Continuation of closed orbits in autonomous systems

(Siegel and Moser, Chapter 21)

Let us consider the system

$$\dot{x} = f(x,\mu), \quad x \in \Omega, \ \mu \in I$$

where Ω is an open subset of $\mathbb{R}^m$ and μ is a parameter lying on the interval $I = [0, \mu_*]$. The vector field $f : \Omega \times I \to \mathbb{R}^m$ will be of class C^1.

Let γ be a closed orbit for $\mu = 0$. The goal is to continue γ for μ small. Let $x^*(t)$ be an associated periodic solution,

$$\gamma = \{ x^*(t) : t \in [0, \tau^*] \}$$

where $\tau^* > 0$ is a period (not necessarily the minimal period). Associated to γ (or x^*) is the variational equation

$$\dot{y} = f'_x(x^*(t), 0)\, y.$$

This is a periodic linear equation and Floquet theory is applicable. Let $\Phi(t)$ be the matrix solution with $\Phi(0) = I_m$, then $M = \Phi(\tau^*)$ is a <u>monodromy matrix</u>.

Now we can be more precise about the goal: to find conditions for local continuation of γ in terms of M. This may seem a useless goal since monodromy matrices cannot be computed for most linear periodic systems.

In some interesting cases the computation of M is possible (and long). Complex notation can help.

Computing a monodromy matrix using complex notation

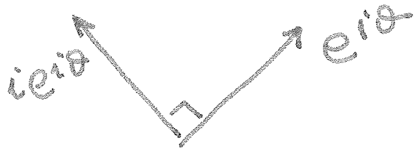
Consider

$$\dot{z} = iz - z(1-|z|^2), \quad z \in \mathbb{C}$$

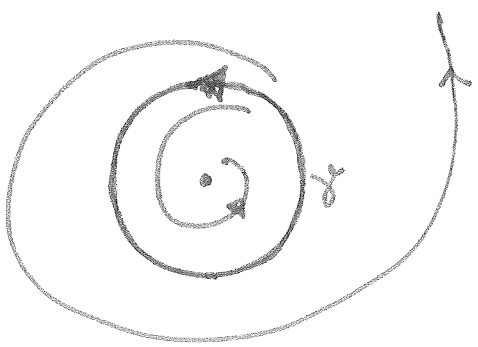
Passing to polar coordinates $z = \rho e^{i\theta}$,

$$\dot{\rho} e^{i\theta} + \rho \dot{\theta} i e^{i\theta} = i\rho e^{i\theta} - \rho e^{i\theta} + \rho^3 e^{i\theta}$$

The vectors $e^{i\theta}$ and $ie^{i\theta}$ are $\mathbb{R}$-linearly independent if $\rho > 0$



$\dot{\rho} = \rho(\rho^2 - 1)$, $\dot{\theta} = 1$. The unit circle $\gamma = \mathbb{S}^1$ is a closed orbit with $z^*(t) = e^{it}$ (a repeller)



To compute the variational equation along $z^*(t)$ we employ the notation $W = \frac{\partial z}{\partial \xi}$, $z = z(t;\xi)$, $z(0;\xi) = \xi$, and express the vector field in terms of z and $\bar{z}$,

$$\dot{z} = (i-1)z + z^2\bar{z}.$$

Differentiating with respect to ζ,

$$\dot{w} = (i-1)w + 2z\bar{z}w + z^2\bar{w}.$$

Letting $z = e^{it}$,

$$\dot{w} = (i-1)w + 2w + e^{2it}\bar{w} = (i+1)w + e^{2it}\bar{w}.$$

To solve it we employ the change of variables $w = e^{it}v$,

$$e^{it}\dot{v} + ie^{it}v = (i+1)e^{it}v + e^{it}\bar{v} \rightsquigarrow$$

$$\dot{v} = v + \bar{v} \quad \text{System of constant coefficients}$$

$$v = v_1 + iv_2, \qquad \dot{v}_1 + i\dot{v}_2 = 2v_1,$$

$$\dot{v} = Av, \quad A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Phi}(t) = e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & 1 \end{pmatrix}$$

Going back to the w-plane

$$\Phi(t) = R[t]e^{At} \quad \text{with } R[t] = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

(e^{it} is identified to Rotation 90° ↺). If $\tau^* = 2\pi$,

$$M = \Phi(2\pi) = R[2\pi]e^{A2\pi} = \begin{pmatrix} e^{4\pi} & 0 \\ 0 & 1 \end{pmatrix}$$

Remark:
An alternative method to solve the variational equation is to observe that $\dot{z}^*(t) = ie^{it}$ is a solution and then reduce the order.

A first attempt: continuation with fixed period

Let $x(t;\xi,\mu)$ be the general solution of the nonlinear system $(x(0)=\xi)$. This function is well defined and of class C^1 in some neighborhood $\mathcal{U} \subset \mathbb{R}\times\mathbb{R}^m\times I$ of $[0,\tau^*]\times\gamma\times\{0\}$. We define the function

$$\Phi(\xi,\mu) = x(\tau^*;\xi,\mu) - \xi$$

so that the zeros of Φ produce periodic solutions with period τ^*. We are assuming that γ is a closed orbit for $\mu=0$ and so

$$f(x,0) \neq 0 \text{ for each } x\in\gamma.$$

when we look for ~~closed orbits around γ and μ is~~ zeros of Φ with ξ close to γ and μ small, we can assume that f does not vanish and so we produce closed orbits and not equilibria.

Given any $\xi^*\in\gamma$ we know that

$$\Phi(\xi^*,0)=0$$

and so one could try to continue this zero $(\xi^*,0)$ to $(\xi(\mu),\mu)$ via the Implicit Function Theorem.

This is not possible because for $\mu=0$ we already have a continuum of zeros (the initial conditions of γ^*). Indeed,

$$\Phi(x^*(t),0)=0, \quad \cancel{t\in\mathbb{R}},\ t\in[0,\tau^*],$$

and, differentiating in t,

$$\partial_\xi \Phi(x^*(t),0)\,\dot{x}^*(t)=0.$$

Letting $t=0$,

$\partial_\xi \Phi(\xi^*,0)\,\dot{x}^*(0)=0$. From the equation

$\partial_\xi \Phi(\xi^*,0)\cdot f(\xi^*,0)=0$ and, since $f(\xi^*,0)\neq 0$, the kernel of $\partial_\xi \Phi(\xi^*,0)$ is non-trivial. Thus

$$\det(\partial_\xi \Phi(\xi^*,0))=0 \quad \text{if } \xi^*\in\gamma.$$

<u>An alternative approach : changing the period</u>

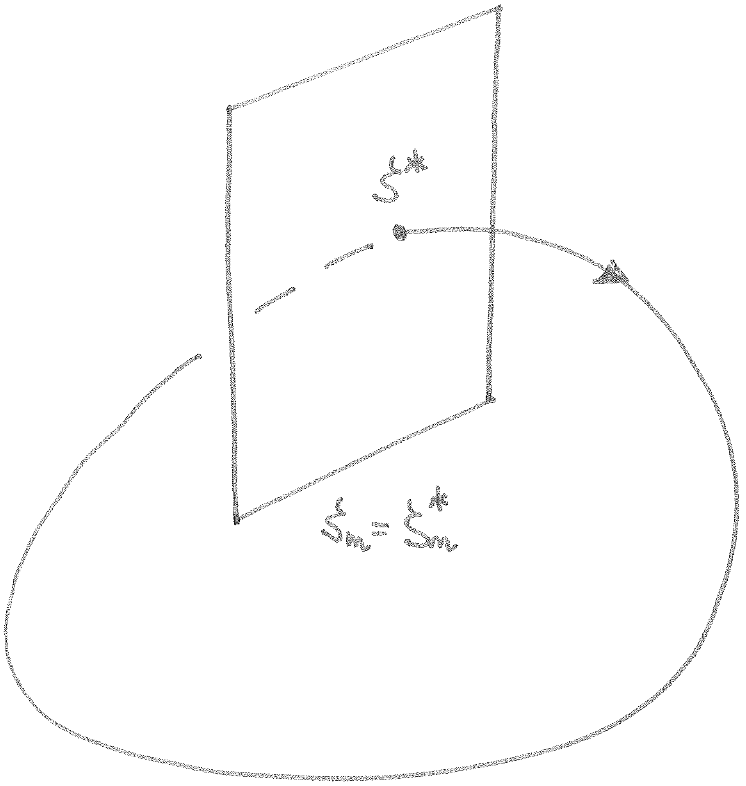
Now we let the period τ to vary and so we try to solve

$$x(\tau;\xi,\mu)=\xi.$$

There are more unknowns than equations: (τ,ξ) $m+1$ unknowns, m equations $x_i=\xi_i$. We fix a coordinate of ξ, say the last one ξ_m.

Let $\hat{\xi}$ denote an arbitrary point of $\mathbb{R}^{m-1}$. Define (locally)

$$\Phi(\tau, (\hat{\xi}, \xi_m^*); \mu) = x(\tau; (\hat{\xi}, \xi_m^*), \mu) - (\hat{\xi}, \xi_m^*)$$



$\mu = 0$

This is not exactly the same as the method of transversal Sections. In that method there is a reduction of dimension and one must solve $m-1$ equations. We are keeping τ as an unknown and so we must solve m equations. The advangtadge is that it is not necessary to assume that the vector field is transversal to $\xi_m = \xi_m^*$. Now $(\tau^*, \hat{\xi}^*; 0)$ is a zero of Φ and we can try to find a continuation $\tau = \tau(\mu)$, $\hat{\xi} = \hat{\xi}(\mu)$ via the Implicit Function Theorem. To this end we need

$$0 \neq \det\left(\frac{\partial \Phi}{\partial(\tau,\hat{\xi})}(\tau^*,\hat{\xi}^*;0)\right) =$$

$$\det\left(f(\xi^*;0) \,\Big|\, \frac{\partial x}{\partial \xi_1}(\tau^*;\xi^*,0)-e_1 \,\Big|\, \dots \,\Big|\, \frac{\partial x}{\partial \xi_{m-1}}(\tau^*;\xi^*,0)-e_{m-1}\right)$$

with $e_k = (0,\dots,\overset{(k}{1},\dots 0)^t$.

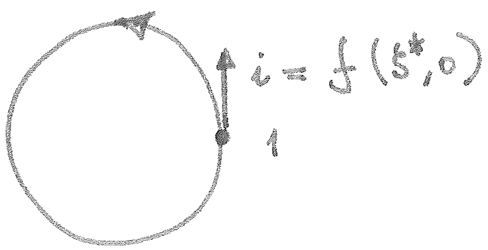
This condition can be verified in some cases, as it is shown by the following example:

$$\dot{z} = iz - z(1-|z|^2) + \mu g(z,\bar{z};\mu)$$

with $g : \mathbb{C}\times[0,\mu_*] \longrightarrow \mathbb{C}$ of class C^1.

For $\mu = 0$ we have the closed orbit $\gamma = \mathbb{S}^1$ with $\tau^* = 2\pi$,

$\xi^* = 1$,



$$\det\left(f(\xi^*,0) \,\Big|\, \frac{\partial z}{\partial \xi_1}(2\pi;\xi^*,0)-e_1\right)$$

$$= \det\begin{pmatrix} 0 & e^{4\pi}-1 \\ 1 & 0 \end{pmatrix} \neq 0.$$

In the previous discussions the choice of the hyperplane $\xi_m = \xi_m^*$ has been arbitrary. Playing with all possible hyperplanes we arrive at the conclusion, Sufficient Condition for Continuation : 1 is algebraically simple as an eigenvalue of M.

Notice that $1 \in \sigma(M)$ since $Mf(\xi^*,0) = f(\xi^*,0)$.

Sketch of the proof :

Lemma If $1 \in \sigma(M)$ is algebraically simple, then

$$\mathbb{R}^m = \operatorname{Ker}(M-I) \oplus \operatorname{Im}(M-I).$$

Proof From the isomorphism theorem we know that

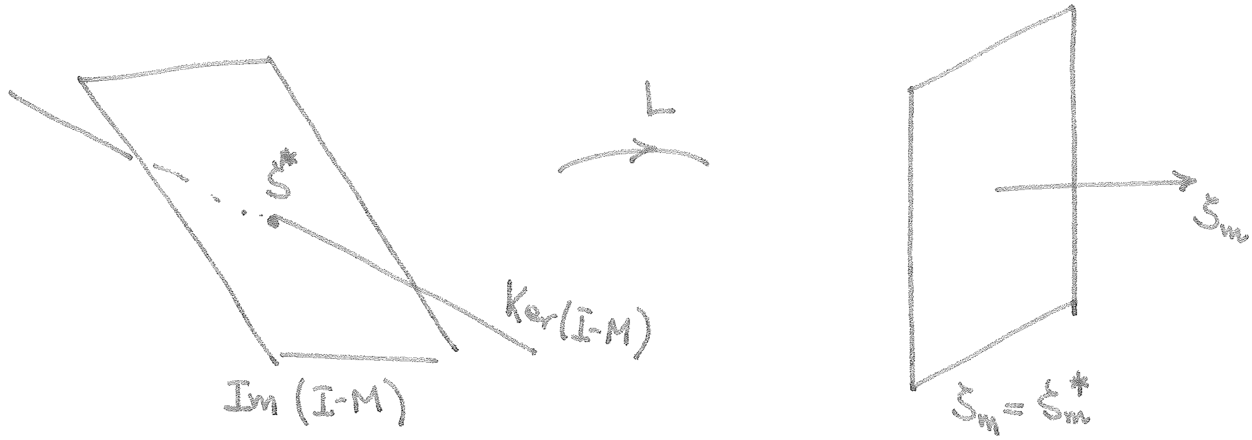
$$\dim \operatorname{Ker}(M-I) + \dim \operatorname{Im}(M-I) = m.$$

Let us prove that $\operatorname{Ker}(M-I) \cap \operatorname{Im}(M-I) = 0$.
Given v in this intersection, there exists w with $(M-I)w = v$. Then $(M-I)^2 w = (M-I)v = 0$ and so w belongs to the iterated kernel $\operatorname{Ker}(M-I)^2$. Since 1 is simple, $w \in \operatorname{Ker}(M-I) = \operatorname{Ker}(M-I)^2$ and so $v = (M-I)w = 0$.

$$\operatorname{Ker}(M-I) \oplus \operatorname{Im}(M-I)$$

spanned by $f(\xi^*,0)$

plays the role of $\mathbb{R}^{m-1} \times \{\xi_m^*\}$



$$X = Lx, \quad \dot{X} = L f(L^{-1}X;\mu) = F(X;\mu)$$

$$\det\left(F(\xi^*,0) \,\Big|\, \frac{\partial X}{\partial \xi_1}(\tau^*;\xi^*,0) - e_1 \,\Big|\, \cdots \,\Big|\, \frac{\partial X}{\partial \xi_{m-1}}(\tau^*;\xi^*,0) - e_{m-1} \right)$$

$= \det\left(e_m \,\Big|\, L\left(\frac{\partial x}{\partial \xi_i} - e_i\right)L^{-1} \right) \neq 0$ because the restriction of $I-M$ to $\mathrm{Im}\,(I-M)$ is an isomorphism.

The impossibility of applying the previous method to a Hamiltonian system

Consider $\dot{x} = J\nabla H(x)$, $x \in \Omega \subset \mathbb{R}^m = \mathbb{R}^{2d}$

$H = H(x)$ of class C^2, $J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$.

$\gamma \subset \Omega$ closed orbit. Notice that

$$\nabla H(\xi) \neq 0 \quad \forall \xi \in \gamma$$

since otherwise γ would contain equilibria.

In these setting 1 cannot be a simple eigenvalue of M and so it is not possible to apply the previous result for continuation of $\dot{x} = J\nabla H(x,\mu)$.

We explain this obstruction in a more general context, autonomous systems with a first integral.

We go back to

$$\dot{x} = f(x), \quad x \in \Omega \subset \mathbb{R}^m.$$

The parameter μ has been eliminated because it plays no role in the discussion. We assume that there is a first integral $\Psi = \Psi(x)$ of class C^1 and such that

$$\nabla\Psi(\xi) \neq 0 \quad \text{for each } \xi \in \gamma.$$

Notice that this is always the case in a Hamiltonian system with $\Psi = H$. In this setting,

<u>1 is not a simple(*) eigenvalue of M</u>

To prove this we start with an <u>algebraic lemma</u>:

$L : \mathbb{R}^m \to \mathbb{R}^m$ linear, $v \in \operatorname{Ker} L$, $w \in \operatorname{Ker} L^*$, $v \perp w$, $v \neq 0$, $w \neq 0$. Then $\operatorname{Ker} L^2 \neq \langle v \rangle$

<u>Proof</u> By contradiction assume that $\operatorname{Ker} L = \operatorname{Ker} L^2 = \langle v \rangle$. Then $\operatorname{Ker} L^* = \langle w \rangle$ since both kernels have the same dimension. By Fredholm's alternative,

---

(*) Simple means $\dim \operatorname{Ker} L^2 = 1$. One-dimensional Jordan box. Simple root of the characteristic polynomial, ...

$$\operatorname{Im} L = (\operatorname{Ker} L^*)^\perp = w^\perp \Longrightarrow v \in \operatorname{Im} L .$$

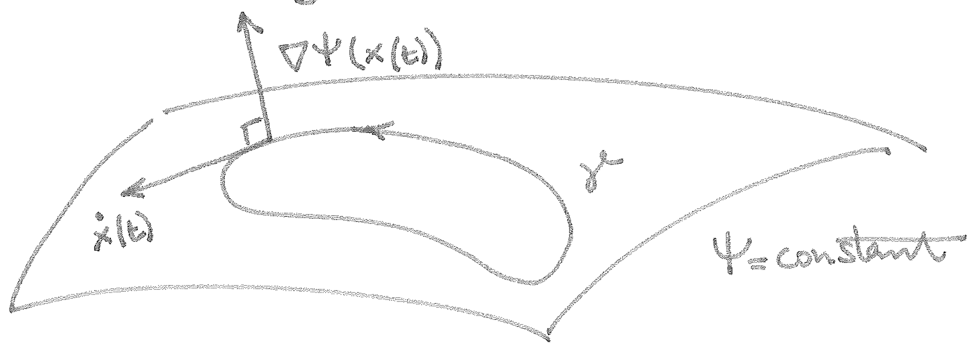
The previous algebraic lemma implies that $\operatorname{Ker} L \cap \operatorname{Im} L = 0$ and so we have reached a contradiction.

Let us go back to our system of differential equations with first integral. We know, for the general solution,

$$\Psi(x(t,\xi)) = \Psi(\xi).$$

Differentiating in t and ξ,

$$\left.\begin{aligned} &\partial\Psi(x(t,\xi))\,\dot{x}(t,\xi) = 0 \\ &\partial\Psi(x(t,\xi))\,\frac{\partial x}{\partial \xi}(t,\xi) = \partial\Psi(\xi) \end{aligned}\right\} \quad \text{with } \partial\Psi = (\nabla\Psi)^t$$



Letting $t = \tau^*$, $\xi = \xi^*$,

$$\nabla\Psi(\xi^*) \perp f(\xi^*), \qquad \nabla\Psi(\xi^*) \in \operatorname{Ker}[M^* - I]$$

Since we know that $f(\xi^*) \in \operatorname{Ker}[M - I]$, we can apply the algebraic lemma.

Systems with a first integral: fixing the period again

Let us go back to the system with parameter

$$\dot{x} = f(x,\mu),\ x \in \Omega,\ \mu \in I$$

having a closed orbit γ for $\mu = 0$. From now on we assume that there exists a first integral $\Psi = \Psi(x,\mu)$, depending on the parameter and of class C^1 on $\Omega \times I$. We also assume that

$$\nabla_x \Psi(x, 0) \neq 0 \quad \text{for each } x \in \gamma.$$

Notice that these assumptions are natural for a Hamiltonian system with $H = H(x,\mu)$.

We will fix the period τ^* of γ and look for solutions of

$$x(\tau^*; \xi, \mu) = \xi$$

with ξ close to some point of γ and μ small. The crucial observation is that we do not need to solve the whole system, finding a solution of $n-1$ coordinates

$$x_i(\tau^*, \xi, \mu) = \xi_i,\quad i \neq h$$

the first integral will imply that the equation for the coordinate h also holds,

$$\Psi(x(\tau^*, \xi, \mu), \mu) = \Psi(\xi, \mu) \Longrightarrow x_h(\tau^*, \xi, \mu) = \xi_h$$

Of course, for this implication to be valid, we need some non-degeneracy on $\frac{\partial \Psi}{\partial x_h}$.

Theorem In the previous conditions assume that 1 is an eigenvalue of the monodromy matrix M with algebraic multiplicity 2 and geometric multiplicity 1,

$$\left[\dim \operatorname{Ker} (M-I) = 1, \ \dim \operatorname{Ker} (M-I)^2 = 2 \right].$$

Then γ admits a local continuation with fixed period τ^*.

Remark This result can be improved: there is a family of closed orbits depending on two parameters $\gamma = \gamma(\tau, \mu)$, the period τ and μ.

Proof. Let J be the Jordan canonical form of M,

$$J = \left(\begin{array}{c|cc} \hat{J} & 0 & \\ \hline 0 & 1 & 1 \\ & 0 & 1 \end{array} \right), \qquad 1 \notin \sigma(\hat{J}).$$

In principle $M = PJP^{-1}$ but, after the change of variables $x = PX$, we can assume that $M = J$. Notice that the system in X has the first integral $\Psi(X,\mu) = \psi(PX,\mu)$ and it satisfies the non-degeneracy condition along the closed orbit. From now on we work with the system in x but assume $M = J$.

First we check that if $\xi^* \in \gamma$ then

$$\frac{\partial \psi}{\partial x_m}(\xi^*, 0) \neq 0.$$

We start from

$$M^* \nabla \psi(\xi^*, 0) = \nabla \psi(\xi^*, 0)$$

and split the gradient as $\nabla \psi = \begin{pmatrix} \nabla \psi_1 \\ \hline \nabla \psi_2 \end{pmatrix}$ with

$\nabla\psi_1 \in \mathbb{R}^{m-2}$, $\nabla\psi_2 \in \mathbb{R}^2$.

$$M^*\nabla\psi = \left(\begin{array}{c|c} \hat{J}^* & 0 \\ \hline 0 & N^* \end{array}\right)\left(\begin{array}{c} \nabla\psi_1 \\ \hline \nabla\psi_2 \end{array}\right) = \left(\begin{array}{c} \hat{J}^*\nabla\psi_1 \\ \hline N^*\nabla\psi_2 \end{array}\right)$$

with $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Since $1 \notin \sigma(\hat{J}^*)$, $\nabla\psi_1 = \hat{J}^*\nabla\psi_1$ implies that $\nabla\psi_1 = 0$. From $(N^* - I)\nabla\psi_2 = 0$ we deduce that the first component of $\nabla\psi_2$ vanishes. The non-degeneracy condition implies that the last component is non-zero.

Next we fix $\xi_{m-1} = \xi^*_{m-1}$ and solve

$$x_1(\tau^*; \xi, \mu) = \xi_1, \dots, \; x_{m-2}(\tau^*, \xi, \mu) = \xi_{m-2}, \; x_{m-1}(\tau^*; \xi, \mu) = \xi^*_{m-1}$$

There are $m-1$ unknowns $(\xi_1, \dots \xi_{m-2}, \xi_m)$ and $m-1$ equations. To apply the Implicit Function Theorem we must check that the Jacobian matrix has non-zero determinant. This matrix is

$$\tilde{M} - \left(\begin{array}{c|c} I_{m-2} & 0 \\ \hline 0 & 0 \end{array}\right)$$

where $\tilde{M}$ is the sub-matrix of M obtained by elimination of ~~the~~ row m and column $m-1$. That is,

$$\tilde{M} = \left(\begin{array}{c|c} \hat{J} & 0 \\ \hline 0 & 1 \end{array}\right).$$

We find $\xi_i = \xi_i(\mu)$, $i = 1, \dots m-2, m$, $\xi_{m-1} = \xi^*_{m-1}$ solving the equations above and $\xi_i(0) = \xi^*_i$.

By continuous dependence, if μ is small, $x(t, \xi(\mu), \mu)$ is close to $x(t, \xi^*, 0)$ for $t \in [0, \tau^*]$. In particular we can assume that

$$\frac{\partial \Psi}{\partial x_m}(x, \mu) \neq 0$$

if x lies in the segment between ξ^* and $x(\tau^*, \xi(\mu), \mu)$

At this point we do not know that $x(t, \xi(\mu), \mu)$ is τ^*-periodic since the last equation $x_m(\tau^*, \xi(\mu), \mu) = \xi(\mu)$ has not been checked.

From the mean value theorem

$$0 = \Psi(x(\tau^*, \xi(\mu), \mu), \mu) - \Psi(\xi(\mu), \mu) =$$

$$\langle \nabla \Psi(\tilde{\xi}(\mu), \mu), x(\tau^*, \xi(\mu), \mu) - \xi(\mu) \rangle$$

where $\tilde{\xi}(\mu)$ lies in the segment between $\xi(\mu)$ and $x(\tau^*, \xi(\mu), \mu)$. We know that the first $m-1$ coordinates of these two vectors coincide so that

$$0 = \underbrace{\frac{\partial \Psi}{\partial x_m}(\tilde{\xi}(\mu), \mu)}_{\neq 0} \Big(x_m(\tau^*, \xi(\mu), \mu) - \xi_m(\mu) \Big).$$

An example without continuation

Consider the second order equation

$$\ddot{x} + x + \mu x^3 = 0, \quad x \in \mathbb{R}$$

or the first order system

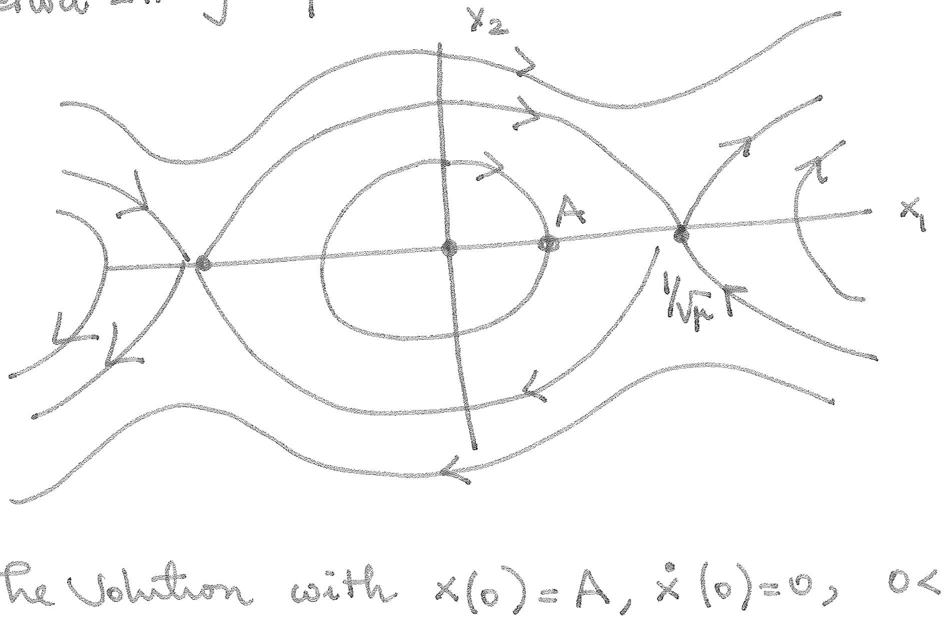
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - \mu x_1^3.$$

It has the first integral

$$\Psi(x,\mu) = \frac{1}{2} x_2^2 + \frac{1}{2} x_1^2 + \frac{1}{4}\mu x_1^4.$$

For $\mu = 0$ we have the closed orbit $\gamma = \mathbb{S}^1$ with period $\tau^* = 2\pi$. The non-degeneracy condition is satisfied (notice that it is a Hamiltonian system).

We will see that there are no closed orbits of period 2π for $\mu < 0$. The phase portrait is



The solution with $x(0) = A$, $\dot{x}(0) = 0$, $0 < A < \frac{1}{\sqrt{\mu}}$ is periodic with minimal period

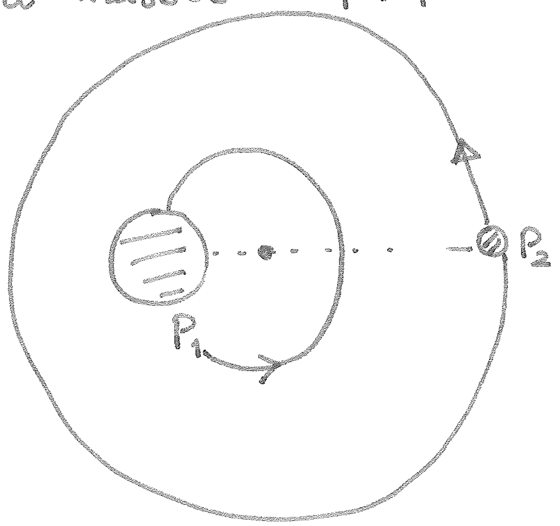
$$T(A,\mu) = \sqrt{2} \int_{-A}^{A} \frac{dx}{\sqrt{\frac{1}{2}(A^2 - x^2) - \frac{\mu}{4}(A^4 - x^4)}}$$

We notice that, for fixed A, $T(A,\cdot)$ is increasing. Also, ~~[illegible]~~ $T(A,0) = 2\pi$ and so $T(A,\mu) > 2\pi$

if $\mu > 0$. Hence, the minimal period of any closed orbit is larger than 2π. In this case the monodromy matrix is $M = I$ and the assumption on the multiplicities in the theorem fails.

Circular RTBP

Suppose that the primaries P_1 and P_2 have circular orbits and masses $1-\mu$, μ with $\mu \in [0, \frac{1}{2}]$



The center of mass is placed at the origin,

$$(1-\mu) P_1(t) + \mu P_2(t) = 0.$$

If we adjust the gravitation constant ($G = 1$), $P_1(t) = -\mu e^{it}$, $P_2(t) = (1-\mu) e^{it}$ is an admissible motion. The third body has a negligible mass ε and so its position satisfies

$$\varepsilon \ddot{w} = \varepsilon (1-\mu) \frac{P_1(t) - w}{|P_1(t) - w|^3} + \varepsilon \mu \frac{P_2(t) - w}{|P_2(t) - w|^3}$$

This is a 2π-periodic system in $\mathbb{C}^2$. We perform the change of variables $w = e^{it}z$

$$\ddot{w} = -e^{it}z + 2ie^{it}\dot{z} + e^{it}\ddot{z}$$

$$\| \quad (1-\mu)\frac{\cancel{+}\mu e^{it}-e^{it}z}{|\mu e^{it}+e^{it}z|^3} + \mu\frac{(1-\mu)e^{it}-e^{it}z}{|(1-\mu)e^{it}-e^{it}z|^3} \rightsquigarrow$$

$$\ddot{z} + 2i\dot{z} - z = -(1-\mu)\frac{\mu+z}{|\mu+z|^3} + \mu\frac{1-\mu-z}{|1-\mu-z|^3}$$

This is an autonomous system with $z \in \mathbb{C}\setminus\{-\mu, 1-\mu\}$.

We notice that the singularities correspond to the position (in the rotating system) of the primaries. For $\mu=0$ there is only one singularity. This is the restricted/restricted problem and P_2 does not affect w.

We can rewrite the system in our framework with $x = (z, \dot{z}) \in \mathbb{R}^4$, $\dot{x} = f(x,\mu)$, $\mu \in [0, \frac{1}{2}]$.

This system has the first integral

$$\Psi(z,\dot{z},\mu) = \frac{1}{2}|\dot{z}|^2 - \frac{1}{2}|z|^2 - \frac{1-\mu}{|\mu+z|} - \frac{\mu}{|1-\mu-z|}.$$

This can be obtained using any of the following observations:

i) the function $\Psi = \frac{1}{2}\|\dot{x}\|^2 + V(x)$ is a first integral of

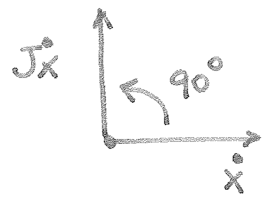
$$\ddot{x} + 2J\dot{x} + \nabla V(x) = 0, \quad x \in \Omega \subset \mathbb{R}^2,$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Indeed, $\frac{d\Psi}{dt} = \langle \dot{x}, \ddot{x} \rangle + \langle \nabla V(x), \dot{x} \rangle$

$$= -2\langle \dot{x}, J\dot{x} \rangle = 0$$

since $\dot{x} \perp J\dot{x}$



ii) The restricted C3BP has a Hamiltonian structure with $q = x$, $p = \dot{x} + Jx$, $H(q,p) = \frac{1}{2}\|p\|^2 - \langle p, Jq \rangle + \tilde{V}(q)$.

We want to do continuation from $\mu = 0$ and to this end we need to know a closed orbit. For $\mu = 0$ the system becomes

$$\ddot{z} + 2i\dot{z} - z = -\frac{z}{|z|^3}, \quad z \in \mathbb{C} \setminus \{0\}$$

This is the Kepler problem observed from a rotating system. The simplest closed orbits are circular. Let us look for solutions of the type

$$z(t) = \rho e^{i\omega t}, \quad \rho > 0, \ \omega \neq 0.$$

Then

$$-\rho\omega^2 - 2\rho\omega - \rho = -\frac{1}{\rho^2} \Longrightarrow$$

$$\boxed{\rho^3 (1+\omega)^2 = 1}$$

We can think that ω moves freely and ρ is a function of ω. Some values of ω must be excluded:

$\omega = 0$ we do not get a closed orbit but a continuum of equilibria

$\omega = -1 \rightsquigarrow \rho = \infty$

$\omega = -2$ This is more subtle. In this case $z(t) = e^{-2it}$, $\dot{z}(t) = -2i e^{-2it}$ is a closed orbit for $\mu = 0$ but we are not in the general framework. The reason is that any neighborhood in $\mathbb{C}^2 \times [0, \frac{1}{2}]$ of $\gamma \times \{0\}$ will intersect the singularity set. Notice that

$$\gamma = \{ (z_1, z_2) \in \mathbb{C}^2 / \ |z_1| = 1, \ z_2 = -2iz_1 \}$$

For $\mu > 0$,

$$\Sigma_\mu = \Sigma_\mu^1 \cup \Sigma_\mu^2, \quad \Sigma_\mu^1 = \{ (-\mu, z_2) : z_2 \in \mathbb{C} \}$$

$$\Sigma_\mu^2 = \{ (1-\mu, z_2) : z_2 \in \mathbb{C} \}$$

Σ_μ^2 converges, as $\mu \downarrow 0$, to a plane intersecting γ

From now on we discuss the continuation of

$$z(t) = (\omega+1)^{-2/3} e^{i\omega t}, \qquad \omega \neq 0, -1, -2$$

The associated orbit is

$$\gamma_\omega = \{ (z_1, z_2) \in \mathbb{C}^2 \,/\, |z_1| = (\omega+1)^{-2/3},\ z_2 = i\omega z_1 \}.$$

With the notation $(z, \dot z) = x \in \mathbb{R}^4$ we notice that

$$\frac{\partial \Psi}{\partial x_3} = x_3, \quad \frac{\partial \Psi}{\partial x_4} = x_4$$

and so the non-degeneracy condition holds.

[Notice that this would be automatic from observation ii) of the Hamiltonian structure].

<u>Computation of the monodromy matrix</u>

$$\mu = 0, \quad z(t) = \rho e^{i\omega t}, \quad \rho^3 (\omega+1)^2 = 1, \quad w = \frac{\partial z}{\partial \xi}$$

$$\ddot z + 2i \dot z - z = - z^{-1/2} \bar z^{-3/2} \xrightarrow{\partial/\partial \xi}$$

$$\ddot w + 2i \dot w - w = \frac{1}{2} z^{-3/2} \bar z^{-3/2} w + \frac{3}{2} z^{-1/2} \bar z^{-5/2} \bar w \rightsquigarrow$$

$$\boxed{\ddot w + 2i \dot w - w = \frac{1}{2} (\omega+1)^2 w + \frac{3}{2} (\omega+1)^2 e^{2i\omega t} \bar w}$$

Variational Equation

Change of variables:

$$\sigma = e^{-i\omega t} w \,, \quad \mu = e^{-i\omega t} \dot{w}$$

$$\dot{\sigma} = e^{-i\omega t}\dot{w} - i\omega e^{-i\omega t} w = \mu - i\omega\sigma$$

$$\dot{\mu} = e^{-i\omega t}\ddot{w} - i\omega e^{-i\omega t}\dot{w} = -i\omega\mu + e^{-i\omega t}\Big[-2i\dot{w} + w$$

$$+\frac{1}{2}(\omega+1)^2 w + \frac{3}{2}(\omega+1)^2 e^{2i\omega t}\,\bar{w}\Big] =$$

$$= -i\omega\mu - 2i\mu + \sigma + \frac{1}{2}(\omega+1)^2\sigma + \frac{3}{2}(\omega+1)^2\bar{\sigma}$$

$$\begin{cases} \dot{\sigma} = -i\omega\sigma + \mu \\ \dot{\mu} = \left[1 + \frac{1}{2}(\omega+1)^2\right]\sigma + \frac{3}{2}(\omega+1)^2\bar{\sigma} - i(\omega+2)\mu \end{cases}$$

In real notation $\sigma = x_1 + ix_2$, $\mu = x_3 + ix_4$ we arrive at $\dot{x} = Ax$ with

$$A = \left(\begin{array}{cc|cc} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ \hline 1+2(\omega+1)^2 & 0 & 0 & \omega+2 \\ 0 & 1-(\omega+1)^2 & -(\omega+2) & 0 \end{array}\right).$$

Using some machine we can compute the eigenvalues

$$\sigma(A) = \{0,\ (\omega+1)i,\ -(\omega+1)i\}.$$

Moreover 0 has algebraic multiplicity 2 and Ker A is spanned by $\begin{pmatrix} 0 \\ -\frac{1}{\omega} \\ 0 \end{pmatrix}$.

In the original variables $w, \dot{w}$ the fundamental matrix with $\Phi(0) = I$ is

$$\Phi(t) = \left(\begin{array}{c|c} R[\omega t] & 0 \\ \hline 0 & R[\omega t] \end{array} \right) e^{At}.$$

For period $\tau^* = \frac{2\pi}{\omega}$ the monodromy matrix is $M = e^{\tau^* A}$ and has eigenvalues

$$\mu_1 = 1, \quad \mu_2 = e^{\frac{2\pi}{\omega}(\omega+1)i} = e^{\frac{2\pi}{\omega}i}, \quad \mu_3 = \overline{\mu_2}.$$

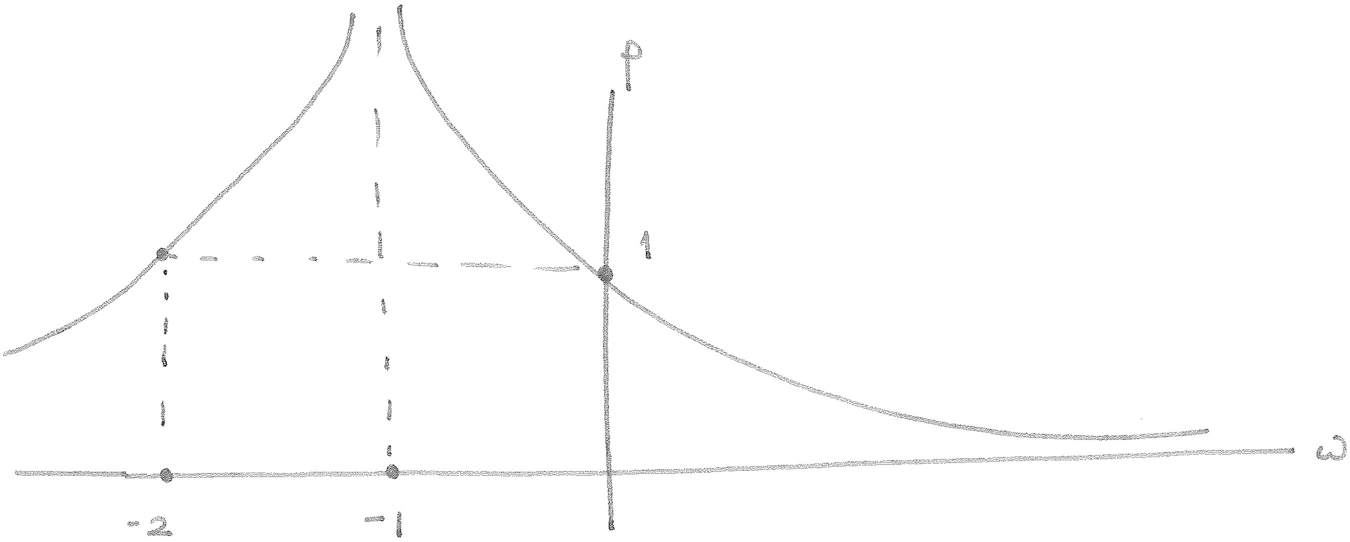
To be in the conditions of the Theorem we need to know that $\mu_2 \neq 1, \mu_3 \neq 1$; that is, $\omega \neq \frac{1}{k}, k \in \mathbb{Z} \setminus \{0\}$.

To sum up, we have proved that there is a continuation of the circular solution with fixed period and small μ if

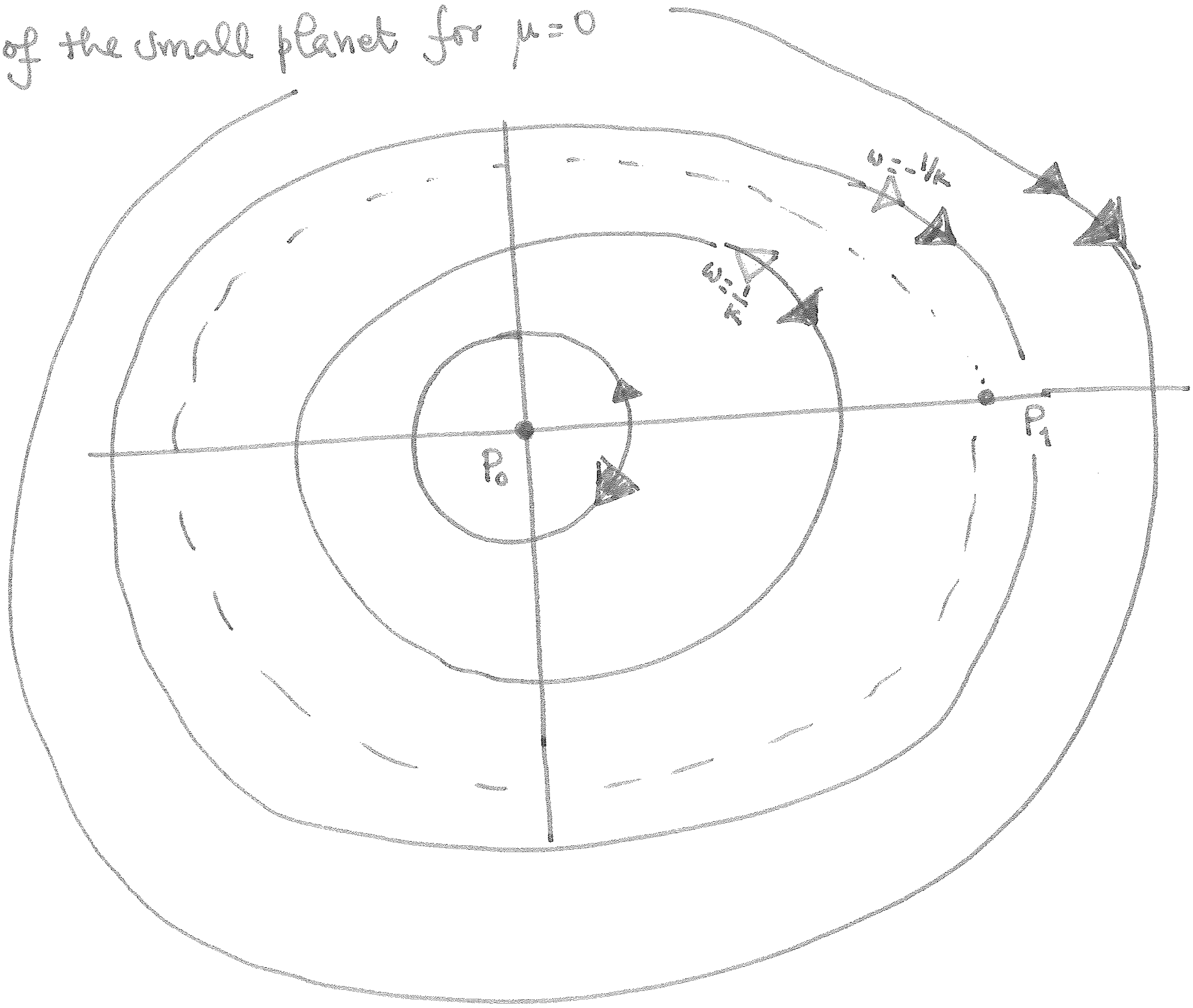
$$\omega \neq 0, -2, \quad \omega \neq \frac{1}{k}, \quad k = \pm 1, \pm 2, \pm 3, \ldots$$

Remarks on the result

① We draw the graph of $\rho = \rho(\omega) = \dfrac{1}{(\omega+1)^{2/3}}$



For $\rho < 1$ we have two orbits, one with positive orientation and one negative. For $\rho > 1$ there are two negative orbits. The excluded values $\omega = \pm \frac{1}{k}$ produce one positive orbit inside $\mathfrak{D}'$ and one negative outside. We draw the "orbit" of the small planet for $\mu = 0$



(2) These solutions are not necessarily periodic in the inertial system

The solution $z(t,\mu,\omega)$ has period $\tau^* = \frac{2\pi}{|\omega|}$ and so

$w(t,\mu,\omega) = e^{it} z(t,\mu,\omega)$ will be periodic if $\omega \in \mathbb{Q}$

and quasi-periodic with two frequencies if $\omega \notin \mathbb{Q}$.
In the last case they will produce invariant tori of dimension 2. These are lower dimensional tori, in this case KAM tori should be 3D.

(3) Understanding the flow around the circular solution ($\mu=0$)

If we let $Z = e^{it} z$, from

$$\ddot{z} + 2i\dot{z} - z = -\frac{z}{|z|^3}$$

we go to the Kepler problem

$$\ddot{Z} = -\frac{Z}{|Z|^3}.$$

The solutions close to circular solutions are elliptic and can be expressed as

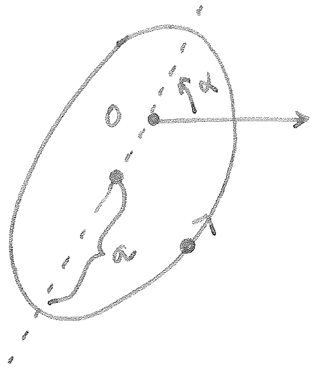
$$Z(t;\tau,\alpha,a,e) = e^{i\alpha} a \left[\cos u - e + i\sqrt{1-e^2}\,\sin u\right]$$

where $u = u(t)$ is the solution of Kepler's equation

$$a^{3/2}(u - e\sin u) = t - \tau$$

$e \in [0,1[$, $e=0$ circular solution, $a>0$ major semi-axis

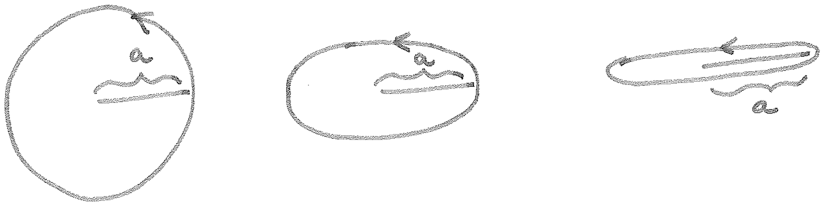
τ time passing through perihelion



"close to circular solution"
$e \approx 0,\ a \approx \rho,\ u \approx a^{-3/2}(t-\tau)$

In the variable Z_1 all these solutions are periodic and there are many with fixed period τ^*,

$\tau^* = 2\pi a^{-3/2}$; e, α, τ free parameters



Typically in the variable z these solutions become quasi-periodic or at least change their period. In this way the circular solution becomes "isolated". This is necessary if we want to apply the Implicit Function Theorem.

In the special case $\omega = \frac{1}{K}$ the solution is $2\pi K$-periodic both in z and Z_1. The same happens to all solutions in a neighborhood with $a = \rho$, $\rho^3(\omega+1)^2 = 1$, $\omega = \frac{1}{K}$. We fix a but e, α and τ are free.

4 Solving the linearized equation

Given a general linear periodic system in $\mathbb{R}^4$, it is very unusual to solve it explicitly. If the system is a variational equation then we know an explicit non-trivial solution,

$x(t)$ periodic solution of $\dot{x} = f(x)$ $\rightsquigarrow$

$\dot{x}(t)$ periodic solution of $\dot{y} = f'(x(t))y$.

By reduction of order it could be reduced to a system in $\mathbb{R}^3$, yet too difficult to solve in general. In our case the secret was the integrability of the nonlinear problem. In general, if $x(t,\lambda)$ is a family of solutions of $\dot{x} = f(x)$ depending smoothly on a parameter λ, then $\frac{\partial x}{\partial \lambda}(t,\lambda)$ is a solution of the variational equation $\dot{y} = f'(x(t,\lambda))y$. This observation could lead to an alternative way of solving the variational equation around a circular solution.

Going back to the ~~inertial~~ solution of the nonlinear problem $\ddot{z} + 2i\dot{z} - z = -\frac{z}{|z|^3}$

$$z(t;\tau,\alpha,a,e) = e^{-it}\, Z(t;\tau,\alpha,a,e),$$

the circular solution $z(t) = re^{i\omega t}$, $r^3(\omega+1)^2 = 1$, appears for $\tau = 0$, $\alpha = 0$, $a^{-3/2} = \omega$, $e = 0$.

We can compute $\frac{\partial z}{\partial \tau}$ and $\frac{\partial z}{\partial \alpha}$ and evaluate them at this point. However this would be useless since we go back (up to a multiplicative constant) to $\dot{z}(t)$.

Let us compute $\frac{\partial z}{\partial a}$ and $\frac{\partial z}{\partial e}$. First we differentiate the anomaly,

$$u - e\sin u = a^{-3/2}(t-\tau) \Longrightarrow \frac{\partial u}{\partial a}(1-e\cos u) = -\frac{3}{2}a^{-5/2}(t-\tau)$$

$$\frac{\partial u}{\partial e}(1-e\cos u) = \sin u$$

Letting $a=r, \tau=0, e=0, u=\omega t, \alpha=0$,

$$\frac{\partial u}{\partial a} = -\frac{3}{2}r^{-5/2}t = -\frac{3}{2}(\omega+1)^{5/3}t \ , \ \frac{\partial u}{\partial e} = \sin\omega t$$

$$\frac{\partial z}{\partial a} = e^{-it}e^{i\alpha}\left[\cos u - e + i\sqrt{1-e^2}\sin u\right] + e^{-it}e^{i\alpha}a$$

$$\left[-\sin u + i\sqrt{1-e^2}\cos u\right]\frac{\partial u}{\partial a}$$

$$\frac{\partial z}{\partial e} = e^{-it}e^{i\alpha}a\left[-\sin u\frac{\partial u}{\partial e} - 1 - i\frac{e}{\sqrt{1-e^2}}\sin u + i\sqrt{1-e^2}\cos u\frac{\partial u}{\partial e}\right]$$

Letting $a=r, e=0, u=\omega t, \ldots$

$$\frac{\partial z}{\partial a} = e^{-it}\left[\cos\omega t + i\sin\omega t\right] + e^{-it}r\left[-\sin\omega t + i\cos\omega t\right]\left(-\frac{3}{2}r^{-5/2}t\right)$$

$$\frac{\partial z}{\partial e} = e^{-it}r\left[-\sin^2\omega t - 1 + i\cos\omega t\sin\omega t\right]$$

Together with $\dot{z}(t) = ir\omega e^{i\omega t}$ we obtain three linearly independent solutions of a system in $\mathbb{R}^4$.

It can be solved by reduction of the order.

Reduction of order in linear systems

$$\dot{x} = A(t)x \,, \quad x \in \mathbb{R}^N$$

$n < N$, $\quad \varphi_1(t), \dots, \varphi_n(t)$ linearly independent solutions

$$P(t) = \big(\varphi_1(t) | \dots | \varphi_n(t) | \underbrace{* | \dots | *}_{\text{arbitrary}}\big) \,, \quad \det P(t) \neq 0$$

$x = P(t)y$,

$$\left.\begin{array}{l} \dot{x} = \dot{P}(t)y + P(t)\dot{y} \\ \shortparallel \\ A(t)P(t)y \end{array}\right\} \quad \dot{y} = \underbrace{P(t)^{-1}\left[A(t)P(t) - \dot{P}(t)\right]}_{B(t)} y$$

$$A(t)P(t) - \dot{P}(t) = \big(\underbrace{0 | \dots | 0}_{n} | * | \dots | *\big)$$

since φ_i is a solution

$$\Rightarrow \quad B(t) = \big(\underbrace{0 | \dots | 0}_{n} | * | \dots | *\big)$$

$y_1(t) = e_1, \dots y_n(t) = e_n$ constant linearly independent solutions of $\dot{y} = B(t)y$

$$\tilde{y} = \begin{pmatrix} y_{n+1} \\ \dots \\ y_N \end{pmatrix} \,, \quad \dot{\tilde{y}} = \tilde{B}(t)\tilde{y} \quad \text{sistem in } \mathbb{R}^{N-n}$$

In particular, if $n = N-1$ the system can be integrated.

R. O.

Lisboa 2009, Milano 2011