Diagonalizing Properties of the Discrete Cosine Transforms

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***Abstract*—There exist eight types of discrete cosine transforms (DCT's). In this paper, we obtain the eight types of DCT's as the complete orthonormal set of eigenvectors generated by a general form of matrices in the same way as the discrete Fourier transform (DFT) can be obtained as the eigenvectors of an arbitrary circulant matrix. These matrices can be decomposed as the sum of a symmetric Toeplitz matrix plus a Hankel or close to Hankel matrix scaled by some constant factors. We also show that all the previously proposed generating matrices for the DCT's are simply particular cases of these general matrix forms. Using these matrices, we obtain, for each DCT, a class of stationary processes verifying certain conditions with respect to which the corresponding DCT has a good asymptotic behavior in the sense that it approaches Karhunen–Loeve transform performance as block size N tends to infinity. As a particular result, we prove that the eight types of DCT's are asymptotically optimal for all finite-order Markov processes. We finally study the decorrelating power of the DCT's, obtaining expressions that show the decorrelating behavior of each DCT with respect to any stationary processes.**

I. Introduction

SINCE its introduction in 1974 by Ahmed *et al.* [1], the discrete cosine transform (DCT) has become a significant tool in many areas of digital signal processing, especially in signal compression [2]. The original motivation for defining the DCT was that its basis set provided a good approximation to the eigenvectors of the class of Toeplitz matrices that constitutes the autocovariance matrix of a first-order stationary Markov process, with the result that it had a better performance than the discrete Fourier transform (DFT) and some other transforms [1], [3], [4] with respect to such kinds of processes. In fact, as shown in [2], the DCT is asymptotically equivalent to the Karhunen–Loeve transform (KLT) of a first-order stationary Markov process as ρ tends to 1, where ρ is the correlation coefficient. Some years later, Jain [5] proposed two new types of DCT, which he called the even discrete cosine transform 2 (EDCT-2) and the odd discrete cosine transform 1 (ODCT-1), and almost simultaneously, Kitajima [6] constructed a symmetric version of the DCT whose basis set approached the eigenvectors of the KLT of a first-order stationary Markov process as block size N tends to infinity. Finally, Wang [7] showed that there exist eight types of DCT's and classified them in even and odd transforms. There exist four even DCT's and four odd DCT's, which he numbered from I to IV with letter E or O indicating whether they were an even or an odd transform. In this way, the original DCT proposed by Ahmed *et al.* [1] is nowadays known as the DCT-*IIE*, the two new DCT's proposed by Jain are the DCT-*IVE* and the DCT-*IVO*, respectively, and the symmetric cosine transform proposed by Kitajima is known as the DCT-*IE*.

Jain [5] first proposed a parametric family of matrices, which is a variation of the tridiagonal Jacobi matrix, whose eigenvectors constituted the basis set for some types of DCT's, in particular, for the DCT-*IIE*, the DCT-*IVE* and the DCT-*IVO*. Kitajima also proposed a generating matrix for his symmetric discrete cosine transform. In this paper, we will obtain the eight types of DCT's as the complete orthonormal set of eigenvectors generated by a general form of matrices in the same way as the discrete Fourier transform can be obtained as the eigenvectors of an arbitrary circulant matrix. These matrices can be decomposed as the sum of a symmetric Toeplitz matrix plus a Hankel or close to Hankel matrix scaled by some constant factors. We will show that all the previously proposed generating matrices for the DCT's are simply particular cases of these general matrix forms. Our development is based on a recent work by Martucci [8], [9], where the relation between symmetric convolution and the discrete sine and cosine transforms is established.

As indicated above, the motivation for originally defining the DCT-*IE* [6] was that its basis set provided a good approximation to the eigenvectors of the autocovariance matrix of the stationary Markov-1 process as $N \to \infty$. Such a good asymptotic behavior with block size N of the DCT with respect to any stationary finite-order Markov process has been proven in the cases of the DCT-*IIE*, the DCT-*IVE*, and the DCT-*IVO* [5]. In this paper, we will obtain, for each DCT, a class of stationary processes verifying certain conditions with respect to which the corresponding DCT has a good asymptotic behavior in the sense that it approaches KLT performance as block size N tends to ∞. As a particular result, we will extend the good asymptotic behavior of the DCT-*IIE*, the DCT-*IVE*, and the DCT-*IVO* with respect to any stationary finite-order Markov process (previously established by Jain [5]) to the rest of the DCT's, concluding that the eight types of DCT's are asymptotically optimal for all finite-order Markov processes.

Apart from showing that all the DCT's have a good asymptotic behavior for stationary processes verifying certain conditions, we are also interested in the rate at which each one

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of those transforms decorrelates a stationary process because this rate will determine quality ranking among the different DCT's [10]. A good measure of the degree of correlation still remaining after the application of a specific transform is given by the norm of the matrix containing the off-diagonal covariance elements of the transformed coefficients [11]. This norm was shown to control the performance degradation resulting from residual correlation in both coding and filtering [11]. We will refer to this norm as residual correlation from now on. Several attempts were made in the past to find analytical expressions for first-order stationary Markov processes that showed the residual correlation as a function of the correlation coefficient ρ and dimension N. Hamidi *et al.* [3] and Kitajima [6] obtained this dependence for the DCT-*IIE* and the DCT-*IE*, respectively. Jain [5] developed expressions for the DCT-*IIE*, the DCT-*IVE*, and the DCT-*IVO* that show how the performance of these transforms depends on ρ, ignoring its dependence on N.

In this paper, we obtain expressions that show how the residual correlation for each one of the DCT's depends on N and the covariance matrix elements $r_{|i-j|}, 0 \leq i, j \leq N-1$ for any stationary process. These expressions allow an analysis of the decorrelation power of each one of the DCT's for any given stationary process and lead us to derive, among other results, that the DCT-*IO* and the DCT-*IIO* have the same decorrelating power for any stationary process and, when those expressions are applied to a first-order stationary Markov process, we obtain that in the same way as the DCT-*IIE* is the best discrete cosine transform for very highly positive correlated processes, the DCT-*IIIO* is the best discrete cosine transform for very highly negative correlated processes for $N > 2$.

The rest of this paper is organized as follows: In Section II, we first present the main results established by Martucci [8], [9], relating the symmetric convolution and the discrete trigonometric transforms, and on that basis, we then obtain a general form of the generating matrices for the eight types of DCT's and show that all the previously proposed generating matrices for the DCT's are simply particular cases of these general matrix forms. Section III contains a study of the asymptotic behavior of the DCT's with stationary processes, and in Section IV, we derive expressions that show the decorrelating power of each DCT for any stationary process. Finally, a brief summary is given in Section V.

II. Generating Matrices for the Discrete Cosine Transforms

It is well known that the DFT can be obtained as the eigenvectors of an arbitrary circulant matrix, the eigenvalues of the matrix given by the DFT of the circulant elements. However, no general matrix forms have been established whose eigenvectors constitute the different DCT's; only some particular cases of matrices diagonalized by certain DCT's have been presented in the past [5], [6], [12]. In this section, we will obtain a general matrix form for each DCT type and show that the previously obtained matrices are simply particular cases of these general matrix forms.

A. Convolution-Multiplication Properties of the Discrete Trigonometric Transforms

Martucci [9] has recently presented the convolution-multiplication properties of the discrete trigonometric transforms (DTT's) that include the eight types of discrete sine transforms (DST's) and the eight types of discrete cosine transforms [7]. In the same way as circular convolution is the type of convolution related to the DFT, the symmetric convolution is the type of convolution related to the DTT's [9].

Let $\boldsymbol{x} = [x_0, x_1, \cdots, x_{N-1}]^T$ and $\boldsymbol{y} = [y_0, y_1, \cdots, y_{N-1}]^T$ be two finite sequences. The convolution-multiplication properties of the DTT's are expressed, according to [9], in the following two equations:

$$w_n = (\varepsilon_a\{x_n\} \circledast \varepsilon_b\{y_n\})\mathcal{R}_n^K \tag{1}$$

$$w_{n-n_0} = \mathcal{T}_c^{-1}\{\mathcal{T}_a\{x_n\} \times \mathcal{T}_b\{y_n\}\} \tag{2}$$

where w_n is the symmetric convolution of sequences $\boldsymbol{x}$ and $\boldsymbol{y}$. In (1), ε_a and ε_b are two symmetric extension operators that convert a finite sequence into the base period of a symmetric-periodic sequence as defined in [9], the symbol $\circledast$ represents the convolution operation that can be either a circular or skew-circular convolution, and $\mathcal{R}_n^K$ is a length-K rectangular window whose purpose is to extract the representative samples. Equation (2) represents an alternative way of determining the symmetric convolution of sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ using transforms. $\mathcal{T}_a$ and $\mathcal{T}_b$ are the corresponding DTT's of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, and $\mathcal{T}_c^{-1}$ is the appropriate inverse transform [9]. The symbol $\times$ indicates element-by-element multiplication. As far as n_0 is concerned, it can have two values: 0 or 1. In the case $n_0 = 1$, that means that from the inverse transform, we get the delayed result of the symmetric convolution. We have to point out that transforms $\mathcal{T}_a$, $\mathcal{T}_b$, and $\mathcal{T}_c^{-1}$ are in convolution form, which is a new formulation for the DTT's proposed by Martucci and different from the orthogonal form previously established by Wang [7] for the DTT's. The convolution form is more suitable for expressing the convolution-multiplication properties of the DTT's, although the transform matrices corresponding to the DTT's in convolution form may no longer be orthogonal. The orthogonal forms of the DTT's are enumerated in [7], and the convolution forms can be found in [9]. The eight types of DCT's in convolution form will be denoted as $\mathcal{C}_{1e}$, $\mathcal{C}_{2e}$, $\mathcal{C}_{3e}$, and $\mathcal{C}_{4e}$ for the even versions and $\mathcal{C}_{1o}$, $\mathcal{C}_{2o}$, $\mathcal{C}_{3o}$, and $\mathcal{C}_{4o}$ for the odd versions.

B. Generating Matrices

There are 40 different types of symmetric convolution listed in [9]. Analyzing that list, we have observed that for each DCT in convolution form $\mathcal{C}_a$, there is a convolution-multiplication expression of the form

$$w_n = \varepsilon_a\{x_n\} \circledast \varepsilon_b\{y_n\} = \mathcal{C}_a^{-1}\{\mathcal{C}_a\{x_n\} \times \mathcal{C}_b\{y_n\}\} \tag{3}$$

where the inverse transform applied $\mathcal{C}_a^{-1}$ is of the same type as one of the direct transforms used $\mathcal{C}_a$ and where transform $\mathcal{C}_b$ can be different from $\mathcal{C}_a$ in the most general case. From now on, we will consider that the rectangular window $\mathcal{R}_n^K$ is implicit and ignore n_0 as it is equal to 0 in all types of convolution-multiplication relations in which we are interested.

Let us express (3) in matrix form. We will use $[\mathcal{C}^a]$ to indicate the matrix expression of transform $\mathcal{C}_a$ and $[\mathcal{C}^a]_{m,n}$ for the specific entry at row m and column n. We have to point out that $[\mathcal{C}^a]\boldsymbol{x} \times [\mathcal{C}^b]\boldsymbol{y}$ is not a matrix operation, the resulting vector having as elements the result of the element-by-element multiplication of vectors $[\mathcal{C}^a]\boldsymbol{x}$ and $[\mathcal{C}^b]\boldsymbol{y}$. We will indicate this including the resulting vector in parenthesis (), obtaining the following equation:

$$\boldsymbol{w} = [\mathcal{Y}^a]\boldsymbol{x} = [\mathcal{C}^a]^{-1}\big([\mathcal{C}^a]\boldsymbol{x} \times [\mathcal{C}^b]\boldsymbol{y}\big). \tag{4}$$

Matrix $[\mathcal{Y}^a]$ is a square matrix whose elements can be expressed as combinations of the elements of sequence $\boldsymbol{y}$. We have built matrix $[\mathcal{Y}^a]$ for each type of symmetric convolution of the form given by (3), and we will show in Section II-B-1 that it can be decomposed as $[\mathcal{Y}^a] = [\mathcal{Y}^a_t] + [\mathcal{Y}^a_h]$, where $[\mathcal{Y}^a_t]$ is a Toeplitz symmetric matrix, and $[\mathcal{Y}^a_h]$ is a Hankel matrix or close to a Hankel matrix. In the cases when $[\mathcal{Y}^a_h]$ is close to a Hankel matrix, all the elements along any cross diagonal are identical except the first or last element, which are equal to zero.

We will next express the term $\big([\mathcal{C}^a]\boldsymbol{x} \times [\mathcal{C}^b]\boldsymbol{y}\big)$ of (4) as a matrix operation. In fact, the operation $\boldsymbol{x} \times \boldsymbol{y} = \boldsymbol{y} \times \boldsymbol{x}$ can be put into matrix form as $[D(\boldsymbol{x})]\boldsymbol{y} = [D(\boldsymbol{y})]\boldsymbol{x}$, where $[D(\boldsymbol{x})]$ is a diagonal matrix whose diagonal elements are the components of vector $\boldsymbol{x}$. Doing so, we have

$$[\mathcal{Y}^a]\boldsymbol{x} = [\mathcal{C}^a]^{-1}[D([\mathcal{C}^b]\boldsymbol{y})][\mathcal{C}^a]\boldsymbol{x}. \tag{5}$$

Given a certain $[\mathcal{Y}^a]$, (5) must be verified for any $\boldsymbol{x}$, which implies that

$$[\mathcal{Y}^a] = [\mathcal{C}^a]^{-1}[D([\mathcal{C}^b]\boldsymbol{y})][\mathcal{C}^a] \tag{6}$$

or, equivalently

$$[\mathcal{C}^a][\mathcal{Y}^a][\mathcal{C}^a]^{-1} = [D([\mathcal{C}^b]\boldsymbol{y})]. \tag{7}$$

Let us now express the relation between the orthogonal and the convolution forms of the DCT's in a form that suits our purposes. Denoting as $[C^A]$ the orthogonal form of transform $\mathcal{C}_a$, we have

$$[C^A] = [D^a_l][\mathcal{C}^a][D^a_r] \tag{8}$$

$$[C^A]^{-1} = [D^a_r]^{-1}[\mathcal{C}^a]^{-1}[D^a_l]^{-1} \tag{9}$$

where $[D^a_l]$ and $[D^a_r]$ are two nonsingular diagonal matrices that depend on the type of DCT being considered.

Using (8) and (9), we can express (7) in terms of transforms in orthogonal form as follows:

$$[D^a_l]^{-1}[C^A][D^a_r]^{-1}[\mathcal{Y}^a][D^a_r][C^A]^{-1}[D^a_l] = [D([\mathcal{C}^b]\boldsymbol{y})] \tag{10}$$

and finally, as the first term of (10) is the product of three diagonal matrices, and diagonal matrices commute, we have

$$[C^A][D^a_r]^{-1}[\mathcal{Y}^a][D^a_r][C^A]^{-1} = [D([\mathcal{C}^b]\boldsymbol{y})]. \tag{11}$$

We obtain, in consequence, that matrix $[Y^A] = [D^a_r]^{-1}[\mathcal{Y}^a][D^a_r] = [D^a_r]^{-1}[[\mathcal{Y}^a_t] + [\mathcal{Y}^a_h]][D^a_r]$ is diagonalized by the DCT given by $[C^A]$. Thus, we can conclude that the DCT's can be obtained as the eigenvectors of such matrices with eigenvalues $\boldsymbol{\lambda}^{C_A}$ given by $[\mathcal{C}^b]\boldsymbol{y}$.

1) Matrix Forms: We have indicated above that matrix $[\mathcal{Y}^a]$ is built from the elements of sequence $\boldsymbol{y}$ and can be decomposed as the sum of a symmetric Toeplitz matrix $[\mathcal{Y}^a_t]$ and a Hankel, or close to a Hankel, matrix $[\mathcal{Y}^a_h]$. We will now show how we have derived this result for the DCT-*IIE*, which is the most popular of the DCT's and the first one to be proposed [1]. A similar procedure can be followed for the rest of the transforms where the only difference is the symmetric extension operators that have to be applied in each case and the kind of convolution operation we have to perform (circular or skew-circular).

As stated above, our development is based on the convolution-multiplication expressions of the form given by (3). In the case of the DCT-*IIE*, we have

$$w_n = \text{HSHS}\{x_n\} \copyright \text{WSWS}\{y_n\} = \mathcal{C}_{2e}^{-1}\{\mathcal{C}_{2e}\{x_n\} \times \mathcal{C}_{1e}\{y_n\}\} \tag{12}$$

where $\copyright$ represents the circular convolution, $\mathcal{C}_{2e}$ is the convolution form of the DCT-*IIE*, $\mathcal{C}_{1e}$ is the convolution form of the DCT-*IE* and, HSHS and WSWS are two symmetric extension operators that, when applied to sequences $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, generate sequences $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ of the form

$$\hat{x}_n = \begin{cases} x_n & n = 0, 1, \cdots, N-1 \\ x_{2N-1-n} & n = N, \cdots, 2N-1 \end{cases} \tag{13}$$

$$\hat{y}_n = \begin{cases} y_n & n = 0, 1, \cdots, N \\ y_{2N-n} & n = N+1, \cdots, 2N-1. \end{cases} \tag{14}$$

The next step is to perform the circular convolution $\copyright$ of sequences $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Let us express this in matrix form:

$$\hat{\boldsymbol{x}} \copyright \hat{\boldsymbol{y}} = \begin{pmatrix} y_0 & \cdots & y_{N-1} & y_N & y_{N-1} & \cdots & y_1 \\ \vdots & \ddots & & \ddots & \ddots & \ddots & \vdots \\ y_{N-1} & & \ddots & & \ddots & \ddots & y_{N-1} \\ y_N & \ddots & & \ddots & & \ddots & y_N \\ y_{N-1} & \ddots & \ddots & & \ddots & & y_{N-1} \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ y_1 & \cdots & y_{N-1} & y_N & y_{N-1} & \cdots & y_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \\ x_{N-1} \\ \vdots \\ x_1 \\ x_0 \end{pmatrix}. \tag{15}$$

The result of that matrix product is a sequence of length $2N$. The result of the symmetric convolution corresponds, in this case, to the first N elements of that sequence that can be expressed in matrix form as in (16), which appears at the bottom of the next page.

Performing the corresponding operations, we would finally obtain the length-N sequence in (17), which is shown at the bottom of the next page

The expression for this length-N sequence can be alternatively expressed in matrix form as the product of an $N \times N$ matrix and the length-N sequence $x_n, n = 0, \ldots, N-1$. This is shown in (18) at the bottom of the next page.

That means we can express the symmetric convolution $\boldsymbol{w}$ as

$$\boldsymbol{w} = [[\mathcal{Y}^{2e}_{(N),t}] + [\mathcal{Y}^{2e}_{(N),h}]]\boldsymbol{x} \tag{19}$$

where we have indicated by a subindex included in parenthesis () the dimension of the corresponding matrix. In the case of the DCT-*IIE*, the diagonal matrix $[D^{IIE}_{(N),r}]$ is equal to the identity matrix $[I_{(N)}]$. Consequently, we finally have that the matrix diagonalized by the DCT-*IIE* is given by

$$[Y^{IIE}_{(N)}] = [\mathcal{Y}^{2e}_{(N),t}] + [\mathcal{Y}^{2e}_{(N),h}] \tag{20}$$

with eigenvalues

$$\lambda^{IIE}_{(N)} = \{[\mathcal{C}^{1e}_{(N+1)}]\boldsymbol{y}\}_{0,\ldots,N-1}. \tag{21}$$

Following a similar procedure with the rest of the DCT's, we have obtained the generating matrices for each transform together with their corresponding eigenvalues, which are listed below. In all cases, the Toeplitz symmetric matrix $[\mathcal{Y}^a_t]$ is the same, and it is equal to the Toeplitz symmetric matrix obtained in the case of the DCT-*IIE*, $[\mathcal{Y}^{2e}_{(N),t}]$, with the only difference being the dimension in some cases. The Hankel, or close to Hankel, matrix $[\mathcal{Y}^a_h]$ is different for each DCT, and for that reason, we will only give explicitly the form of this matrix in the listing below.

DCT-*IE*

$$y_n, 0 \le n \le N$$

$$[D^{1e}_{(N+1),r}] = \text{diag}(1, 1/\sqrt{2}, \ldots, 1/\sqrt{2}, 1)$$

$$[\mathcal{Y}^{1e}_{(N+1),h}] = \begin{pmatrix} 0 & y_1 & \cdots & y_{N-1} & 0 \\ 0 & \vdots & \cdots & y_N & 0 \\ 0 & y_{N-1} & \cdots & y_{N-1} & 0 \\ 0 & y_N & \cdots & \vdots & 0 \\ 0 & y_{N-1} & \cdots & y_1 & 0 \end{pmatrix}$$

$$[Y^{IE}_{(N+1)}] = [D^{1e}_{(N+1),r}]^{-1}[[\mathcal{Y}^{1e}_{(N+1),t}] + [\mathcal{Y}^{1e}_{(N+1),h}]][D^{1e}_{(N+1),r}]$$

$$\lambda^{IE}_{(N+1)} = [\mathcal{C}^{1e}_{(N+1)}]\boldsymbol{y}$$

DCT-*IIE*

$$y(n), 0 \le n \le N$$

$$\boldsymbol{w} = \{\hat{\boldsymbol{x}} \circledcirc \hat{\boldsymbol{y}}\}_{0,\ldots,N-1} = \begin{pmatrix} y_0 & \cdots & y_{N-1} & y_N & y_{N-1} & \cdots & y_1 \\ \vdots & \ddots & & \ddots & \ddots & \ddots & \vdots \\ y_{N-2} & & \ddots & & \ddots & \ddots & y_{N-1} \\ y_{N-1} & y_{N-2} & \cdots & y_0 & \cdots & y_{N-1} & y_N \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \\ x_{N-1} \\ \vdots \\ x_1 \\ x_0 \end{pmatrix} \tag{16}$$

$$\begin{aligned} \boldsymbol{w} &= \begin{pmatrix} \sum_{n=0}^{N-1} y_n x_n + \sum_{n=1}^{N} y_n x_{n-1} \\ \vdots \\ \sum_{n=0}^{i} y_{i-n}x_n + \sum_{n=1}^{N-1-i} y_n x_{n+i} + \sum_{n=0}^{i} y_{N-(i-n)}x_{N-1-n} + \sum_{n=1}^{N-1-i} y_{N-n}x_{N-1-(n+i)} \\ \vdots \\ \sum_{n=0}^{N-1} y_{N-1-n}x_n + \sum_{n=1}^{N} y_n x_{N-n} \end{pmatrix} \\ &= \begin{pmatrix} (y_0+y_1)x_0 + (y_1+y_2)x_1 + \cdots + (y_{N-2}+y_{N-1})x_{N-2} + (y_{N-1}+y_N)x_{N-1} \\ \vdots \\ \sum_{n=0}^{N-1}(y_{|i-n|} + y_{N-|i+n-(N-1)|})x_n \\ \vdots \\ (y_{N-1}+y_N)x_0 + (y_{N-2}+y_{N-1})x_1 + \cdots + (y_1+y_2)x_{N-2} + (y_0+y_1)x_{N-1} \end{pmatrix} \end{aligned} \tag{17}$$

$$\begin{aligned} \boldsymbol{w} &= \begin{pmatrix} y_0+y_1 & y_1+y_2 & \cdots & y_{N-2}+y_{N-1} & y_{N-1}+y_N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_i + y_{N-|i-(N-1)|} & y_{|i-1|} + y_{N-|i+1-(N-1)|} & \cdots & y_{|i-(N-2)|} + y_{N-|i-1|} & y_{|i-(N-1)|} + y_{N-i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{N-1}+y_N & y_{N-2}+y_{N-1} & \cdots & y_1+y_2 & y_0+y_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{pmatrix} \\ &= \left[\begin{pmatrix} y_0 & y_1 & \cdots & y_{N-2} & y_{N-1} \\ y_1 & \ddots & \ddots & & y_{N-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ y_{N-2} & & \ddots & \ddots & y_1 \\ y_{N-1} & y_{N-2} & \cdots & y_1 & y_0 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 & \cdots & y_{N-1} & y_N \\ y_2 & & \cdots & \cdots & y_{N-1} \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ y_{N-1} & \cdots & \cdots & & y_2 \\ y_N & y_{N-1} & \cdots & y_2 & y_1 \end{pmatrix} \right] \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{pmatrix} \end{aligned} \tag{18}$$

$$[D^{2e}_{(N),r}] = [I]$$

$$[\mathcal{Y}^{2e}_{(N),h}] = \begin{pmatrix} y_1 & y_2 & \cdots & y_{N-1} & y_N \\ y_2 & & \cdots & \cdots & y_{N-1} \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ y_{N-1} & \cdots & \cdots & & y_2 \\ y_N & y_{N-1} & \cdots & y_2 & y_1 \end{pmatrix}$$

$$[Y^{IIE}_{(N)}] = [\mathcal{Y}^{2e}_{(N),t}] + [\mathcal{Y}^{2e}_{(N),h}]$$

$$\boldsymbol{\lambda}^{IIE}_{(N)} = \{[\mathcal{C}^{1e}_{(N+1)}]\boldsymbol{y}\}_{0,\ldots,N-1}$$

DCT-*IIIE*

$$y_n, 0 \le n \le N-1$$

$$[D^{3e}_{(N),r}] = \mathrm{diag}(1, 1/\sqrt{2}, \cdots, 1/\sqrt{2})$$

$$[\mathcal{Y}^{3e}_{(N),h}] = \begin{pmatrix} 0 & y_1 & \cdots & y_{N-2} & y_{N-1} \\ 0 & \vdots & \cdots & \cdots & 0 \\ 0 & y_{N-2} & \cdots & \cdots & -y_{N-1} \\ 0 & y_{N-1} & \cdots & \cdots & \vdots \\ 0 & 0 & -y_{N-1} & \cdots & -y_2 \end{pmatrix}$$

$$[Y^{IIIE}_{(N)}] = [D^{3e}_{(N),r}]^{-1}[[\mathcal{Y}^{3e}_{(N),t}] + [\mathcal{Y}^{3e}_{(N),h}]][D^{3e}_{(N),r}]$$

$$\boldsymbol{\lambda}^{IIIE}_{(N)} = [\mathcal{C}^{3e}_{(N)}]\boldsymbol{y}$$

DCT-*IVE*

$$y_n, 0 \le n \le N-1$$

$$[D^{4e}_{(N),r}] = [I]$$

$$[\mathcal{Y}^{4e}_{(N),h}] = \begin{pmatrix} y_1 & y_2 & \cdots & y_{N-1} & 0 \\ y_2 & & \cdots & \cdots & -y_{N-1} \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ y_{N-1} & \cdots & \cdots & & -y_2 \\ 0 & -y_{N-1} & \cdots & -y_2 & -y_1 \end{pmatrix}$$

$$[Y^{IVE}_{(N)}] = [\mathcal{Y}^{4e}_{(N),t}] + [\mathcal{Y}^{4e}_{(N),h}]$$

$$\boldsymbol{\lambda}^{IVE}_{(N)} = [\mathcal{C}^{3e}_{(N)}]\boldsymbol{y}$$

DCT-*IO*

$$y_n, 0 \le n \le N-1$$

$$[D^{1o}_{(N),r}] = \mathrm{diag}(1, 1/\sqrt{2}, \ldots, 1/\sqrt{2})$$

$$[\mathcal{Y}^{1o}_{(N),h}] = \begin{pmatrix} 0 & y_1 & \cdots & y_{N-2} & y_{N-1} \\ 0 & \vdots & \cdots & \cdots & y_{N-1} \\ 0 & y_{N-2} & \cdots & \cdots & y_{N-2} \\ 0 & y_{N-1} & \cdots & \cdots & \vdots \\ 0 & y_{N-1} & y_{N-2} & \cdots & y_1 \end{pmatrix}$$

$$[Y^{IO}_{(N)}] = [D^{1o}_{(N),r}]^{-1}[[\mathcal{Y}^{1o}_{(N),t}] + [\mathcal{Y}^{1o}_{(N),h}]][D^{1o}_{(N),r}]$$

$$\boldsymbol{\lambda}^{IO}_{(N)} = [\mathcal{C}^{1o}_{(N)}]\boldsymbol{y}$$

DCT-*IIO*

$$y_n, 0 \le n \le N-1$$

$$[D^{2o}_{(N),r}] = \mathrm{diag}(1, \ldots, 1, \sqrt{2})$$

$$[\mathcal{Y}^{2o}_{(N),h}] = \begin{pmatrix} y_1 & \cdots & y_{N-2} & y_{N-1} & 0 \\ \vdots & \cdots & \cdots & y_{N-1} & 0 \\ y_{N-2} & \cdots & \cdots & y_{N-2} & 0 \\ y_{N-1} & \cdots & \cdots & \vdots & 0 \\ y_{N-1} & y_{N-2} & \cdots & y_1 & 0 \end{pmatrix}$$

$$[Y^{IIO}_{(N)}] = [D^{2o}_{(N),r}]^{-1}[[\mathcal{Y}^{2o}_{(N),t}] + [\mathcal{Y}^{2o}_{(N),h}]][D^{2o}_{(N),r}]$$

$$\boldsymbol{\lambda}^{IIO}_{(N)} = [\mathcal{C}^{1o}_{(N)}]\boldsymbol{y}$$

DCT-*IIIO*

$$y_n, 0 \le n \le N-1$$

$$[D^{3o}_{(N),r}] = \mathrm{diag}(1, 1/\sqrt{2}, \ldots, 1/\sqrt{2})$$

$$[\mathcal{Y}^{3o}_{(N),h}] = \begin{pmatrix} 0 & y_1 & \cdots & y_{N-2} & y_{N-1} \\ 0 & \vdots & \cdots & \cdots & -y_{N-1} \\ 0 & y_{N-2} & \cdots & \cdots & -y_{N-2} \\ 0 & y_{N-1} & \cdots & \cdots & \vdots \\ 0 & -y_{N-1} & -y_{N-2} & \cdots & -y_1 \end{pmatrix}$$

$$[Y^{IIIO}_{(N)}] = [D^{3o}_{(N),r}]^{-1}[[\mathcal{Y}^{3o}_{(N),t}] + [\mathcal{Y}^{3o}_{(N),h}]][D^{3o}_{(N),r}]$$

$$\boldsymbol{\lambda}^{IIIO}_{(N)} = [\mathcal{C}^{3o}_{(N)}]\boldsymbol{y}$$

DCT-*IVO*

$$y_n, 0 \le n \le N-1$$

$$[D^{4o}_{(N-1),r}] = [I]$$

$$[\mathcal{Y}^{4o}_{(N-1),h}] = \begin{pmatrix} y_1 & y_2 & \cdots & y_{N-2} & y_{N-1} \\ y_2 & & \cdots & \cdots & -y_{N-1} \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ y_{N-2} & \cdots & \cdots & & -y_3 \\ y_{N-1} & -y_{N-1} & \cdots & -y_3 & -y_2 \end{pmatrix}$$

$$[Y^{IVO}_{(N-1)}] = [\mathcal{Y}^{4o}_{(N-1),t}] + [\mathcal{Y}^{4o}_{(N-1),h}]$$

$$\boldsymbol{\lambda}^{IVO}_{(N-1)} = \{[\mathcal{C}^{3o}_{(N)}]\boldsymbol{y}\}_{0,\ldots,N-2}.$$

The generation of the eight types of DCT's as the eigenvectors of these general matrix forms facilitates the study of the statistical properties of the different DCT's or, more specifically, their performance as substitutes of the KLT. Using these matrix forms, we will obtain, in a very simple and straightforward way, processes that include the finite-order Markov processes for which the different DCT's have a good asymptotic behavior with block size N; we will also obtain analytical expressions that show the decorrelating behavior of each DCT for any stationary process.

C. Previous Results

We will now show that the different types of matrices presented in the past that were diagonalized by the DCT's are simply particular cases of the general expressions obtained in the previous subsection.

The first to propose a parametric family of matrices whose eigenvectors constituted different DCT's was Jain [5]. He considered the parametric family of matrices

$$J = J(k_1, k_2, k_3, k_4) = \begin{pmatrix} 1-k_1\alpha & -\alpha & 0 & \cdots & 0 & k_3\alpha \\ -\alpha & 1 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & & \ddots & \vdots \\ \vdots & \ddots & & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 1 & -\alpha \\ k_4\alpha & 0 & \cdots & 0 & -\alpha & 1-k_2\alpha \end{pmatrix} \tag{22}$$

which is a variation of the tridiagonal Jacobi matrix.

The DCT-*IIE*, DCT-*IVE*, and DCT-*IVO* are obtained according to [5] as the eigenvectors of matrices $J(1,1,0,0)$, $J(1,-1,0,0)$, and $J(1,0,0,0)$, respectively. In fact, these matrices are simply particular cases of matrices $[Y_{(N)}^{IIE}]$, $[Y_{(N)}^{IVE}]$, and $[Y_{(N-1)}^{IVO}]$ described in Section II-B, where $y_0 = 1, y_1 = -\alpha, y_2 = \cdots = y_N = 0$ in the case of the DCT-*IIE* and $y_0 = 1, y_1 = -\alpha, y_2 = \cdots = y_{N-1} = 0$ in the cases of the DCT-*IVE* and DCT-*IVO*.

As far as the DCT-IE is concerned, Kitajima [6] defined it as the eigenvectors of matrix A,

$$A = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & \cdots & \cdots & 0 \\ 1/\sqrt{2} & \ddots & 1/2 & \ddots & & \vdots \\ 0 & 1/2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1/2 & 0 \\ \vdots & & \ddots & 1/2 & \ddots & 1/\sqrt{2} \\ 0 & \cdots & \cdots & 0 & 1/\sqrt{2} & 0 \end{pmatrix}. \tag{23}$$

This matrix can also be obtained from matrix $[Y_{(N+1)}^{IE}]$ simply by making $y_0 = 0, y_1 = 1/2, y_2 = \cdots = y_N = 0$.

Finally, Hou [12] obtains two new matrices A and B, which are diagonalized by the DCT-*IIIE* and the DCT-*IVE*, respectively.

$$A = \begin{pmatrix} 2 & \sqrt{2} & 0 & \cdots & 0 \\ \sqrt{2} & \ddots & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 1 & 0 & \cdots & 0 \\ 1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}. \tag{24}$$

It can be also shown that these matrices are equal to $[Y_{(N)}^{IIIE}]$ and $[Y_{(N)}^{IVE}]$ if we make $y_0 = 2, y_1 = 1, y_2 = \cdots = y_{N-1} = 0$.

Regarding transforms DCT-*IO*, DCT-*IIO*, and DCT-*IIIO*, no generating matrices had been previously proposed.

III. Asymptotic Behavior of the Discrete Cosine Transforms

A. Definitions

We follow here the formalism developed by Yemini and Pearl [10]. Let $[A_{(N)}]$ be a symmetric matrix of dimension $N \times N$ with elements given by a_{ij} and eigenvalues $\{\lambda_i\}_{i=0}^{N-1}$. The weak norm of $[A_{(N)}]$ is defined.

$$|[A_{(N)}]| = \frac{1}{N}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}|a_{ij}|^2 = \frac{1}{N}\sum_{i=0}^{N-1}|\lambda_i|^2. \tag{25}$$

This norm is invariant under unitary transforms, i.e., if $[T_{(N)}]$ is unitary, then

$$|[A_{(N)}]| = |[T_{(N)}][A_{(N)}][T_{(N)}]^H| \tag{26}$$

where the superscript H indicates Hermitian transpose. In order to consider sequences of matrices, several terms will be defined [10]. A net is a strongly bounded sequence of matrices $[A_{(N)}], N = 1, 2, \ldots, \infty$ denoted by $\alpha = [A_{(N)}]_{N=1}^{\infty}$. A matrix class is a collection of nets. We will denote it by $\mathcal{A}$. Finally, an N section is the collection of $N \times N$ matrices that belong to the nets in a class. We will denote it by $\mathcal{A}_N$.

We define a net equivalence relation in $\mathcal{A}$ and say that two nets $\alpha = [A_{(N)}]_{N=1}^{\infty}$ and $\beta = [B_{(N)}]_{N=1}^{\infty}$ are asymptotically equivalent if $|[A_{(N)}] - [B_{(N)}]| \overset{N\to\infty}{\longrightarrow} 0$. In order to define a matrix class equivalence, we will first define the concept of asymptotic cover. Let $\mathcal{A}$ and $\mathcal{B}$ be two matrix classes; $\mathcal{A}$ is said to be an asymptotic cover of $\mathcal{B}$ if for any net $\beta \in \mathcal{B}$ there is a net $\alpha \in \mathcal{A}$ such that α and β are asymptotically equivalent. We will use the notation $\mathcal{A} \supseteq \mathcal{B}$ to indicate this. Two matrix classes $\mathcal{A}$ and $\mathcal{B}$ are asymptotically equivalent $\mathcal{A} \leftrightarrow \mathcal{B}$ if both $\mathcal{A} \supseteq \mathcal{B}$ and $\mathcal{B} \supseteq \mathcal{A}$.

Using the previous definitions, let us now focus on the problem of diagonalization of a given signal covariance matrix. Let $\tau = [T_{(N)}]_{N=1}^{\infty}$ be a net of unitary transform matrices, $\mathcal{S}$ be a class of signal covariance matrices, and $\mathcal{D}$ be the diagonal class that contains all nets of diagonal matrices $\delta = [D_{(N)}]_{N=1}^{\infty}$, and let the transformed signal covariance class be denoted by $\tau\mathcal{S}\tau^H$. τ has good behavior on signal class $\mathcal{S}$ in the sense that it approximately diagonalizes the class $\mathcal{S}$ if every net $\tau\varsigma\tau^H$ in $\tau\mathcal{S}\tau^H$ is asymptotically equivalent to a diagonal net, i.e., if $\mathcal{D}$ is an asymptotic cover of $\tau\mathcal{S}\tau^H$. Taking into account the invariance of the weak norm under

TABLE I
NORM OF THE DIFFERENCE OF A TOEPLITZ SYMMETRIC NET $[R_{(N)}]_{N=1}^{\infty}$ AND A NET OF THE CLASS DIAGONAL IN EACH DCT $[Y_{(N)}^{A}]_{N=1}^{\infty}$

DCT-A	$[Y_{(N)}^{A}]$	$\|[R_{(N)}] - [Y_{(N)}^{A}]\|$
DCT-*IE*	$y_n = r_n$ $0 \le n \le N-1$	$\frac{1}{N}\left[12\sum_{n=1}^{N-2} r_n^2 - 8\sqrt{2}\sum_{n=1}^{N-2} r_n^2 + \sum_{n=2}^{N-1}(n-1)r_n^2 + \sum_{n=2}^{N-2}(n-1)r_n^2\right]$
DCT-*IIE*	$y_n = r_n$ $0 \le n \le N-1$ $y_n = 0$ $n = N$	$\frac{1}{N}\left[2\sum_{n=1}^{N-1} n r_n^2\right]$
DCT-*IIIE*	$y_n = r_n$ $0 \le n \le N-1$	$\frac{1}{N}\left[6\sum_{n=1}^{N-1} r_n^2 - 4\sqrt{2}\sum_{n=1}^{N-1} r_n^2 + 2\sum_{n=2}^{N-1}(n-1)r_n^2\right]$
DCT-*IVE*	$y_n = r_n$ $0 \le n \le N-1$	$\frac{1}{N}\left[2\sum_{n=1}^{N-1} n r_n^2\right]$
DCT-*IO*	$y_n = r_n$ $0 \le n \le N-1$	$\frac{1}{N}\left[6\sum_{n=1}^{N-1} r_n^2 - 4\sqrt{2}\sum_{n=1}^{N-1} r_n^2 + \sum_{n=2}^{N-1}(n-1)r_n^2 + \sum_{n=1}^{N-1} n r_n^2\right]$
DCT-*IIO*	$y_n = r_n$ $0 \le n \le N-1$	$\frac{1}{N}\left[6\sum_{n=1}^{N-1} r_n^2 - 4\sqrt{2}\sum_{n=1}^{N-1} r_n^2 + \sum_{n=2}^{N-1}(n-1)r_n^2 + \sum_{n=1}^{N-1} n r_n^2\right]$
DCT-*IIIO*	$y_n = r_n$ $0 \le n \le N-1$	$\frac{1}{N}\left[6\sum_{n=1}^{N-1} r_n^2 - 4\sqrt{2}\sum_{n=1}^{N-1} r_n^2 + \sum_{n=2}^{N-1}(n-1)r_n^2 + \sum_{n=1}^{N-1} n r_n^2\right]$
DCT-*IVO*	$y_n = r_n$ $0 \le n \le N-1$ $y_n = 0$ $n = N$	$\frac{1}{N}\left[\sum_{n=1}^{N-1} n r_n^2 + \sum_{n=2}^{N-1}(n-1)r_n^2\right]$

unitary transforms, we have [10]

$$\mathcal{D} \supseteq \tau \mathcal{S} \tau^H \quad \text{if and only if} \quad \tau^H \mathcal{D} \tau \supseteq \mathcal{S}. \tag{27}$$

That means that if we want to find out whether a certain unitary transform has a good asymptotic behavior when applied to a given signal covariance class, what we have to do is to determine whether our signal covariance class is asymptotically covered by the class diagonal in τ, i.e., by the class formed by those matrices that are diagonalized by τ.

B. Stationary Processes

Let $[R_{(N)}]_{N=1}^{\infty}$ be a Toeplitz net that constitutes the autocovariance net of any stationary process. The elements of theses matrices $[R_{(N)}]_{i,j}$ are given by the covariance elements $r_{|i-j|}, 0 \le i, j \le N-1$. In order to simplify the notation, we will refer to $r_{|i-j|}$ as r_n with $0 \le n \le N-1$. For each DCT, we will now find a net of the class of matrices diagonalized by that DCT that is asymptotically equivalent to $[R_{(N)}]_{N=1}^{\infty}$ for stationary processes verifying certain conditions. The nets built from the autocovariance matrices of such processes will constitute a signal class that is asymptotically covered by the class diagonal in the corresponding DCT, which is equivalent to saying that the corresponding DCT has a good asymptotic performance with such a signal class.

In Section II-B, we obtained eight general forms of matrices that were diagonalized by the DCT's. Using those matrix forms, we have built for each DCT a net of the class of matrices diagonalized by that DCT: $[Y_{(N)}^{A}]_{N=1}^{\infty}$. These matrices are shown in Table I together with the norm $|[R_{(N)}] - [Y_{(N)}^{A}]|$.

Analyzing Table I, we can then conclude that we have a good asymptotic performance of the following:

- The DCT-*IE* and DCT-*IIIE* for those stationary processes for which $\sum_{n=1}^{\infty} r_n^2 < \infty$ and $\sum_{n=2}^{\infty}(n-1)r_n^2 < \infty$
- the DCT-*IO*, DCT-*IIO*, and DCT-*IIIO* for those verifying $\sum_{n=1}^{\infty} r_n^2 < \infty$, $\sum_{n=2}^{\infty}(n-1)r_n^2 < \infty$, and $\sum_{n=1}^{\infty} n r_n^2 < \infty$
- the DCT-*IIE* and DCT-*IVE* if $\sum_{n=1}^{\infty} n r_n^2 < \infty$
- the DCT-*IVO* if $\sum_{n=2}^{\infty}(n-1)r_n^2 < \infty$ and $\sum_{n=1}^{\infty} n r_n^2 < \infty$.

We have to point out that the condition $\sum_{n=1}^{\infty} n r_n^2 < \infty$ for the DCT-*IIE* had been already obtained by Yemini *et al.* [10] using numerical quadrature theory.

In the case of a finite-order Markov process, all three conditions $\sum_{n=1}^{\infty} r_n^2 < \infty$, $\sum_{n=2}^{\infty}(n-1)r_n^2 < \infty$, and $\sum_{n=1}^{\infty} n r_n^2 < \infty$ are verified because r_n is asymptotically exponential with n. Consequently, the eight types of discrete cosine transforms have a good asymptotic behavior with these kinds of processes.

IV. RESIDUAL CORRELATION

In Section III, we studied the behavior of the DCT's with stationary processes and showed that for stationary processes verifying certain conditions, there is a good asymptotic behavior in the sense that they approach KLT performance as $N \to \infty$. However, although all of them have a good asymptotic performance, we are interested in how the decorrelating power of each DCT depends on both dimension N and the covariance matrix elements $r_n, 0 \le n \le N-1$. A good measure of the degree of correlation still remaining after the application of a specific transform $[T_{(N)}]$ is given by the norm of the matrix containing the off-diagonal covariance elements of the transformed coefficients [11], i.e., the residual correlation (RC) is given by

$$RC = \frac{1}{N} \sum_{i \ne j} |[T_{(N)}][R_{(N)}][T_{(N)}]^H|^2_{ij}. \tag{28}$$

This norm was shown [11] to control the performance degradation resulting from residual correlation in both coding and filtering. From now on, we will refer to it as RC.

This residual correlation can also be expressed in a different way [11]. Let $[\bar{R}_{(N)}] = [T_{(N)}][R_{(N)}][T_{(N)}]^H$ be the autocovariance matrix in the transform domain and $[D([\bar{R}_{(N)}]_{j,j})]$ be the diagonal matrix representing the diagonal elements of $[\bar{R}_{(N)}]$. Then, an autocovariance matrix $[R'_{(N)}]$ that is diagonalized by transform $[T_{(N)}]$ can be obtained by inverse transforming $[D([\bar{R}_{(N)}]_{j,j})]$ such that

$$[R'_{(N)}] = [T_{(N)}]^H [D([\bar{R}_{(N)}]_{j,j})][T_{(N)}] \tag{29}$$

and the residual correlation can be alternatively expressed as [11]

$$RC = |[R_{(N)}] - [R'_{(N)}]|. \tag{30}$$

A. Development

Let $[T_{(N)}] = [C^A_{(N)}]$, where $[C^A_{(N)}]$ is any of the eight types of DCT's in orthogonal form. In order to calculate the previous norm, we first have to determine matrix $[R'_{(N)}]$. This can be easily done by using the expressions obtained in Section II for the generating matrices of the DCT's and their eigenvalues. By determining $[R'_{(N)}]$, what we want is to obtain the form of the matrix diagonalized by $[C^A_{(N)}]$ whose eigenvalues are given by the diagonal elements of matrix $[D([\bar{R}_{(N)}]_{j,j})]$. Putting these diagonal elements in vector form, we have $\boldsymbol{v} = [[\bar{R}_{(N)}]_{0,0}, [\bar{R}_{(N)}]_{1,1}, \ldots, [\bar{R}_{(N)}]_{N-1,N-1}]$. We will first express $\boldsymbol{v}$ as $\boldsymbol{v} = [V_{(N)}]\boldsymbol{r}$ with $\boldsymbol{r} = [r_0, r_1, \ldots r_{N-1}]^T$ and $[V_{(N)}]$ given by (31), which appears at the bottom of the page.

From Section II, we know that the eigenvalues of the matrix $[Y^A_{(N)}]$, which is diagonalized by the discrete cosine transform $[C^A_{(N)}]$, are given by $[\mathcal{C}^b]\boldsymbol{y}$; consequently, we would only have to apply the inverse $\mathcal{C}_b$ to our vector $\boldsymbol{v}$ in order to obtain the elements of vector $\boldsymbol{y}$. Doing so, we have

$$\boldsymbol{y} = [\mathcal{C}^b]^{-1}[V_{(N)}]\boldsymbol{r}. \tag{32}$$

Once we know $\boldsymbol{y}$, we only have to construct the corresponding matrix $[R'_{(N)}] = [Y^A_{(N)}]$ as given in Section II.

For the sake of brevity, we will not give the expression of matrix $[\mathcal{C}^b]^{-1}[V_{(N)}]$ for each transform. We will simply indicate that in the cases of transforms $[C^{IIE}_{(N)}]$ and $[C^{IVO}_{(N-1)}]$, the corresponding matrices $[\mathcal{C}^b]^{-1}[V_{(N)}]$ are not square matrices. This is due to the fact that in the case of the DCT-*IIE*, the eigenvalues of matrix $[Y^{IIE}_{(N)}]$ are given by the first N components of $[\mathcal{C}^{1e}_{(N+1)}]\boldsymbol{y}$ with $\boldsymbol{y} = [y_0, y_1, \ldots y_N]^T$. Matrix $[V_{(N)}]$ is then a $(N+1) \times N$ matrix with the last row having all elements equal to zero, and consequently, $[\mathcal{C}^{1e}_{(N+1)}]^{-1}[V_{(N)}]$ is also a $(N+1) \times N$ matrix. In the case of the DCT-*IVO*, the eigenvalues of matrix $[Y^{IVO}_{(N-1)}]$ are given by the first $N-1$ components of $[\mathcal{C}^{3o}_{(N)}]\boldsymbol{y}$ with $\boldsymbol{y} = [y_0, y_1, \ldots y_{N-1}]^T$, and then, matrix $[V_{(N)}]$, as well as $[\mathcal{C}^{3o}_{(N)}]^{-1}[V_{(N)}]$, are $N \times (N-1)$ matrices.

Once we have matrix $[R'_{(N)}]$ for each transform, we simply have to calculate the norm (30). In Table II, we give the expression of the residual correlation for each discrete cosine transform. Symbols RC-*IE*, RC-*IO*, and RC-*IIIO* that appear in Table II are defined as follows:

$$\begin{aligned}
&\text{RC-}IE = \\
&\frac{1}{N(N-1)^2}\Bigg[\sum_{n=1}^{N-2}(4((N-1)^2 - 4(N-2)) \\
&+ 2(N-2-n)(4(N-3) + (n-1)(N-5)))r_n^2 \\
&+ 2(N-2)^2 r_{N-1}^2 - \sum_{n=1}^{N-2} 8(N-2-n)(N-3)\sqrt{2}r_n^2 \\
&+ \sum_{m=1}^{\lfloor\frac{N+1}{2}\rfloor-2}\ \sum_{n=m+1}^{\lfloor\frac{N+1}{2}\rfloor-1} 32(1-(N-2n-1)m)r_{2m-1}r_{2n-1} \\
&- \sum_{m=\lfloor\frac{N+1}{2}\rfloor}^{\lfloor\frac{N}{2}\rfloor}\ \sum_{n=1}^{\lfloor\frac{N}{2}\rfloor-1} 16n r_{2m-1}r_{2n-1} \\
&+ \sum_{m=1}^{\lfloor\frac{N}{2}\rfloor-2}\ \sum_{n=m+1}^{\lfloor\frac{N}{2}\rfloor-1} (32-(N-2-2n)(32m+16))r_{2m}r_{2n} \\
&- \sum_{m=\lfloor\frac{N}{2}\rfloor}^{\lfloor\frac{N-1}{2}\rfloor}\ \sum_{n=1}^{\lfloor\frac{N}{2}\rfloor-1} (16n+8)r_{2m}r_{2n} \\
&+ \sum_{m=1}^{\lfloor\frac{N+1}{2}\rfloor-2}\ \sum_{n=m+1}^{\lfloor\frac{N+1}{2}\rfloor-1}
\end{aligned}$$

$$[V_{(N)}]_{m,n} = \begin{cases} \sum_{k=0}^{N-1}[C^A_{(N)}]^2_{m,k} & 0 \le m < N, n = 0 \\ \sum_{k=0}^{N-1-n}[C^A_{(N)}]_{m,k+n}[C^A_{(N)}]_{m,k} + \sum_{k=n}^{N-1}[C^A_{(N)}]_{m,k-n}[C^A_{(N)}]_{m,k} & 0 \le m < N, 1 \le n < N \end{cases} \tag{31}$$

TABLE II
RESIDUAL CORRELATION (RC) EXPRESSIONS FOR EACH DCT

DCT-A	RC
DCT-IE	RC-IE
DCT-IIE	$\frac{1}{N^3}\left[\sum_{n=1}^{N-1} 2(N-2)n(N-n)r_n^2 - \sum_{m=1}^{N-1}\sum_{n=m+1}^{N-1} 8(N-n)mr_mr_n\right]$
DCT-$IIIE$	$\frac{1}{N^2}\left[\sum_{n=1}^{N-1} 2(n-1+(N-1-n)(2+n))r_n^2 - \sum_{n=1}^{N-2} 4(N-1-n)\sqrt{2}r_n^2\right]$
DCT-IVE	$\frac{1}{N^3}\left[\sum_{n=1}^{N-1} 2Nn(N-n)r_n^2\right]$
DCT-IO	RC-IO
DCT-IIO	Same as DCT-IO
DCT-$IIIO$	RC-$IIIO$
DCT-IVO	$\frac{1}{N(2N+1)}\left[\sum_{n=1}^{N-1} 2(N-n)(2n-1)r_n^2\right]$

$$
\begin{aligned}
&(-32m + (N-2n-1)16)\sqrt{2}r_{2m-1}r_{2n-1}\\
&+ \sum_{m=\lfloor\frac{N+1}{2}\rfloor}^{\lfloor\frac{N}{2}\rfloor}\sum_{n=1}^{\lfloor\frac{N}{2}\rfloor-1} 8\sqrt{2}r_{2m-1}r_{2n-1}\\
&+ \sum_{m=1}^{\lfloor\frac{N}{2}\rfloor-2}\sum_{n=m+1}^{\lfloor\frac{N}{2}\rfloor-1} (16(N-2-2n)\\
&\quad - (32m+16))\sqrt{2}r_{2m}r_{2n}\\
&+ \left.\sum_{m=\lfloor\frac{N}{2}\rfloor}^{\lfloor\frac{N-1}{2}\rfloor}\sum_{n=1}^{\lfloor\frac{N}{2}\rfloor-1} 8\sqrt{2}r_{2m}r_{2n}\right]
\end{aligned}
$$

$$
\begin{aligned}
\text{RC-}IO &=\\
&\frac{1}{N(2N-1)^2}\left[\sum_{n=1}^{N-1}((2N-3)(4N-6)\right.\\
&+ 2(N-1-n)(10(N-2)+(n-1)(4N-10)+3))r_n^2\\
&- \sum_{n=1}^{N-2} 8(N-1-n)(3+2(N-3))\sqrt{2}r_n^2\\
&- \sum_{m=1}^{N-1}\sum_{n=m+1}^{N-2} 16(2m+(2m+1)(N-2-n))r_mr_n\\
&+ \sum_{n=1}^{N-2} 16r_nr_{N-1} + \sum_{m=1}^{N-1}\sum_{n=m+1}^{N-1} 4(-6+4(N-1-n)\\
&\left. -4(m-1))\sqrt{2}r_mr_n\right]
\end{aligned}
$$

$$
\begin{aligned}
\text{RC-}IIIO &=\\
&\frac{1}{N(2N-1)^2}\left[\sum_{n=1}^{N-1}((2N-3)(4N-6)\right.\\
&+2(N-1-n)(10(N-2)\\
&+ (n-1)(4N-10)+3))r_n^2\\
&- \sum_{n=1}^{N-2} 8(N-1-n)(3+2(N-3))\sqrt{2}r_n^2\\
&+ \sum_{m=1}^{N-1}\sum_{n=m+1}^{N-2} (-1)^{m+n+1}\\
&\quad 16(2m+(2m+1)(N-2-n))r_mr_n\\
&+ \sum_{n=1}^{N-2}(-1)^{n+N-1}16r_nr_{N-1}\\
&+ \sum_{m=1}^{N-1}\sum_{n=m+1}^{N-1} (-1)^{m+n}4(-6+4(N-1-n)\\
&\left. -4(m-1))\sqrt{2}r_mr_n\right]
\end{aligned}
$$

B. Discussion

In Section IV-A, we obtained expressions that indicate the rate at which the different DCT's decorrelate a stationary process. Analyzing those expressions, one conclusion is immediately obtained: The DCT-IO and the DCT-IIO have the same decorrelating power for any stationary process. This, of course, also applies to any stationary Markov process of first order. The covariance matrix in this case is a Toeplitz matrix with $r_n = \rho^n$, where $\rho, |\rho| < 1$ is the correlation coefficient. We simply have to substitute in the expressions of Table II, and we will obtain the residual correlation for each DCT as a function of both the block size N and the correlation coefficient ρ. Such expressions for first-order Markov processes had been previously obtained by Hamidi *et al.* [3] and Kitajima [6] only for the DCT-*IIE* and the DCT-*IE*, respectively; meanwhile, Jain obtained some expressions for the performance of the DCT-*IIE*, the DCT-*IVE*, and the DCT-*IVO*, where the dependence on N was omitted.

Analyzing those expressions, we can conclude that the DCT-*IE*, DCT-*IIIE*, DCT-*IVE*, and DCT-*IVO* have the same decorrelation performance for positive and negative ρ; meanwhile, in the cases of the DCT-*IIE*, the DCT-*IO*, and the DCT-*IIIO*, there is a different decorrelation behavior for positive and negative ρ. In particular, the residual correlation of DCT-*IO* for positive ρ is equal to the residual correlation of DCT-*IIIO* for negative ρ and vice versa.

In Section III, we studied the asymptotic behavior of the DCT's with dimension N. Expressions in Table II allow us now to study the asymptotic behavior of the different DCT's as $\rho \rightarrow 1$ or $\rho \rightarrow -1$. We can obtain expressions for the asymptotic behavior of each DCT when $\rho \rightarrow 1$, which are functions of N except for the DCT-*IIE*, which tends to 0 as $\rho \rightarrow 1$ independently of N. This result had been previously stated; in fact, it was considered in the original derivation of the DCT-*IIE* [1] that was conceived as asymptotically equivalent to the KLT of a first-order Markov process as $\rho \rightarrow 1$. That means that for a given N, the DCT-*IIE* is the best transform for highly positive correlated Markov-1 processes, as is well known. It can now be easily shown that in the case of a highly negative correlated first-order Markov process, the DCT-*IIIO* gives the best performance for $N > 2$. We have to point out that unlike the DCT-*IE*, the DCT-*IIE*, and the DCT-*IIIE*, the rest of the transforms do not diagonalize symmetric Toeplitz matrices of dimension $N = 2$.

V. Summary

We have obtained the eight types of DCT's established by Wang [7] as the complete orthonormal set of eigenvectors generated by a general form of matrices that can be decomposed as the sum of a symmetric Toeplitz matrix plus a Hankel or close to Hankel matrix scaled by some constant factors. We have also shown that all the previously proposed generating matrices for the DCT's are simply particular cases of these general matrix forms.

Using these matrices, we have obtained for each DCT a class of stationary processes verifying certain conditions with respect to which the corresponding DCT has a good asymptotic behavior in the sense that it approaches KLT performance as block size N tends to infinity. As a particular result, we have proven that the eight types of DCT's are asymptotically optimal for all finite-order Markov processes.

We have finally studied the decorrelating power of the DCT's, obtaining expressions that show the decorrelating behavior of each DCT with respect to any stationary process. These expressions allow us to conclude that the DCT-*IO* and the DCT-*IIO* have the same decorrelating power for any stationary process and, when those expressions are applied to a first-order stationary Markov process, we obtain that, in the same way as the DCT-*IIE* is the best discrete cosine transform for very highly positive correlated processes, the DCT-*IIIO* is the best discrete cosine transform for very highly negative correlated processes for $N > 2$.

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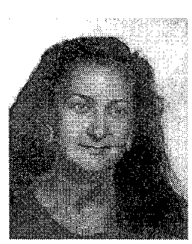

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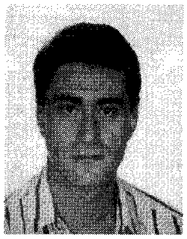

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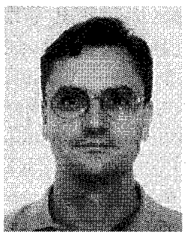

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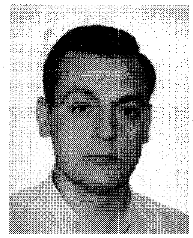

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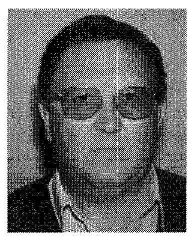

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