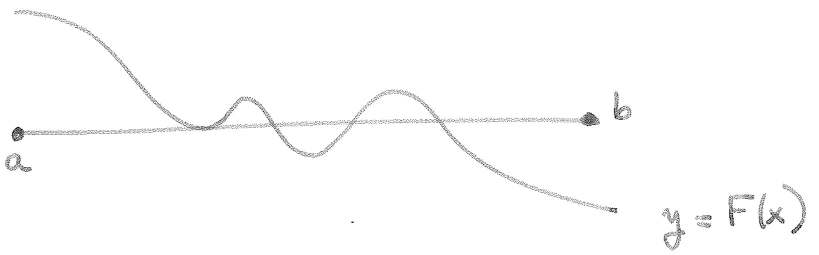
Changes of sign and closed orbits

Rafael Ortega

Given a continuous function $F : [a,b] \to \mathbb{R}$ and a change of sign,

$$F(a) \cdot F(b) \leq 0,$$

there exists at least one zero of F (Bolzano Th).



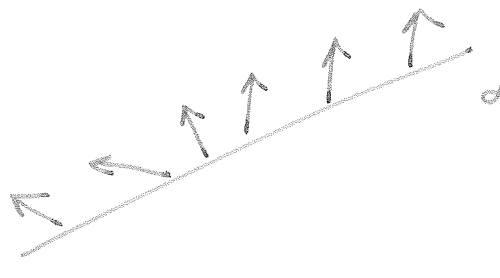
Fixed points in 1D can be deduced from here.

<u>1D Brouwer's Fixed point Th</u> : $g : [a,b] \to \mathbb{R}$ continuous, $g([a,b]) \subset [a,b]$ then g has a fixed point ($g(x) = x$)

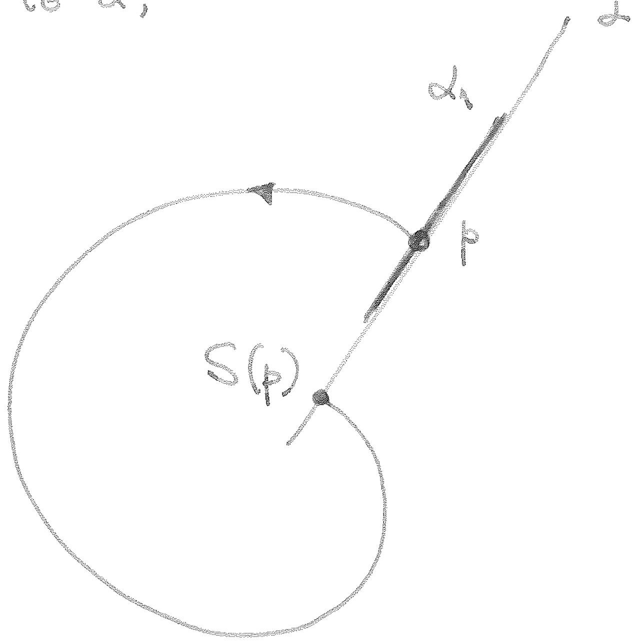
[Proof: $F = g - id$]

<u>Fried egg Th</u> : $g : [a,b] \to \mathbb{R}$ continuous, $[a,b] \subset g([a,b])$ then g has a fixed point.

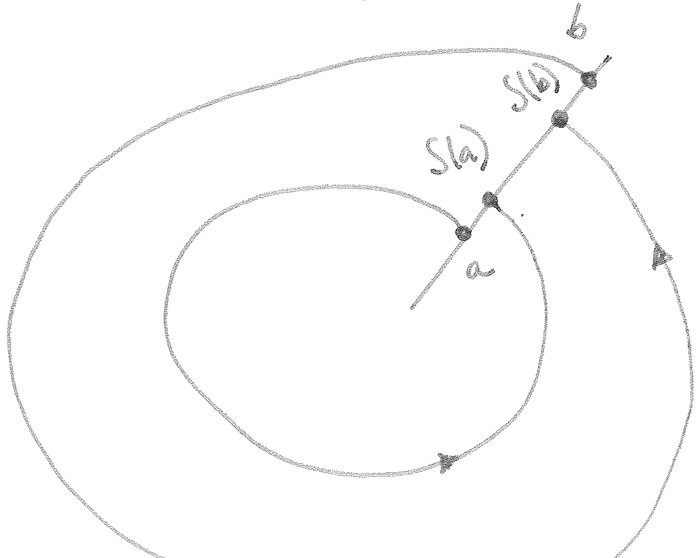
These results lead to the existence of closed orbits in 2D autonomous differential equations. Given a vector field in the plane assume that α is a segment transversal to the vector field



and suppose that there is a smaller segment $\alpha_1 \subset \alpha$ with a return to α,



By continuous dependence the function $S : \alpha_1 \to \alpha$ is continuous. Assume that there are two points $a, b \in \alpha_1$ with $a \leq S(a) \leq S(b) \leq b$



Then, according to Brouwer's Th, there exists a fixed point of S in $[a, b]$. This leads to a closed orbit.

An analogous result is obtained via the fried egg theorem when $S(a) \leq a < b \leq S(b)$.

Example

$$\begin{cases} \dot{x} = -y + x\,F(x,y) \\ \dot{y} = x + y\,F(x,y) \end{cases}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous, bounded and such that there is uniqueness for the initial value problem. Notice that the above assumptions imply global extendability. The origin is an equilibrium and we pass to polar coordinates

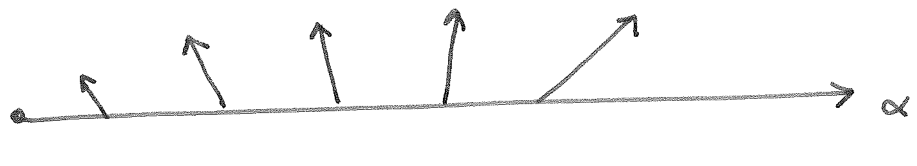
$$x = r\cos\theta, \quad y = r\sin\theta$$

so that the system becomes

$$\dot{\theta} = 1, \quad \dot{r} = r\,\Phi(\theta, r)$$

where $\Phi(\theta, r) = F(r\cos\theta, r\sin\theta)$.

Since $\dot{y} = +x$ when $y = 0$ we notice that $\alpha =]0, \infty[\times \{0\}$ is a transversal segment



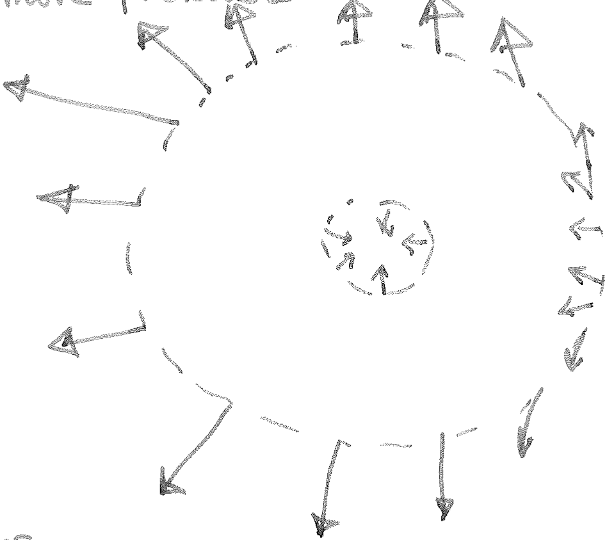
Moreover, since $\dot{\theta} = 1$, we can take $\alpha_1 = \alpha$ and there is a return for $\theta = 2\pi$. We impose two conditions on Φ,

(A1) $\quad \Phi(\theta, r) < 0 \quad$ if $r \in]0, \delta]$

(A2) $\quad \displaystyle\int_0^{2\pi} \underline{\Phi}(\theta, +\infty)\, d\theta > 0$

where $\underline{\Phi}(\theta, +\infty) = \liminf_{r \to +\infty} \Phi(\theta, r)$.

Assumption (A1) indicates that the vector field points towards the origin on small circles $r = \text{constant} \to 0$. On circles $r = \text{constant} \to \infty$, the vector field can switch the sense, pointing sometimes towards the origin and sometimes towards infinity. Nevertheless we still have some information: (A2) says that it is more probable that it points towards infinity



Let $r(\theta, r_0)$ be the solution of

$$\frac{dr}{d\theta} = r\,\Phi(\theta, r), \; r(0) = r_0.$$

[We assume uniqueness for this i.v.p.]

From (A1) we deduce that $r(\theta, r_0)$ is decreasing in θ when $r_0 \in]0, \delta]$. Hence

$$S(\delta) = r(2\pi, \delta) < r(0, \delta) = \delta.$$

The solution also satisfies the integral equation

$$r(\theta, r_0) = r_0 \exp\left\{ \int_0^\theta \Phi(\Theta, r(\Theta, r_0))\, d\Theta \right\}.$$

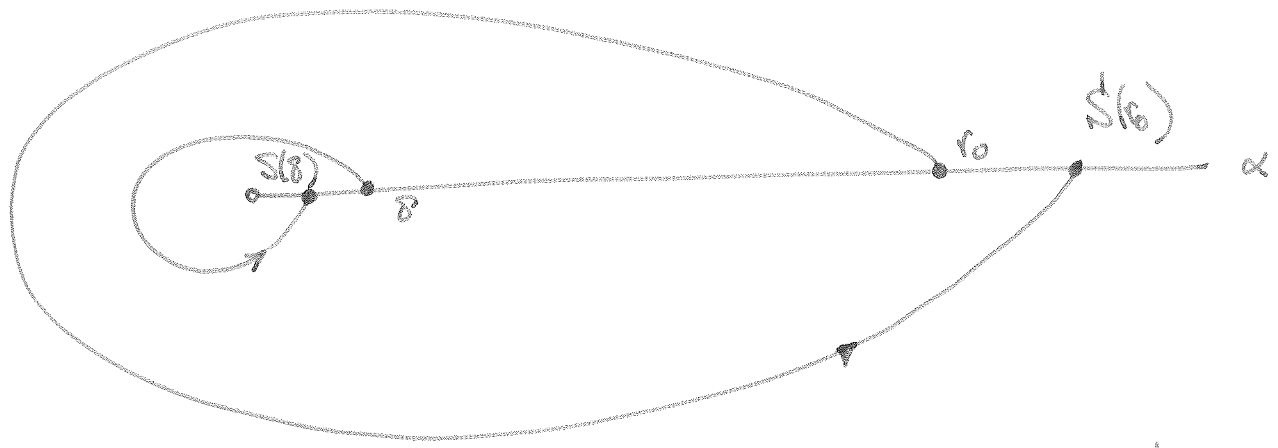
Since Φ is bounded, there are fixed numbers $A > a > 0$ such that

$$A r_0 \geq r(\theta, r_0) \geq a r_0 \quad \text{if } \theta \in [0, 2\pi].$$

Thus $r(\theta, r_0) \longrightarrow +\infty$ as $r_0 \to +\infty$ (uniformly in $[0, 2\pi]$) and, by Fatou's Lemma,

$$\liminf_{r_0 \to +\infty} \int_0^{2\pi} \Phi(\theta, r(\theta, r_0))\, d\theta \geq \int_0^{2\pi} \underline{\Phi}(\theta, +\infty)\, d\theta > 0$$

by (A2). For large r_0, $S(r_0) = r(2\pi, r_0) > r_0$



By the fried egg th we deduce that there exists a closed orbit.

<u>Exercise</u> Generalization to systems of the type

$$\begin{cases} \dot{x} = -y + x F(x,y) \\ \dot{y} = x + y\, G(x,y) \end{cases}$$

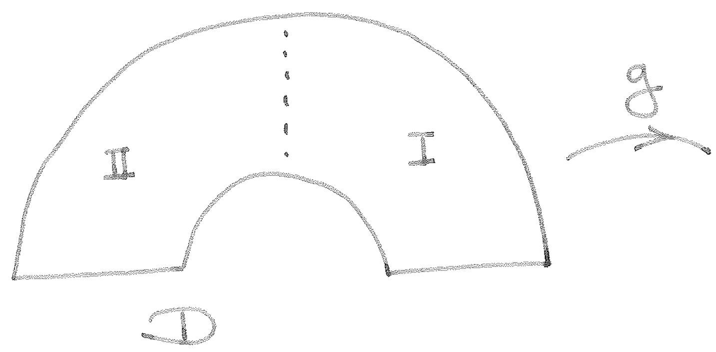
with $F \neq G$.

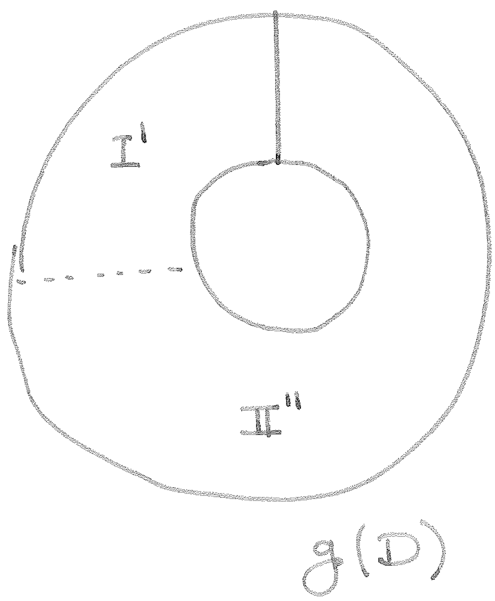
The connaisseur already has noticed that the previous results also follow from Poincaré-Bendixson theory. Next we present extensions to higher dimensions.

2D Fixed points theorems and the torus principle

Brouwer's fixed point th is also valid in 2D. Assume that D is a topological disk(*) in $\mathbb{R}^2$ and $g: D \to \mathbb{R}^2$ continuous with $g(D) \subseteq D$. Then g has a fixed point.

Such an extension is not valid for the fried egg th. Next we draw an example of a continuous map $g: D \to \mathbb{R}^2$ with $g(D) \supseteq D$ and without fixed points.

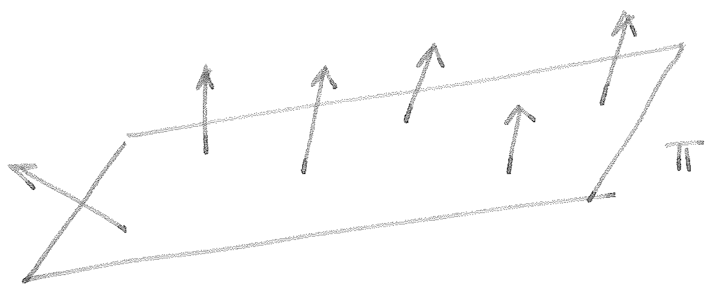




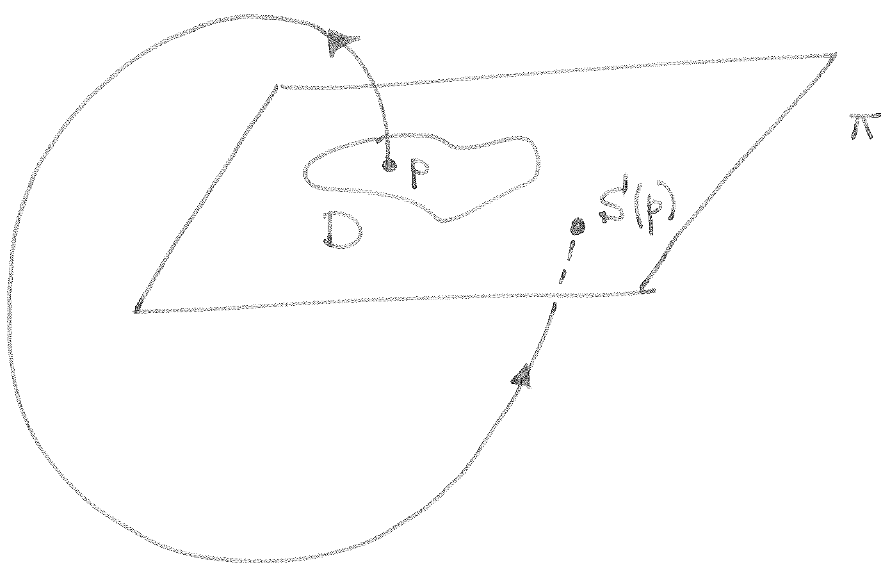
Rotate 90° I so that it becomes I' and then enlarge II to II' and glue it to the border of I'.

(*) D is homeomorphic to $\{z \in \mathbb{C} \mid |z| \leq 1\}$

We can derive the torus principle from Brouwer's Th.
Assume that we have a smooth vector field in $\mathbb{R}^3$ and a planar section which is transversal



Assume that there is a disk $D \subset \pi$ with a return to π,



If we assume that $S(D) \subseteq D$ then a fixed point exists (and hence a closed orbit).
The situation $D \subseteq S(D)$ also produces a fixed point. It is not possible to apply now the fried egg th. Instead we apply Brouwer's Th to the inverse map S^{-1} defined on D. Notice that S is a one-to-one map by the uniqueness of the initial value problem.

2D Bolzano's Theorem and a consequence

A more subtle principle can be derived from the following <u>2D Bolzano's Theorem</u>:

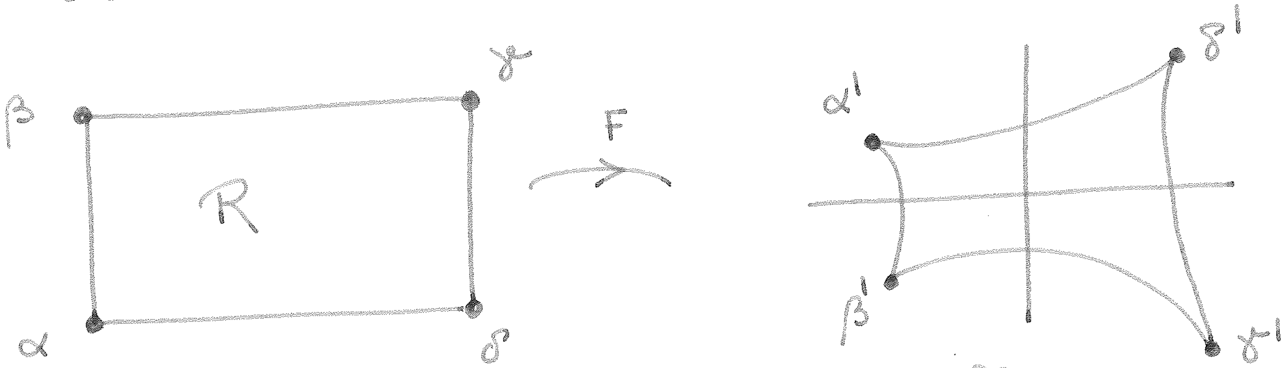
$$R = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq A,\ b \leq y \leq B\}$$

$$F : R \longrightarrow \mathbb{R}^2, \quad F = (F_1, F_2) \text{ continuous}$$

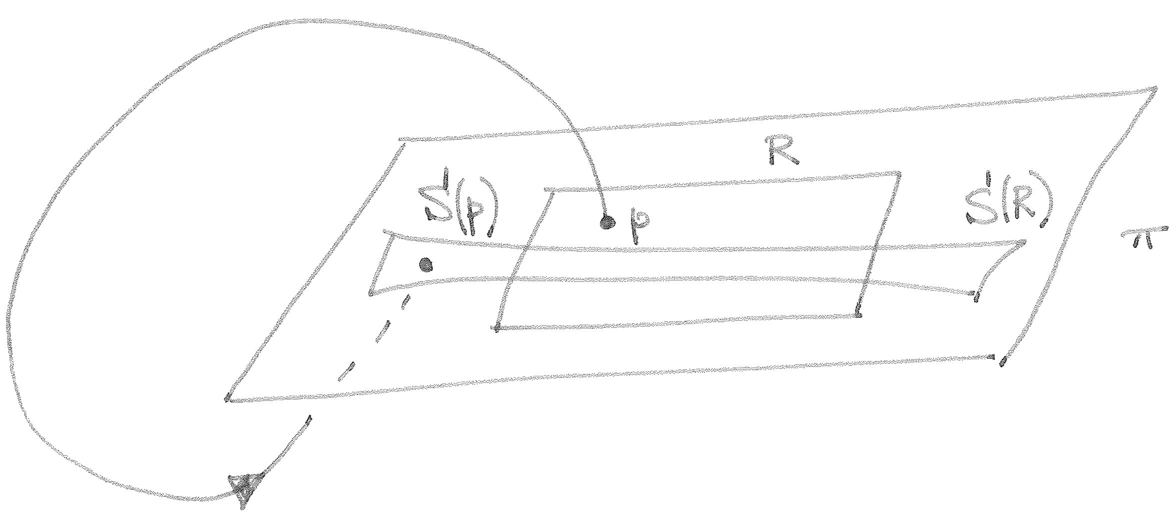
$$F_1(a,y) \leq 0 \leq F_1(A,y), \quad y \in [b,B]$$

$$F_2(x,b) \geq 0 \geq F_2(x,B), \quad x \in [a,A]$$

Then the system $F_1(x,y) = 0$, $F_2(x,y) = 0$ has a solution ~~inside~~ R



Assume the following situation for a 3D flow



Then $F = S - id$ is in the above conditions.

Example
$$\begin{cases} \dot{x} = -y + xF(x,y,z) \\ \dot{y} = +x + yF(x,y,z) \\ \dot{z} = Z(x,y,z) \end{cases}$$

with $F, Z : \mathbb{R}^3 \to \mathbb{R}$ continuous, bounded and such that there is uniqueness for the initial value problem. Again global extendability is automatic.

Assume that there are numbers $0 < \delta < \Delta$ and $R > 0$ such that

(A1) $\quad F(x,y,z) < 0$ if $0 < x^2+y^2 < \delta^2$, $z \in \mathbb{R}$
$\quad F(x,y,z) > 0$ if $x^2+y^2 > \Delta^2$, $z \in \mathbb{R}$

(A2) $\quad z \cdot Z(x,y,z) < 0$ if $|z| \geq R$.

We will prove that there exists a closed orbit.

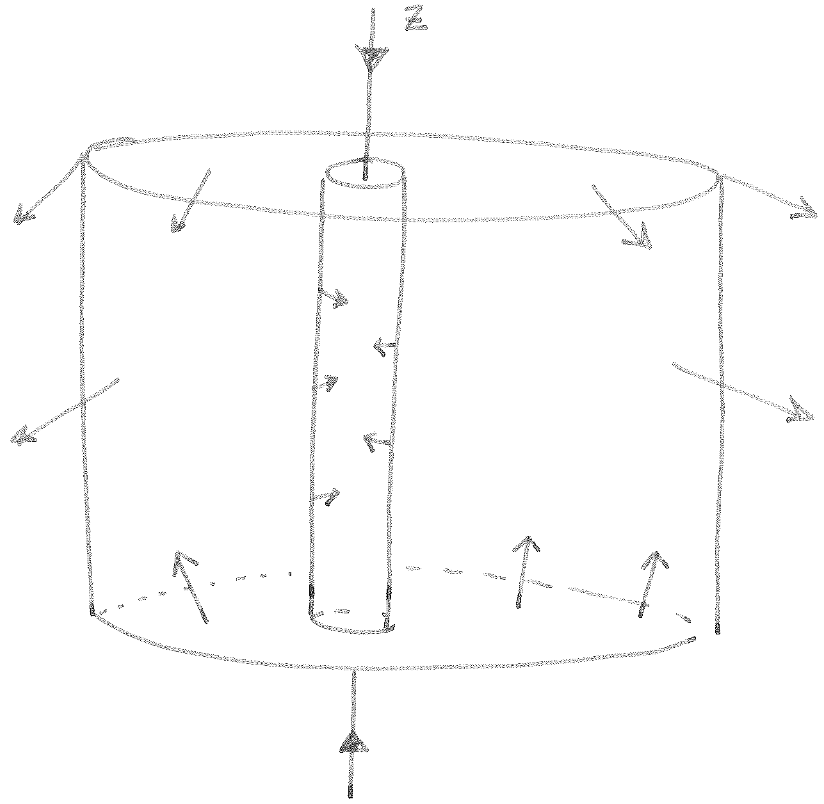
To interpret the assumptions first notice that the line $x = y = 0$ is invariant. We pass to cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

and the system becomes

$$\dot{\theta} = 1, \quad \dot{r} = r\mathcal{F}(\theta, r, z), \quad \dot{z} = \mathcal{Z}(\theta, r, \cancel{\theta} z).$$

The assumptions (A-1) and (A2) are illustrated in the figure below



The region $\Pi = \{(x, 0, z) : x > 0,\ z \in \mathbb{R}\}$ is a transversal section since the vector field at Π has a non-zero second coordinate. From $\dot{\theta} = 1$ we deduce that Π has a return. Let $(r(\theta; r_0, z_0), z(\theta; r_0, z_0))$ be the solution of

$$\frac{dr}{d\theta} = \mathcal{R}(\theta, r, z),\quad \frac{dz}{d\theta} = \mathcal{Z}(\theta, r, z),\quad r(0) = r_0,\ z(0) = z_0.$$

We notice that $S(r_0, z_0) = (r(2\pi; r_0, z_0), z(2\pi; r_0, z_0))$

From (A1) we deduce that

$$r(2\pi; r_0, z_0) < r_0 \quad \text{if} \quad r_0 = \delta \quad \text{and}$$

$$r(2\pi; r_0, z_0) > r_0 \quad \text{if} \quad r_0 = \Delta$$

From (A2),

$$z(2\pi; r_0, z_0) < z_0 \quad \text{if} \quad r_0 \to +\infty$$

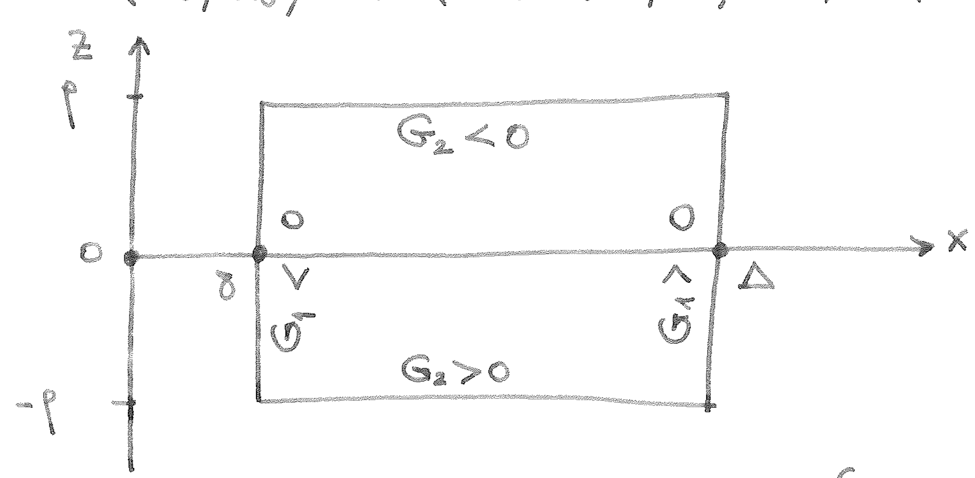
$$z(2\pi; r_0, z_0) > z_0 \quad \text{if} \quad r_0 \to -\infty$$

We can now apply 2D Bolzano Th to $\mathcal{S}$-id with

$R: \ \delta \leq r_0 \leq \Delta, \ -\rho \leq z_0 \leq \rho$ (ρ large enough).

The coordinates of $\mathcal{S}$-id are

$$G : (r_0, z_0) \longmapsto \left(r(2\pi; r_0, z_0) - r_0 \, , \, z(2\pi; r_0, z_0) - z_0 \right)$$



<u>Exercise</u> Generalization to
$$\begin{cases} \dot{x} = -y + x F(x,y,z) \\ \dot{y} = x + y\, G(x,y,z) \\ \dot{z} = Z(x,y,z) \end{cases}$$

<u>Question</u> : How to interpret "change of sign" in 2D?

<u>Acknowledgement</u> My thanks to Juan Campos. He read a preliminary version, found several mistakes and made ~~interesting~~ suggestions.