

Degree theory and boundary value problems

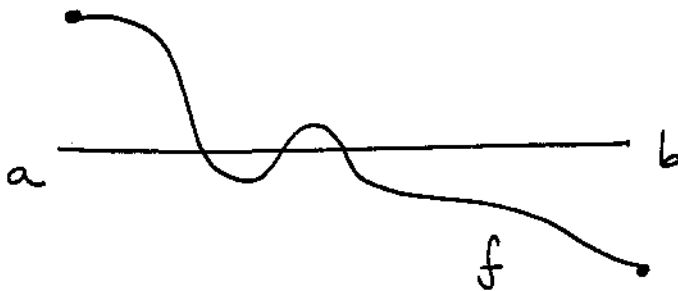
Rafael Ortega

August 06

Bolzano's Theorem

$f: [a, b] \rightarrow \mathbb{R}$ continuous, $f(a) \cdot f(b) < 0 \Rightarrow$

f has a zero in $]a, b[$



The shooting method

$$\text{Dirichlet problem} \quad \begin{cases} \ddot{u} = F(t, u) \\ u(0) = u(1) = 0 \end{cases}$$

$F: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ continuous /

- uniqueness for the initial value problem (i.v.p)
- every solution of the i.v.p is well defined in $[0, 1]$

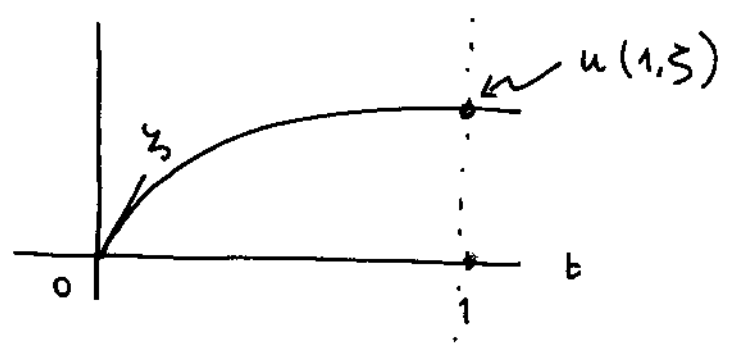
For each $\xi \in \mathbb{R}$, $u(t, \xi)$ is the solution with

$$u(0) = 0, \quad \dot{u}(0) = \xi$$

By continuous dependence with respect to initial conditions, $(t, \xi) \in [0, 1] \times \mathbb{R} \mapsto u(t, \xi)$ is continuous.

Define $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(\xi) = u(1, \xi)$

To solve the Dirichlet problem we look for zeros of f



We test the method: proving the following result:

If F is bounded then the Dirichlet problem has a solution.

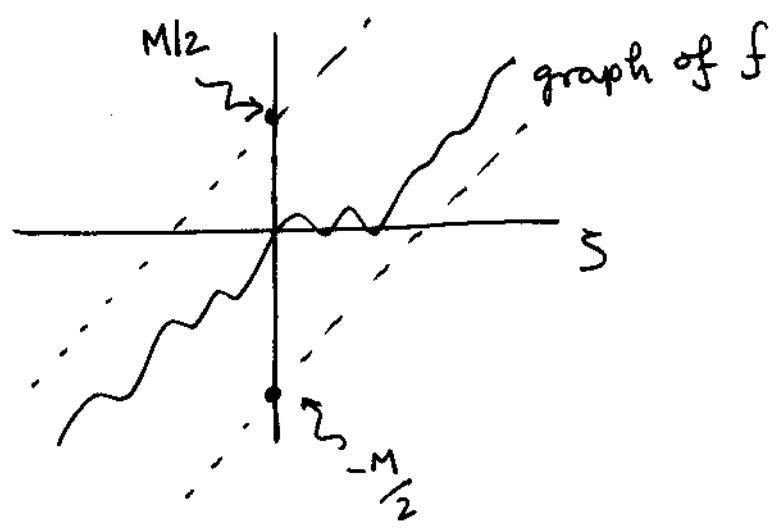
Proof $u(t, \xi)$ satisfies

$$u(t, \xi) = \xi t + \int_0^t (t-s) F(s, u(s, \xi)) ds$$

If $|F(t, u)| \leq M \quad \forall (t, u)$,

$$|u(t, \xi) - \xi t| \leq M \int_0^t (t-s) ds \leq \frac{M}{2}$$

$$\Rightarrow |f(\xi) - \xi| \leq \frac{M}{2} \quad \forall \xi \in \mathbb{R}$$

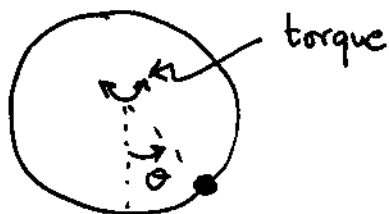


As $\lim_{\xi \rightarrow \pm\infty} f(\xi) = \pm\infty$, f has a zero.

Example The forced pendulum

$$\ddot{\theta} + \sin \theta = f(t), \quad f: [0,1] \rightarrow \mathbb{R} \text{ continuous}$$

"torque"



Given any torque f , there is a motion starting and finishing ($t=0, t=1$) at the lowest position

Next we present a result about multiplicity of solutions

Assume that F is bounded, of class C^1 ,

$$F(t,0) = 0, \quad \pi^2 < -F_u(t,0) < 4\pi^2 \quad \forall t$$

Then the Dirichlet problem has at least 3 solutions (one of them is $u=0$)

Proof The theorem of differentiability w.r. to initial conditions implies that $(t, \xi) \mapsto u(t, \xi)$ is C^1 .

Moreover, $\frac{\partial u}{\partial \xi}(t, \xi)$ is the solution of

$$\ddot{v} = F_u(t, u(t, \xi))v, \quad v(0) = 0, \quad \dot{v}(0) = 1.$$

We know that $u(t,0) \equiv 0$ and so $f(0) = 0$. Let us compute $f'(0)$. It is given by

$$f'(0) = \frac{\partial u}{\partial \xi}(1,0) = v(1)$$

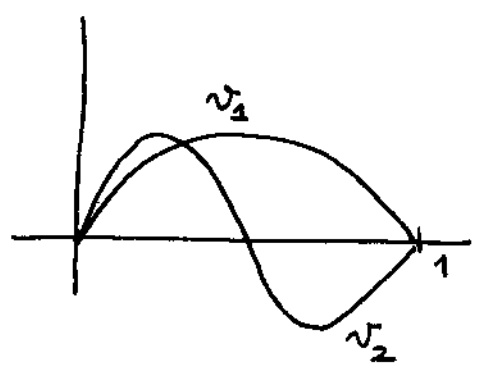
where

$$\ddot{v} = F_u(t,0)v, \quad v(0) = 0, \quad \dot{v}(0) = 1.$$

By Sturm comparison theory the zeros of $v(t)$

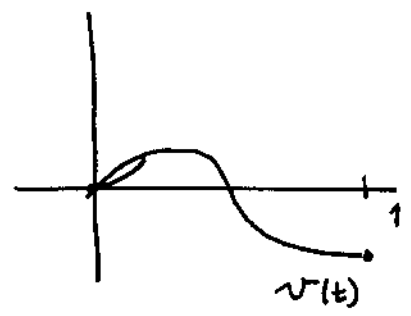
can be compared to those of $\ddot{v}_1 + \pi^2 v_1 = 0$ and $\ddot{v}_2 + 4\pi^2 v_2 = 0$,

$v_1(t) = \sin \pi t, v_2(t) = \sin 2\pi t$



$\Rightarrow v(t)$ has one zero in $]0, 1[$

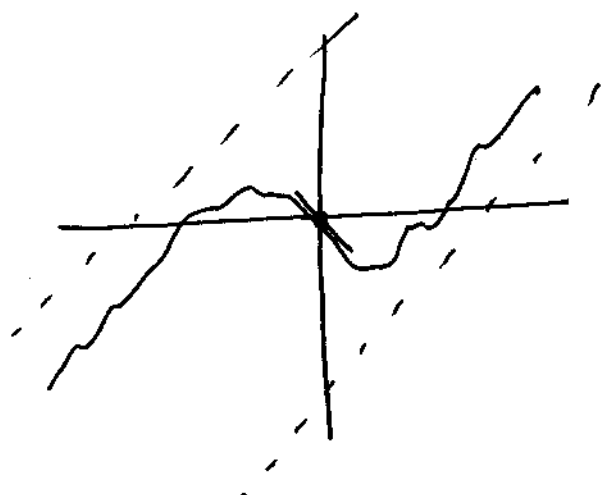
$\Rightarrow v(1) < 0$ (Notice that the zeros of $v(t)$ are simple)



Thus $f'(0) < 0$.

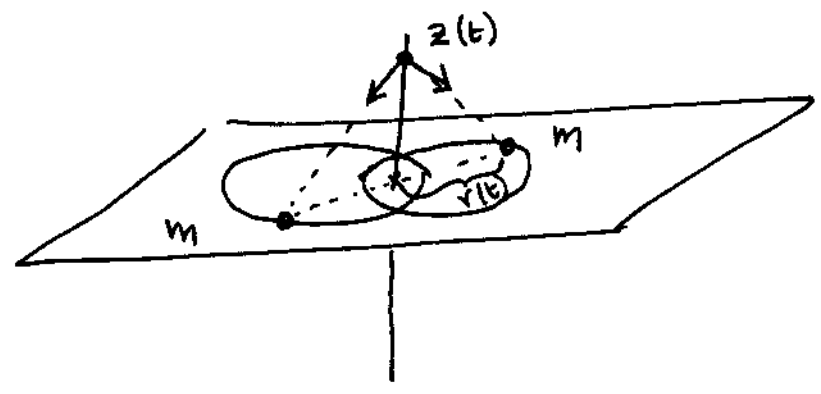
We know from the previous proof (F is bounded)

that $\lim_{\xi \rightarrow \pm \infty} f(\xi) = \pm \infty$



f has at least one positive zero and one negative

Example The Sibnikov problem



$$\ddot{z} = \frac{-2z}{(z^2 + r(t)^2)^{3/2}}$$

$r(t)$ distance of the primaries to the origin

$z \equiv 0$ equilibrium

$$F_z(t, 0) = \frac{-2}{r(t)^3} \quad \text{If } \pi^2 < \frac{2}{r(t)^3} < 4\pi^2 \quad \forall t$$

the asteroid can move in such a way that the initial and final position are at the center of mass.

Exercise The Neumann problem

$$\begin{cases} \ddot{u} = F(t, u) \\ \dot{u}(0) = \dot{u}(1) = 0 \end{cases} \quad \begin{array}{l} F: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \\ \text{continuous, bounded,} \\ \text{uniqueness i. s. p.} \end{array}$$

$$\exists \rho > 0: F(t, R) < 0 < F(t, -R) \text{ if } R \geq \rho$$

Then \exists solution of the Neumann problem

Problems which cannot be treated with the same

technique (Bolzano Th)

① The Dirichlet problem for systems

$$\begin{cases} \ddot{u}_i = F_i(t, u_1, \dots, u_N), \quad i=1, \dots, N \\ u_i(0) = u_i(1) = 0 \end{cases}$$

$$\text{Now } f: \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad f(\xi) = u(1, \xi)$$

② The periodic problem

$$\begin{cases} \ddot{u} = F(t, u) \\ u(0) = u(1), \dot{u}(0) = \dot{u}(1) \end{cases}$$

$u(t, \xi, \eta)$ is the solution with $u(0) = \xi, \dot{u}(0) = \eta$

Under assumptions of uniqueness and extendability for the initial value problem, we look for zeros of

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(\xi, \eta) = (u(1, \xi, \eta) - \xi, \dot{u}(1, \xi, \eta) - \eta)$$

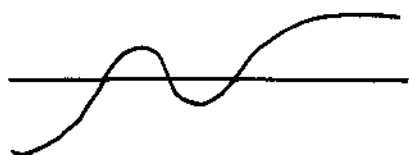
Question How to prove the existence of zeros for systems?

How to generalise Bolzano's Th. to higher dimensions?

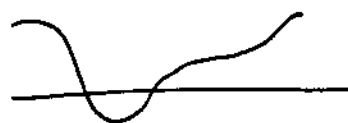
Degree in \mathbb{R}

$f: [a, b] \rightarrow \mathbb{R}$ continuous, $f(x) \neq 0$ if $x = a, b$

$$\deg(f,]a, b[) = \begin{cases} 1 & \text{if } f(a) < 0 < f(b) \\ -1 & \text{if } f(b) < 0 < f(a) \\ 0 & \text{otherwise} \end{cases}$$



deg = 1



deg = 0

Re-statement of Bolzano's Theorem :

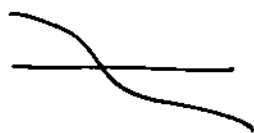
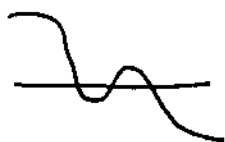
If $\deg(f,]a, b[) \neq 0 \Rightarrow f$ has a zero in $]a, b[$

Given any open and bounded set $\Omega \subset \mathbb{R}$,
it can be expressed as a disjoint and countable
union of open intervals, $\Omega = \bigcup_n I_n$

$f: \bar{\Omega} \rightarrow \mathbb{R}$ continuous, $f(x) \neq 0 \forall x \in \partial\Omega$

$$\deg(f, \Omega) = \sum_n \deg(f, I_n)$$

Example: $\Omega =]0, 1[\cup]2, 3[$



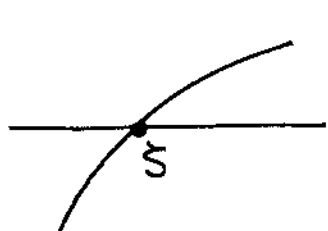
deg = -2

We observe that f can vanish only at a finite
number of components I_n and so the sum is always
meaningful.

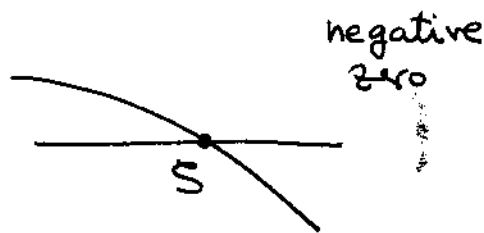
[Proof. Assume by contradiction that $f(x_n) = 0$ with $x_n \in I_{\sigma(n)}$ ~~pairwise~~ all of them different. After extracting a subsequence, $x_n \rightarrow x_*$. By continuity $f(x_*) = 0$ and so $x_* \notin \partial\Omega$. Thus $x_* \in I_m$ for some m and so $x_n \in I_m$ for n large. This is a contradiction.]

An important formula

Assume now that $f \in C^1[a, b]$, $f(x) \neq 0$ if $x = a, b$. Given a zero $\xi \in]a, b[$ we say that it is simple if $f'(\xi) \neq 0$



positive
zero



negative
zero

Assume now that all the zeros of f are simple, then there is a finite number of them and they can be labeled as

$$a < \xi_1 < \xi_2 < \dots < \xi_n < b.$$

We observe that positive and negative zeros alternate, leading to

$$\deg(f,]a, b[) = \sum_{i=1}^n \text{sign } f'(\xi_i)$$

We are using the convention

$$\sum_{\emptyset} = 0.$$

Degree in \mathbb{R}^2

Lemma 1 Assume that $\alpha: [a, b] \rightarrow \mathbb{R}^2 - \{0\}$ is continuous.

Then there exists a continuous function $\theta: [a, b] \rightarrow \mathbb{R}$ such that

$$\alpha(t) = r(t) (\cos \theta(t), \sin \theta(t)), \quad t \in [a, b]$$

with $r(t) = \|\alpha(t)\|$ (euclidean norm).

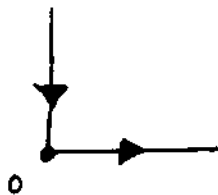
Moreover $\theta(t)$ is unique up to an additive constant 2π

Lemma 2 Assume that $\alpha_n: [a, b] \rightarrow \mathbb{R}^2 - \{0\}$ is a sequence of continuous functions converging uniformly to

$\alpha: [a, b] \rightarrow \mathbb{R}^2 - \{0\}$. If $\theta_n(a) \rightarrow \theta(a)$ then

$\theta_n \rightarrow \theta$ uniformly.

Remark Notice that the condition $\alpha([a, b]) \subset \mathbb{R}^2 - \{0\}$ is essential to find a continuous argument. This is shown by the curve



which is continuous but has a discontinuous argument.

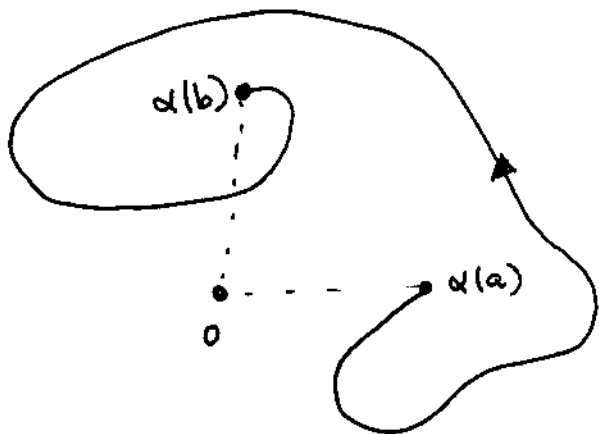
Given $\alpha: [a, b] \rightarrow \mathbb{R}^2 - \{0\}$ continuous we observe that the number

$$\frac{1}{2\pi} [\theta(b) - \theta(a)]$$

is independent of the chosen argument. We call it

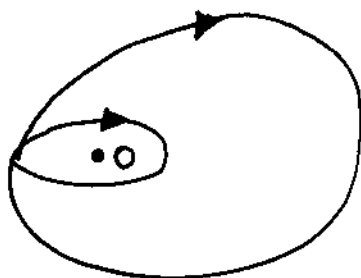
the winding number of the curve $\alpha(t)$ around

the origin.



$$\text{winding number} = \frac{1}{4}$$

If the curve is closed ($\alpha(a) = \alpha(b)$) the winding number is an integer. Intuitively speaking $+n$ means n revolutions around the origin counterclockwise, $-n$ if it is clockwise.



$$-2$$

Assume now that Ω is a Jordan domain in \mathbb{R}^2 . This means that Ω is the bounded component of $\mathbb{R}^2 - \Gamma$, where Γ is a Jordan curve. We assume



that $\alpha: [0, 1] \rightarrow \mathbb{R}^2$ is a positive parameterization of Γ ,

$$\alpha(0) = \alpha(1)$$

$$\alpha|_{[0, 1[} \text{ one-to-one}$$

$$\alpha([0, 1]) = \Gamma$$

Given $f: \bar{\Omega} \rightarrow \mathbb{R}^2$ continuous,

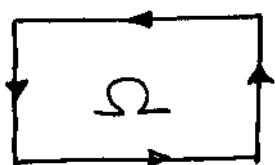
$$f(x) \neq 0 \quad \forall x \in \partial\Omega = \Gamma$$

We define

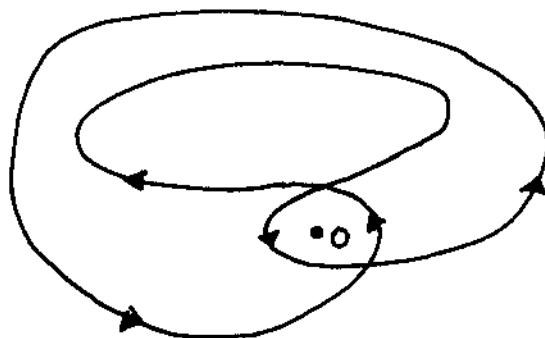
$\deg(f, \Omega) =$ winding number of $f \circ \alpha$ around the origin

Examples

i)



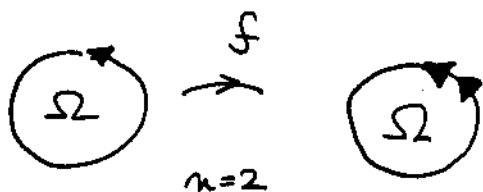
f



$$\deg(f, \Omega) = 2$$

ii) $\mathbb{C} \cong \mathbb{R}^2$, $f(z) = z^n$, $n = 4, 2, \dots$

$\Omega =$ unit disk, $\alpha(t) = e^{2\pi i t}$

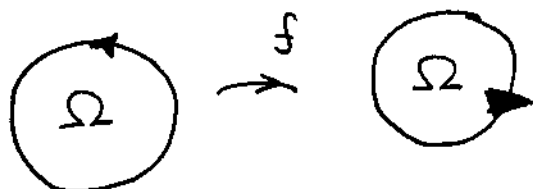


$$f(\alpha(t)) = e^{2\pi i n t}$$

$$\deg(f, \Omega) = n$$

iii) $f(z) = \bar{z}$, Ω unit disk

$$f(\alpha(t)) = e^{-2\pi i t}$$



$$\deg(f, \Omega) = -1$$

Lemma 2 implies the continuity of the degree. If

$f_n, f: \bar{\Omega} \rightarrow \mathbb{R}^2$ continuous $f_n(x) \neq 0, f(x) \neq 0 \quad \forall x \in \partial\Omega$

$f_n \rightarrow f$ uniformly on $\partial\Omega$. Then $\deg(f_n, \Omega) \rightarrow \deg(f, \Omega)$.

As $\deg(f, \Omega)$ is an integer it means that it must be eventually constant. From here we deduce the important

Homotopy property $H: \bar{\Omega} \times [0, 1] \rightarrow \mathbb{R}^2, H = H(x, \lambda)$ continuous

$H(x, \lambda) \neq 0$ if $x \in \partial\Omega, \lambda \in [0, 1]$.

Then $\deg(H(\cdot, \lambda), \Omega)$ is independent of λ .

Proof. $\lambda \in [0, 1] \mapsto \deg(H(\cdot, \lambda), \Omega)$ is continuous and takes integer values.

The existence property $f: \bar{\Omega} \rightarrow \mathbb{R}^2$ continuous

$f(x) \neq 0 \quad \forall x \in \partial\Omega$

$\deg(f, \Omega) \neq 0 \Rightarrow f$ has a zero in Ω

Proof. Ω unit ball $H(x, \lambda) = f(\lambda x)$.

By contradiction, if f has no zeros H is a homotopy implying that

$$\deg(H(\cdot, 0), \Omega) = \deg(H(\cdot, 1), \Omega) = \deg(f, \Omega) \neq 0$$

But $H(x, 0) = f(0)$ and a constant function has degree zero (Notice that $f \circ \alpha$ has constant argument).

If Ω is any Jordan domain, $H(x, \lambda) = f(\Phi(x, \lambda))$

where $\Phi: \bar{\Omega} \times [0, 1] \rightarrow \bar{\Omega}$ continuous,

$\Phi(\cdot, 1) = \text{identity}, \Phi(\cdot, 0) = \text{constant}$

This is always possible since $\bar{\Omega}$ is homeomorphic to the closed unit ball.

Appendix: proofs of the lemmas

Lemma 1 First we prove the uniqueness. If $\theta_1, \theta_2: [a, b] \rightarrow \mathbb{R}$ are continuous and $(\cos \theta_1(t), \sin \theta_1(t)) = (\cos \theta_2(t), \sin \theta_2(t))$ then $\theta_2(t) = \theta_1(t) + 2\pi k(t)$ for each t , with $k(t) \in \mathbb{Z}$. As $t \mapsto k(t)$ is continuous and integer-valued, k is constant.

Existence for $\alpha \in C^1([a, b], \mathbb{R}^2)$

Before the proof assume that the result is valid and let us try to find a formula for $\dot{\theta}$

$$\alpha = r(\cos \theta, \sin \theta) \Rightarrow \dot{\alpha} = \dot{r}(\cos \theta, \sin \theta) + r \dot{\theta}(-\sin \theta, \cos \theta)$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ rotation of } 90^\circ \curvearrowright$$

$$\langle \dot{\alpha}, J\alpha \rangle = r^2 \dot{\theta}$$

Now we can start the proof: define

$$\theta(t) = \theta_0 + \int_a^t \frac{-\alpha_2(s) \dot{\alpha}_1(s) + \dot{\alpha}_2(s) \alpha_1(s)}{\alpha_1(s)^2 + \alpha_2(s)^2} ds,$$

with θ_0 chosen so that $\alpha(a) = \|\alpha(a)\| (\cos \theta_0, \sin \theta_0)$.

The function θ is C^1 and

$x_1 = \cos \theta, x_2 = \sin \theta$ is a solution of the system of differential equations

$$\dot{x}_1 = -\dot{\theta}(t)x_2, \quad \dot{x}_2 = \dot{\theta}(t)x_1$$

Next we prove that $y_1 = \frac{\alpha_1}{r}, y_2 = \frac{\alpha_2}{r}$ is also a solution of this system.

$$r^2 = \alpha_1^2 + \alpha_2^2 \Rightarrow r \dot{r} = \alpha_1 \dot{\alpha}_1 + \alpha_2 \dot{\alpha}_2$$

$$\begin{aligned} \dot{y}_1 &= \frac{\dot{\alpha}_1 r - \alpha_1 \dot{r}}{r^2} = \frac{\dot{\alpha}_1 r^2 - \alpha_1 r \dot{r}}{r^3} = \frac{\dot{\alpha}_1 r^2 - \alpha_1^2 \dot{\alpha}_1 - \alpha_1 \alpha_2 \dot{\alpha}_2}{r^3} \\ &= \frac{\alpha_2^2 \dot{\alpha}_1 - \alpha_1 \alpha_2 \dot{\alpha}_2}{r^3} = \frac{\alpha_2 \dot{\alpha}_1 - \alpha_1 \dot{\alpha}_2}{r^2} \frac{\alpha_2}{r} = -\dot{\theta}(t) y_2 \end{aligned}$$

u8

With a similar computation, $\dot{y}_2 = \dot{\Theta}(t) y_1$.

The solutions x_1, x_2 and y_1, y_2 satisfy the same initial condition at $t=a$. By uniqueness of the Cauchy problem,

$$x_1 = y_1, \quad x_2 = y_2.$$

An estimate $|\theta_1 - \theta_2| \leq \frac{\pi}{4} \Rightarrow |\theta_1 - \theta_2| \leq \sqrt{2} |e^{i\theta_1} - e^{i\theta_2}|$

It is enough to prove

$$0 \leq \theta \leq \frac{\pi}{4} \Rightarrow \theta \leq \sqrt{2} |e^{i\theta} - 1| \quad (\theta = \theta_1 - \theta_2)$$

$$|e^{i\theta} - 1| \geq |\cos\theta - 1| + |\sin\theta| \geq |\sin\theta| \\ \geq |\cos\zeta| \theta, \quad \text{for some } \zeta \in [0, \frac{\pi}{4}]$$

$$\Rightarrow |e^{i\theta} - 1| \geq \frac{1}{\sqrt{2}} \theta$$

Existence for α continuous

We construct a sequence $\alpha_n \in C^1([a, b], \mathbb{R}^2)$ such that $\alpha_n \rightarrow \alpha$ uniformly. For n large enough $\alpha_n(t) \neq 0 \forall t \in [a, b]$. We construct the argument function θ_n according to the above step. We find N such that

$$\left\| \frac{\alpha_n(t)}{\|\alpha_n(t)\|} - \frac{\alpha_m(t)}{\|\alpha_m(t)\|} \right\| < \frac{\pi}{8\sqrt{2}}, \quad n, m \geq N, \quad t \in [a, b]$$

We can also assume that $\theta_n(a)$ is convergent to θ_0 , so that $|\theta_n(a) - \theta_m(a)| < \frac{\pi}{4}$. From the estimate,

$$|\theta_n(t) - \theta_m(t)| < \frac{\pi}{8}$$

as soon as $|\theta_n(t) - \theta_m(t)| < \frac{\pi}{4}$. This implies

that $|\Theta_n(t) - \Theta_m(t)| < \frac{\pi}{4} \quad \forall t, n, m \geq N.$

Going back to the estimate,

$$|\Theta_n(t) - \Theta_m(t)| \leq \sqrt{2} \left\| \frac{\alpha_n(t)}{\|\alpha_n(t)\|} - \frac{\alpha_m(t)}{\|\alpha_m(t)\|} \right\|$$

and so $\{\Theta_n(t)\}$ is uniformly a Cauchy sequence. (*)

Thus $\Theta_n \rightarrow \Theta$ uniformly and, by a passage to the limit, Θ is the searched argument.

Lemma 2 The proof is a repetition of the last arguments.

(*) Notice that $\|\alpha_n(t)\| \geq \delta_0, n \geq N, \forall t$

Applications of the degree in the plane

The fundamental theorem of Algebra

Let $p \in \mathbb{C}[z]$, $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$ be a polynomial with $n \geq 1$. Then p has a root.

Proof First we recall the estimate for any possible root ξ ,

$$|\xi| \leq \max \{ 1, |a_0| + \dots + |a_{n-1}| \}.$$

Define

$$H(z, \lambda) = z^n + \lambda a_{n-1}z^{n-1} + \dots + \lambda a_1z + \lambda a_0$$

and let Ω be a ball centered at the origin with radius $R > \max \{ 1, |a_0| + \dots + |a_{n-1}| \}$.

Then $H(z, \lambda) \neq 0 \quad \forall z \in \partial\Omega, \lambda \in [0, 1]$

and so

$$\begin{aligned} \deg(p, \Omega) &= \deg(H(\cdot, 1), \Omega) \\ &= \deg(H(\cdot, 0), \Omega) = n > 0. \end{aligned}$$

Then existence property implies that p has a zero

Brouwer's fixed point theorem

$$B = \{ x \in \mathbb{R}^2 : \|x\| \leq 1 \}$$

$\Phi: B \rightarrow B$ continuous. Then Φ has a fixed point

Proof Assume that $\Phi(x) \neq x \quad \forall x \in \partial B$, for otherwise the theorem is already proven. Define

$$H(x, \lambda) = x - \lambda \Phi(x)$$

and observe that if $\|x\| = 1, \lambda \in [0, 1[$,

$$\|H(x, \lambda)\| \geq \|x\| - \lambda \|\Phi(x)\| \geq 1 - \lambda > 0.$$

Thus $H(x, \lambda) \neq 0$ if $x \in \partial B, \lambda \in [0, 1]$ and so

$$\begin{aligned} \deg(\text{id} - \Phi, \overset{\circ}{B}) &= \deg(H(\cdot, 1), \overset{\circ}{B}) \\ &= \deg(H(\cdot, 0), \overset{\circ}{B}) = \deg(\text{id}, \overset{\circ}{B}) = 1 \end{aligned}$$

Thus $\text{id} - \Phi$ has a zero and this is a fixed point of Φ .

Exercise Extend Brouwer's Theorem to any space which is homeomorphic to B .

A difference between 1 and 2 dimensions

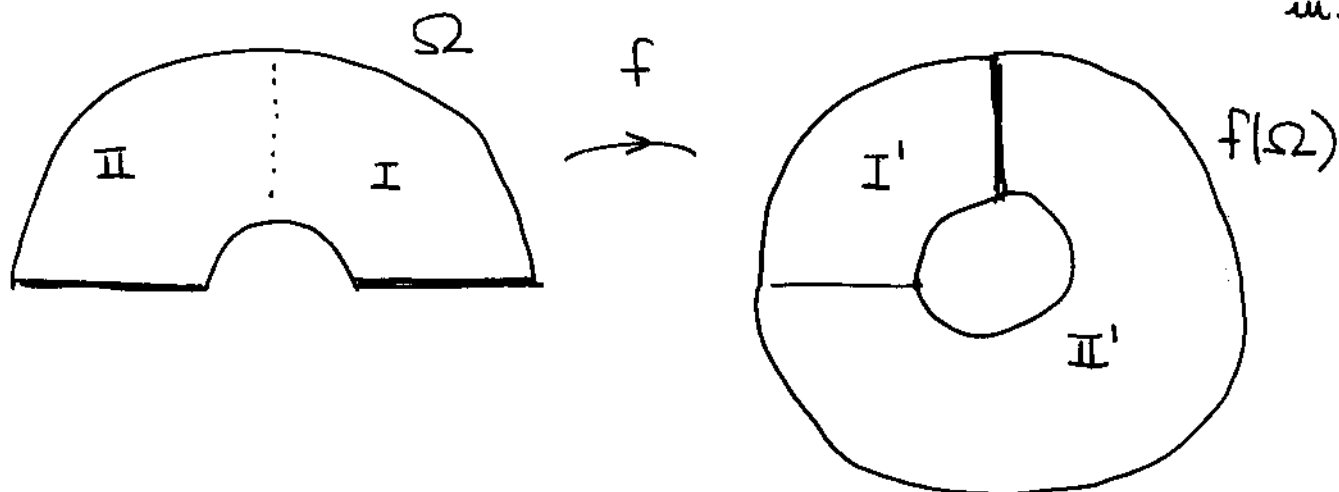
$$\Phi: [a, b] \rightarrow \mathbb{R} \text{ continuous, } [a, b] \subset \Phi([a, b])$$

$\Rightarrow \Phi$ has a fixed point

[Use Bolzano]

$$\Phi: \overline{\Omega} \rightarrow \mathbb{R}^2 \text{ continuous, } \Omega \text{ Jordan domain}$$

$$\overline{\Omega} \subset \Phi(\overline{\Omega}) \not\Rightarrow \Phi \text{ has a fixed point}$$



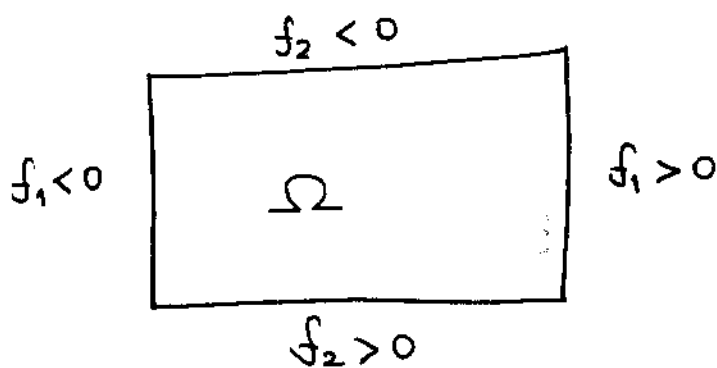
A Generalization of Bolzano's Th. (Poincaré-Miranda)

$$\Omega =]-a, a[\times]-b, b[$$

$$f_1(a, y) > 0 > f_1(-a, y)$$

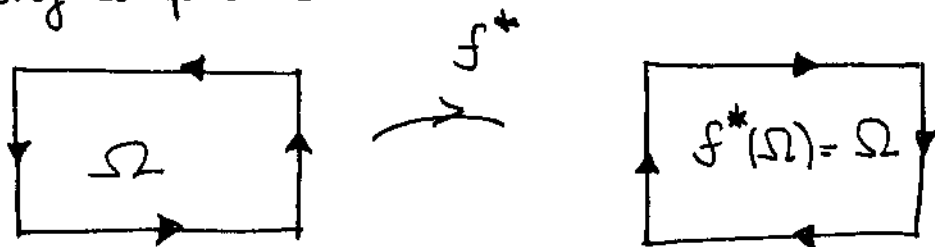
$$f_2(x, b) < 0 < f_2(x, -b)$$

$$f = (f_1, f_2): \overline{\Omega} \rightarrow \mathbb{R}^2 \text{ continuous.}$$



Then f has a zero in Ω .

Proof We first observe that the map $f^*(x, y) = (x, -y)$ is in the assumptions of the theorem. The degree is easily computed



$$\deg(f^*, \Omega) = -1$$

Define

$$H(x, y, \lambda) = (\lambda f_1(x, y) + (1-\lambda)x, \lambda f_2(x, y) - (1-\lambda)y)$$

We observe that for each $\lambda \in [0, 1]$, $H(\cdot, \cdot, \lambda)$ is in the conditions of the Theorem and so

$$H(x, y, \lambda) \neq 0 \quad \text{if } x = \pm a \text{ or } y = \pm b \\ \Leftrightarrow (x, y) \in \partial \Omega$$

Thus,

$$\begin{aligned} \deg(f, \Omega) &= \deg(H(\cdot, \cdot, 1), \Omega) = \deg(H(\cdot, \cdot, 0), \Omega) \\ &= \deg(f^*, \Omega) = -1. \end{aligned}$$

Exercise Construct variants of the above theorem

Periodic solutions of a system

$$\dot{u}_1 = F_1(t, u_2), \quad \dot{u}_2 = F_2(t, u_1)$$

$F_1, F_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ continuous, uniqueness for i. v. p.

F_2 is bounded,

$$F_1(t, \rho) < 0 < F_1(t, -\rho) \quad \text{if } t \in [0, 1], \rho \geq \mathbb{R}.$$

$$F_2(t, \rho) > 0 > F_2(t, -\rho)$$

Then there is a 1-periodic solution.

Proof First we observe that the solutions of the initial value problem are defined in $[0, 1]$. Indeed, as F_2 is bounded, $u_2(t)$ is bounded and so it cannot blow up. From there we know that $F_1(t, u_2(t))$ remains bounded and so $u_1(t)$ is also bounded.

We can consider

$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f(\xi, \eta) = (u_1(1, \xi, \eta) - \xi, u_2(1, \xi, \eta) - \eta)$ and we are going to prove that it has a zero using a variant of P.M. th.

$$\Omega =]-a, a[\times]-b, b[, \quad \begin{array}{c} f_1 < 0 \\ \square \\ f_1 > 0 \end{array} \quad \begin{array}{c} f_2 < 0 \\ \square \\ f_2 > 0 \end{array}$$

Let $M > 0$: $|F_2| \leq M$ everywhere and $M^* > 0$:

$$|F_1| \leq M^* \quad \text{if} \quad |u_2| \leq R + 2M.$$

Define $a = R + M^*$, $b = R + M$. From the second equation

$$|u_2(t, \xi, \eta) - \eta| \leq M.$$

$$\text{If } (\xi, \eta) \in \bar{\Omega}, \quad |u_2(t, \xi, \eta)| \leq R + 2M \Rightarrow$$

$$|F_1(t, u_2(t, \xi, \eta))| \leq M^*. \quad \text{From here,}$$

$$|u_1(t, \xi, \eta) - \xi| \leq M^*.$$

$$\text{If } (\xi, \eta) \in \bar{\Omega} \quad \text{with} \quad \eta = b = R + M, \quad u_2(t, \xi, \eta) \geq \eta - M = R$$

$$\Rightarrow F_1(t, u_2(t, \xi, b)) < 0 \Rightarrow$$

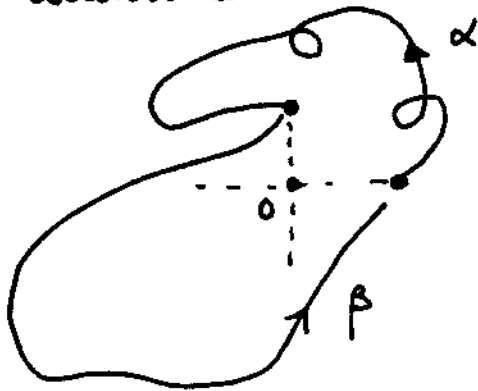
$$f_1(\xi, \eta) = u_1(1, \xi, \eta) - \xi = \int_0^1 F_1(t, u_2(t, \xi, b)) dt < 0$$

In a similar way, $f_2(5, -b) > 0$ and

$$f_2(a, \eta) > 0 > f_2(-a, \eta).$$

Some properties of the degree

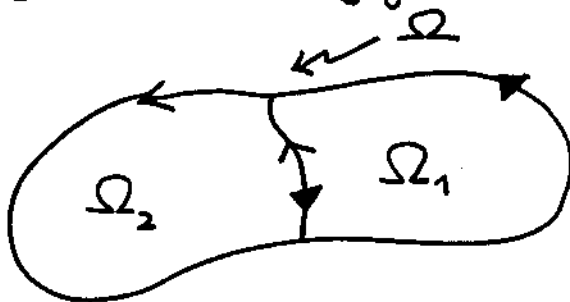
Given two continuous functions $\alpha_1: [a, b] \rightarrow \mathbb{R}^2 - \{0\}$,
 $\alpha_2: [b, c] \rightarrow \mathbb{R}^2 - \{0\}$, we can juxtapose and obtain
 the new function $\alpha_1 * \alpha_2: [a, c] \rightarrow \mathbb{R}^2 - \{0\}$. It is
 clear from the definition that the winding number
 is additive



winding number

$$\left. \begin{array}{l} \alpha \rightarrow \frac{1}{4} \\ \beta \rightarrow \frac{3}{4} \end{array} \right] \Rightarrow \alpha * \beta \rightarrow 1$$

Assume now that we have three Jordan domains as
 indicated in the figure



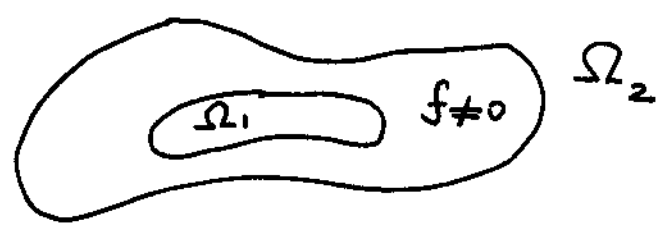
and $f: \bar{\Omega} \rightarrow \mathbb{R}^2$, $f(x) \neq 0 \quad \forall x \in \partial\Omega_1 \cup \partial\Omega_2$,
 f continuous. Then

$$\deg(f, \Omega) = \deg(f, \Omega_1) + \deg(f, \Omega_2)$$

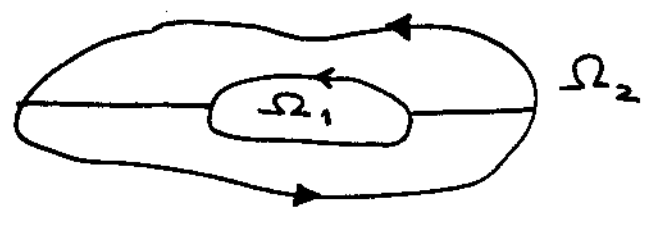
[Notice that $\partial\Omega \subset \partial\Omega_1 \cup \partial\Omega_2$]

With a similar argument we can deduce the
excision property:

Ω_1, Ω_2 Jordan domains with $\overline{\Omega_1} \subset \Omega_2$,
 $f: \overline{\Omega_2} \rightarrow \mathbb{R}^2$ continuous, $f(x) \neq 0 \quad \forall x \in \overline{\Omega_2} - \Omega_1$
 $\Rightarrow \deg(f, \Omega_1) = \deg(f, \Omega_2)$



Proof



Use additivity with 3 domains and the existence property

The degree of a linear map

Assume that L is a 2×2 matrix with $\det L \neq 0$ and let Ω be the unit ball centered at the origin. We observe that $Lx \neq 0 \quad \forall x \in \partial\Omega$ and we want to compute $\deg(L, \Omega)$. We recall that the group

$$GL(\mathbb{R}^2) = \{L \in M_2(\mathbb{R}) : \det L \neq 0\}$$

has two connected components: matrices with positive [resp negative] determinant.

Assume now that $\det L > 0$. Then we can define a continuous map $\lambda \in [0, 1] \mapsto L_\lambda \in GL_+(\mathbb{R}^2)$

with $L_0 = L, L_1 = I$. The map

$$H(x, \lambda) = L_\lambda x$$

is a homotopy since $L_2 x = 0 \Rightarrow x = 0$ and so

$$\deg(L, \Omega) = \deg(\text{Id}, \Omega) = 1$$

If $\det L < 0$ we make a homotopy with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and we know that this map has degree -1 ($z \mapsto \bar{z}$).

Summing up,

$$\deg(L, \Omega) = \begin{cases} 1 & \text{if } \det L > 0 \\ -1 & \text{if } \det L < 0 \end{cases} = \text{sign}(\det L).$$

Linearization and degree

Assume now that Ω is an arbitrary Jordan domain and $f: \bar{\Omega} \rightarrow \mathbb{R}^2$ is C^1 . Moreover f has a unique zero at $x_* \in \Omega$ and this zero is simple (this means that $\det(f'(x_*)) \neq 0$). Then

$$\deg(f, \Omega) = \deg(f'(x_*), \Omega) = \text{sign}(\det f'(x_*)).$$

Proof The inverse function theorem implies that there exists a small open ball centered at x_* , B , such that f is a diffeomorphism from B onto its image. By excision,

$$\deg(f, \Omega) = \deg(f, B).$$

For simplicity we assume $x_* = 0$ and define the homotopy on \bar{B} ,

$$H(x, \lambda) = \begin{cases} \frac{1}{\lambda} f(\lambda x) & \text{if } \lambda \in]0, 1] \\ f'(0)x & \text{if } \lambda = 0. \end{cases}$$

It is easy to check that H is continuous and has no zeros on ∂B (in fact the only zero is $x=0$). Thus, $\deg(H(\cdot, 1), B) = \deg(H(\cdot, 0), B)$ and this completes the proof.

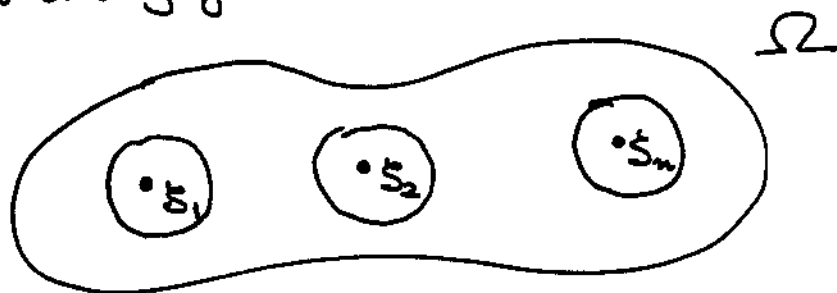
An important formula

Assume that Ω is a Jordan domain and $f: \bar{\Omega} \rightarrow \mathbb{R}^2$ is C^1 with $f(x) \neq 0 \forall x \in \partial\Omega$. In addition all its zeros are simple. We label them as $\xi_1, \dots, \xi_n \in \Omega$, then

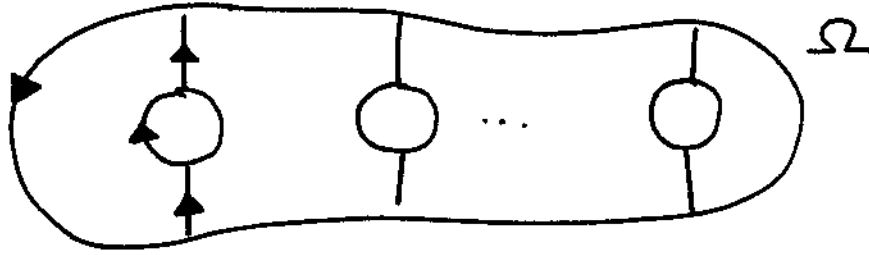
$$\deg(f, \Omega) = \sum_{i=1}^n \text{sign}(\det f'(\xi_i)).$$

Remark Notice that a simple zero is isolated (\Leftarrow inverse function th.) Since $\bar{\Omega}$ is compact $f^{-1}(0)$ must be finite.

Proof We find small disks around ξ_1, \dots, ξ_n as in the figure



Then we apply additivity and the linearization principle



Exercise Compute the degree of $f(x,y) = (e^{x+y}-1, e^{x-y}-1)$ in a neighborhood of the origin.

Degree in \mathbb{R}^d (The Nagumo approach)

Given $\Omega \subset \mathbb{R}^d$ non-empty, bounded and open
and $f: \bar{\Omega} \rightarrow \mathbb{R}^d$ continuous with

$$f(x) \neq 0 \quad \forall x \in \partial\Omega$$

we want to define $\deg(f, \Omega)$ so that the existence, homotopy and excision properties hold.

The idea will be to use the above formula as the definition of degree. We do the definition in three steps.

Step 1 f is C^1 and all its zeros are simple

$$\deg(f, \Omega) = \sum_{i=1}^n \text{sign}(\det f'(S_i))$$

with $f^{-1}(0) = \{S_1, \dots, S_n\}$. Convention $\sum_{\emptyset} = 0$.

Example $d=2$, $\mathbb{R}^2 \cong \mathbb{C}$, $f(z) = z^2 - \varepsilon$, $\varepsilon > 0$

Ω unit ball

$$f^{-1}(0) = \{\sqrt{\varepsilon}, -\sqrt{\varepsilon}\}, \quad f'(x, y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$$\deg(f, \Omega) = 2$$

Step 2 f is C^1 but its zeros are not necessarily simple

We apply Sard's Theorem which says that almost all vectors $v \in \mathbb{R}^d$ are regular values for f . This

means that if $f(\xi) = v$ then $\det f'(\xi) \neq 0$. May be the zero vector $v = 0$ is not a regular value but certainly we can find a sequence of vectors $v_n \in \mathbb{R}^d$ such that they are regular values and converge to 0. We can apply Step 1 to the map $f_n(x) = f(x) - v_n$ and define (*)

$$\deg(f, \Omega) = \lim_{n \rightarrow \infty} \deg(f_n, \Omega).$$

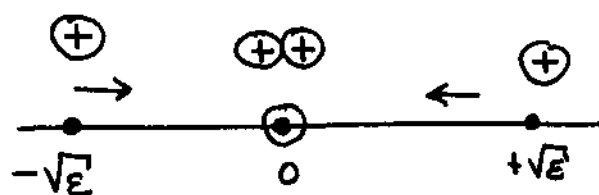
For this definition to make sense one must prove that the sequence $\deg(f_n, \Omega)$ becomes eventually constant and also that the limit is independent of the choice of v_n .

Exercise Prove the previous assertions for $d = 2$ and Ω a Jordan domain.

Example $d = 2, \mathbb{R}^2 \cong \mathbb{C}, f(z) = z^2, \Omega$ unit ball

$$\deg(f, \Omega) = \lim_{\varepsilon \downarrow 0} \deg(f_\varepsilon, \Omega) = 2$$

$$f_\varepsilon(z) = z^2 - \varepsilon$$



(*) Remark If $f(x) \neq 0 \forall x \in \partial\Omega \Rightarrow$

$f_n(x) \neq 0 \forall x \in \partial\Omega$ and n large

Step 3 (General Case) f is continuous

We can approximate f by a sequence of functions f_n of class C^1 . This means that $f_n \rightarrow f$ uniformly in $\bar{\Omega}$. The maps f_n satisfy $f_n(x) \neq 0 \forall x \in \partial\Omega$ if n is large and so $\deg(f_n, \Omega)$ is defined according to Step 2.

By definition,

$$\deg(f, \Omega) = \lim_{n \rightarrow \infty} \deg(f_n, \Omega).$$

It must be proven that this limit exists and is independent of the choice of f_n . Again we know how to prove this for $d=2$.

Example $\mathbb{D} \ni f(x_1, \dots, x_d) = (x_1, x_2, \dots, x_d)$

$\Omega =$ unit ball in \mathbb{R}^d

$$f_\varepsilon(x_1, \dots, x_d) = (\sqrt{x_1^2 + \varepsilon}, x_2, \dots, x_d)$$

According to Step 1,

$$\deg(f_\varepsilon, \Omega) = 0 \text{ since it has no zeros}$$

$$\Rightarrow \deg(f, \Omega) = 0$$

Now we can generalize the result of the first lecture on the Dirichlet problem.

The Dirichlet problem for systems

$$\begin{cases} \ddot{u}_1 = F_1(t, u_1, \dots, u_N) & , \quad u_1(0) = u_1(1) = 0 \\ \dots & \dots \\ \ddot{u}_N = F_N(t, u_1, \dots, u_N) & , \quad u_N(0) = u_N(1) = 0 \end{cases}$$

$F_i: [0, 1] \times \mathbb{R}^N \rightarrow \mathbb{R}$ continuous and bounded

Uniqueness of the initial value problem

\Rightarrow There is a solution of the Dirichlet problem.

The proof is an easy consequence of

Lemma $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ continuous, $\exists M > 0$:

$$\|f(x) - x\| \leq M \quad \forall x \in \mathbb{R}^d.$$

Then f has a zero.

Proof $f(x) = x + \Phi(x)$ with $\|\Phi\| \leq M$

Define the homotopy

$$H(x, \lambda) = \lambda f(x) + (1-\lambda)x, \quad \lambda \in [0, 1]$$

$$H(x, \lambda) = 0 \iff x + \lambda \Phi(x) = 0 \implies \|x\| \leq$$

$$\|\Phi(x)\| \leq M.$$

Let Ω be a ball centered at the origin

with radius $> M$.

The homotopy property implies that

$$\begin{aligned} \deg(f, \Omega) &= \deg(H(\cdot, 1), \Omega) \\ &= \deg(H(\cdot, 0), \Omega) = \deg(\text{id}, \Omega) \stackrel{\text{Step 1}}{=} 1 \end{aligned}$$

For applications to p.d.e.'s it is important to define the degree in spaces of infinite dimensions. This is not immediate and cannot be done for all continuous mappings. We finish with an example of a map $f: H \rightarrow H$ which is continuous, H is a Hilbert space and $f(B) \subset B$, $B = \{x \in H: \|x\| \leq 1\}$ and such that f has no fixed points. This shows that Brouwer's Fixed Point Theorem cannot be extended to infinite dimensions and the map

$\text{id} - f$ is homotopic to id but has no zeros.

Example $H = \ell^2$, $\| \{x_n\} \| = \sqrt{\sum x_n^2}$

$$f(\{x_0, x_1, \dots, x_n, \dots\}) = \{ \cancel{x_0}, \cancel{x_1}, \dots, \cancel{x_n}, \dots \}$$

$$= \{ \sqrt{1 - \|x\|^2}, x_0, x_1, \dots \}$$

$f(B) \subset \partial B \subset B$, f has no fixed points