdots,N-2}.$$

DST-*IIIO*:

$$y_n, \qquad 0 \le n \le N-1$$

$$[D^{3o}_{(N-1),r}] = [I]$$

$$[\mathcal{Y}^{3o}_{(N-1),h}] = \begin{pmatrix} -y_2 & -y_3 & \cdots & -y_{N-1} & y_{N-1} \\ -y_3 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & y_{N-2} \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ -y_{N-1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & y_2 \\ y_{N-1} & y_{N-2} & \cdots & y_2 & y_1 \end{pmatrix}$$

$$[Y^{IIIO}_{(N-1)}] = [\mathcal{Y}^{3o}_{(N-1),t}] + [\mathcal{Y}^{3o}_{(N-1),h}]$$

$$\lambda^{IIIO}_{(N-1)} = \{[\mathcal{C}^{3o}_{(N)}]\boldsymbol{y}\}_{1,\cdots,N-1}.$$

DST-*IVO*:

$$y_n, \qquad 0 \le n \le N-1$$

$$[D^{4o}_{(N),r}] = \mathrm{diag}(1, \cdots, 1, \sqrt{2})$$

$$[\mathcal{Y}^{4o}_{(N),h}] = \begin{pmatrix} -y_1 & \cdots & -y_{N-2} & -y_{N-1} & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & y_{N-1} & 0 \\ -y_{N-2} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & y_{N-2} & 0 \\ -y_{N-1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots & 0 \\ y_{N-1} & y_{N-2} & \cdots & y_1 & 0 \end{pmatrix}$$

$$[Y^{IVO}_{(N)}] = [D^{4o}_{(N),r}]^{-1}[[\mathcal{Y}^{4o}_{(N),t}] + [\mathcal{Y}^{4o}_{(N),h}]][D^{4o}_{(N),r}]$$

$$\lambda^{IVO}_{(N)} = [\mathcal{C}^{3o}_{(N)}]\boldsymbol{y}.$$

It can now be shown that the generating matrices that were proposed by Jain [1], [3] for the DST-*IE*, DST-*IIE*, DST-*IVE*, DST-*IO*, DST-*IIO* and DST-*IIIO* are simply particular cases of the matrices presented above if we make $y_0 = 1, y_1 = -\alpha, y_2 = \cdots = y_{N-1} = y_N = 0$ in all cases.

Using these matrices, we can as well study the asymptotic behavior of the DST's with block length N for any stationary process. Following the formalism developed by Yemini and Pearl [11] and the same kind of development as that presented in [9] for the discrete cosine transforms, we have found for each DST a class of stationary processes verifying certain conditions with respect to which the corresponding DST has a good asymptotic behavior in the sense that it approaches Karhunen–Loeve transform performance as block size N tends to infinity. If $r_{|i-j|} = r_n$ are the elements of the autocovariance matrix of a stationary process, we have a good asymptotic performance for the following:

1) the DST-*IE* for those stationary processes for which $\Sigma_{n=2}^{\infty}(n-1)r_n^2 < \infty$
2) the DST-*IIE* and DST-*IVE* for those verifying $\Sigma_{n=1}^{\infty}\ nr_n^2 < \infty$
3) the DST-*IIIE* for those verifying $\Sigma_{n=1}^{\infty}\ r_n^2 < \infty, \Sigma_{n=2}^{\infty}(n-1)r_n^2 < \infty$ and $\Sigma_{n=1}^{\infty}\ nr_n^2 < \infty$
4) the DST-*IO*, DST-*IIO* and DST-*IIIO* for those verifying $\Sigma_{n=2}^{\infty}(n-1)r_n^2 < \infty$ and $\Sigma_{n=1}^{\infty}\ nr_n^2 < \infty$
5) the DST-*IVO* if $\Sigma_{n=1}^{\infty}\ r_n^2 < \infty$ and $\Sigma_{n=1}^{\infty}\ nr_n^2 < \infty$.

All three conditions $\Sigma_{n=1}^{\infty}\ r_n^2 < \infty, \Sigma_{n=2}^{\infty}(n-1)r_n^2 < \infty$ and $\Sigma_{n=1}^{\infty}\ nr_n^2 < \infty$ are verified for finite-order Markov processes; therefore, we can conclude that as in the case of the discrete cosine transforms, the discrete sine transforms also have a good asymptotic behavior with these kinds of processes.

REFERENCES

[1] A. K. Jain, "A fast Karhunen–Loeve transform for a class of random processes," *IEEE Trans. Commun.*, vol. COM-24, pp. 1023–1029, Sept. 1976.

[2] H. B. Kekre and J. K. Solanka, "Comparative performance of various trigonometric unitary transforms for transform image coding," *Int. J. Electron.*, vol. 44, pp. 305–315, 1978.

[3] A. Jain, "A sinusoidal family of unitary transforms," *IEEE Trans. Patt. Anal. Machine Intell.*, pp. 356–365, Sept. 1979.

[4] Z. Wang and B. Hunt, "The discrete W transform," *Applied Math. Computat.*, vol. 16, pp. 19–48, Jan. 1985.

[5] S. Cheng, "Application of the sine transform method in time of flight positron emission image reconstruction algorithms," *IEEE Trans. Biomed. Eng.*, vol. BME-32, pp. 185–192, Mar. 1985.

[6] K. Rose, A. Heiman, and I. Dinstein, "DCT/DST alternate transform image coding," in *Proc. GLOBECOM 87*, vol. 1, Nov. 1987, pp. 426–30.

[7] J. L. Wang and Z. Q. Ding, "Discrete sine transform domain LMS adaptive filtering," in *Proc. Int. Conf. Acoust., Speech, Signal Processing*, 1985, pp. 260–263.

[8] Z. Wang and L. Wang, "Interpolation using the fast discrete sine transform," *Signal Processing*, vol. 26, pp. 131–137, 1992.

[9] V. Sánchez, P. García, A. M. Peinado, J. C. Segura, and A. J. Rubio, "Diagonalizing properties of the discrete cosine transforms," *IEEE Trans. Signal Processing*, pp. 2631–2641, Nov. 1995.

[10] S. A. Martucci, "Symmetric convolution and the discrete sine and cosine transforms," *IEEE Trans. Signal Processing*, pp. 1038–1051, May 1994.

[11] Y. Yemini and J. Pearl, "Asymptotic properties of discrete unitary transforms," *IEEE Trans. Patt. Anal. Machine Intell.*, pp. 366–371, Oct. 1979.