The classical variational problem

Space of virtual paths : $\mathcal{D}$

$\mathcal{H} = H^1(\mathbb{R}/_{T\mathbb{Z}}, \mathbb{C})$, $x : \mathbb{R} \to \mathbb{C}$, T-periodic, absolutely continuous

$\dot{x} \in L^2_{loc}(\mathbb{R}, \mathbb{C})$

$$\langle x, y \rangle_{\mathcal{H}} = \int_0^T \langle x(t), y(t) \rangle dt + \int_0^T \langle \dot{x}(t), \dot{y}(t) \rangle dt$$

Hilbert space , $\mathcal{H} \subset C(\mathbb{R}/_{T\mathbb{Z}}, \mathbb{C})$

$\mathcal{D} = \{ x \in \mathcal{H} : x(t) \neq 0 \ \forall t \in \mathbb{R} \}$ open in $\mathcal{H}$

Action functional

$$\mathcal{A} : \mathcal{D} \longrightarrow \mathbb{R}$$

$$\mathcal{A}[x] = \int_0^T \Big\{ \underbrace{\frac{1}{2} |\dot{x}(t)|^2 + \frac{1}{|x(t)|} + U(t, x(t))}_{\text{Lagrangian } (T-V)} \Big\} dt$$

$x \in \mathcal{D}$, $\mathcal{A}'[x] = 0 \iff$

$x(t)$ is a classical T-periodic solution of

$$\ddot{x} = -\frac{x}{|x|^3} + \nabla_x U(t, x)$$

Change of variables in $\mathcal{A}$ (I)

LC: $x = z^2$, $ds = \frac{dt}{|x|}$

$x(t)$ T-periodic $\Longrightarrow$ $z(s)$ σ-periodic

$$\sigma = \int_0^T \frac{dt}{|x(t)|}, \quad \sigma = \sigma_x$$

$H_{\pm 1}$ Sobolev space, $z = z(\tau)$, $z(\tau+1) = \pm z(\tau)$

1-periodic / 1 antiperiodic

Definition $z \in H_{\pm 1}$, $t_z(\tau) = \frac{T}{\|z\|^2} \int_0^\tau |z(\xi)|^2 d\xi$

$\|\cdot\|$ L^2-norm in $[0,1]$

$$t_z(\tau+1) = t_z(\tau) + T$$

$\forall z \in H_{\pm 1} \setminus \{0\}$, t_z is well defined, non-decreasing

Change of variables in $\mathcal{A}$ (II)

$$\hat{\mathcal{D}} = \{ z \in H_{\pm 1} : z(\tau) \neq 0 \ \forall \tau \in \mathbb{R} \} \qquad z \in \hat{\mathcal{D}} \Rightarrow t_z \text{ diffeomorphism}$$

Normalized LC : $\quad x \circ t_z = z^2$

$$\mathcal{B} : \hat{\mathcal{D}} \to \mathbb{R}, \quad \mathcal{B}(z) := \mathcal{A}[x]$$

$$(\dot{x} \circ t_z)\, t_z' = 2 z z' \Longrightarrow \dot{x} \circ t_z = \frac{2\|z\|^2}{T} \frac{z'}{\bar{z}}$$

$$\mathcal{A}[x] \underset{t = t_z(\tau)}{=} \int_0^1 \left\{ \frac{1}{2} |\dot{x} \circ t_z|^2 + \frac{1}{|x \circ t_z|} + U(t_z, x \circ t_z) \right\} \frac{T|z|^2}{\|z\|^2} d\tau$$

$$\boxed{\mathcal{B}(z) = \frac{2}{T} \|z\|^2 \|z'\|^2 + \frac{T}{\|z\|^2} + \frac{T}{\|z\|^2} \int_0^1 |z|^2 U(t_z, z^2)\, d\tau}$$

$\mathcal{SB}$ is not a standard functional

$$\mathcal{SB}(z) = \int_0^1 L_z \, d\tau$$

$$L_z = \frac{2}{T}\|z\|^2 \, |z'|^2 + \frac{T}{\|z\|^2}\left(1 + |z|^2 V\left(t_z(\tau), z^2\right)\right.$$

Lagrangian

non-local terms

$$L_z = L(\tau, z, z', P_z)$$

$$P_z(\tau) = \int_0^\tau |z(\zeta)|^2 \, d\zeta$$

Regularization of $\mathcal{A}$

$$\mathcal{B}(z) = \frac{2}{T}\|z\|^2\|z'\|^2 + \frac{T}{\|z\|^2}\left(1+\int_0^1 V\big(t_z(\tau), z(\tau)^2\big)\,|z(\tau)|^2\,d\tau\right)$$

$\mathcal{B}$ is well defined on $H_{\pm 1}\setminus\{0\}$

$$\begin{tikzcd} H_{\pm 1}\setminus\{0\} \supset \hat{\mathcal{D}} \arrow[r, "\widetilde{LC}"] \arrow[rr, bend right, "\mathcal{B}"'] & \mathcal{D} \arrow[r, "\mathcal{A}"] & \mathbb{R} \end{tikzcd}$$

$$\widetilde{LC}(z) = z^2 \circ t_z^{-1}$$

Differentiability of $\mathcal{B}$

$$U \in C^1 \Longrightarrow \mathcal{B} : H_{\pm 1} \setminus \{0\} \to \mathbb{R} \text{ is } C^1$$

Proof $\mathcal{B}(z) = \frac{2}{T} \|z\|^2 \|z'\|^2 + \frac{T}{\|z\|^2} \left(1 + \mathcal{R}(z)\right)$

$$\mathcal{R} : H_{\pm 1} \to \mathbb{R}, \quad \mathcal{R}(z) = \int_0^1 |z(\tau)|^2 U\left(t_z(\tau), z(\tau)^2\right) d\tau$$

$\mathcal{R}$ is Gateaux differentiable if z is not the zero function

$$\mathcal{R}'(z)w = \frac{d}{d\varepsilon} \mathcal{R}(z + \varepsilon w)\Big|_{\varepsilon = 0} = \int_0^1 \left\{ \frac{\partial P}{\partial t} \delta + \langle \nabla P, w \rangle \right\} d\tau$$

$$P(t,z) = |z|^2 U(t, z^2), \quad \delta(\tau) = \frac{2T}{\|z\|^4} \left[\left(\int_0^1 |z|^2\right)\left(\int_0^\tau \langle z, w\rangle\right) - \left(\int_0^\tau |z|^2\right)\left(\int_0^1 \langle z, w \rangle\right) \right]$$

$$\mathcal{R}' : H_{\pm 1} \setminus \{0\} \to H^*_{\pm 1} \text{ is continuous} \Longrightarrow$$

$\mathcal{R}$ Fréchet differentiable and C^1

Consistency of the functional

$$\delta \mathcal{B}'(z) = 0$$

$z \in \hat{\mathcal{D}} \iff x = \tilde{L}C(z)$ T-periodic solution in the classical sense

chain rule

$z \in H_{\pm 1} \setminus \{0\} \iff x = \tilde{L}C(z)$ T-periodic solution in the generalized sense

delicate

An example $U \equiv 0$

$$\mathcal{B}(z) = \frac{2}{T}\|z\|^2\|z'\|^2 + \frac{T}{\|z\|^2}$$

$\mathcal{B} : H_{+1} \setminus \{0\} \to \mathbb{R}$ does not reach its infimum

$\mathcal{B}(z) > 0 \quad \forall z \in H_{+1}$, $\{z_n\}$ constant functions, $|z_n| \to \infty$

$$\inf_{H_{+1}\setminus\{0\}} \mathcal{B} = 0 \qquad \mathcal{B}(z_n) \to 0$$

$\mathcal{B} : H_{-1} \setminus \{0\} \to \mathbb{R}$ reaches its infimum $\quad \inf_{H_{-1}\setminus\{0\}} \mathcal{B} > 0$

Key inequality $\quad \|z'\|^2 \geq \pi^2 \|z\|^2 \quad \forall z \in H_{-1}$

$$\mathcal{B} : H_{-1} \setminus \{0\} \to \mathbb{R}, \quad \mathcal{B}(z) = \frac{2}{T}\|z\|^2 \|z'\|^2 + \frac{T}{\|z\|^2}$$

$\{z_n\}$ minimizing sequence, $\mathcal{B}(z_n) \to \inf\limits_{H_{-1}\setminus\{0\}} \mathcal{B} =: c_{-1} \geq 0$

For large n, $c_{-1} + 7 \geq \mathcal{B}(z_n) \geq \frac{2\pi^2}{T}\|z_n\|^4 + \frac{T}{\|z_n\|^2} \Longrightarrow$

Key inequality

$0 < a \leq \|z_n\| \leq A < \infty$

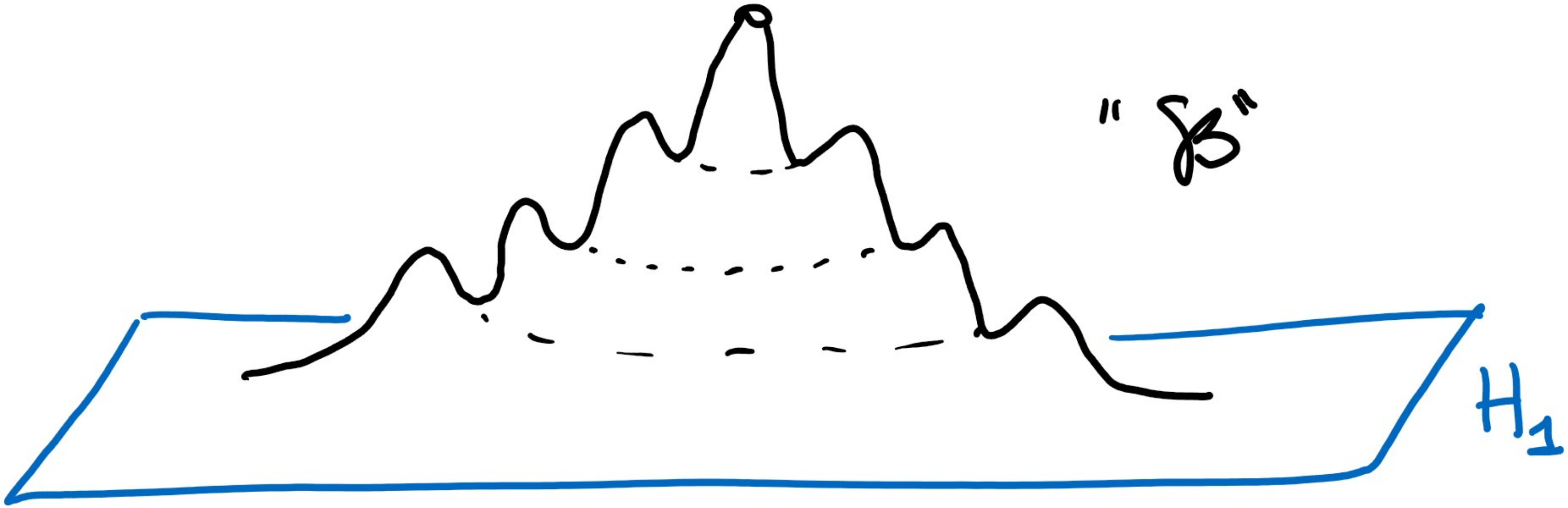
L^2- bounds

$c_{-1} + 7 \geq \mathcal{B}(z_n) \geq \frac{2}{T} a^2 \|z_n'\|^2 \Longrightarrow \{z_n\}$ bounded in H_{-1}

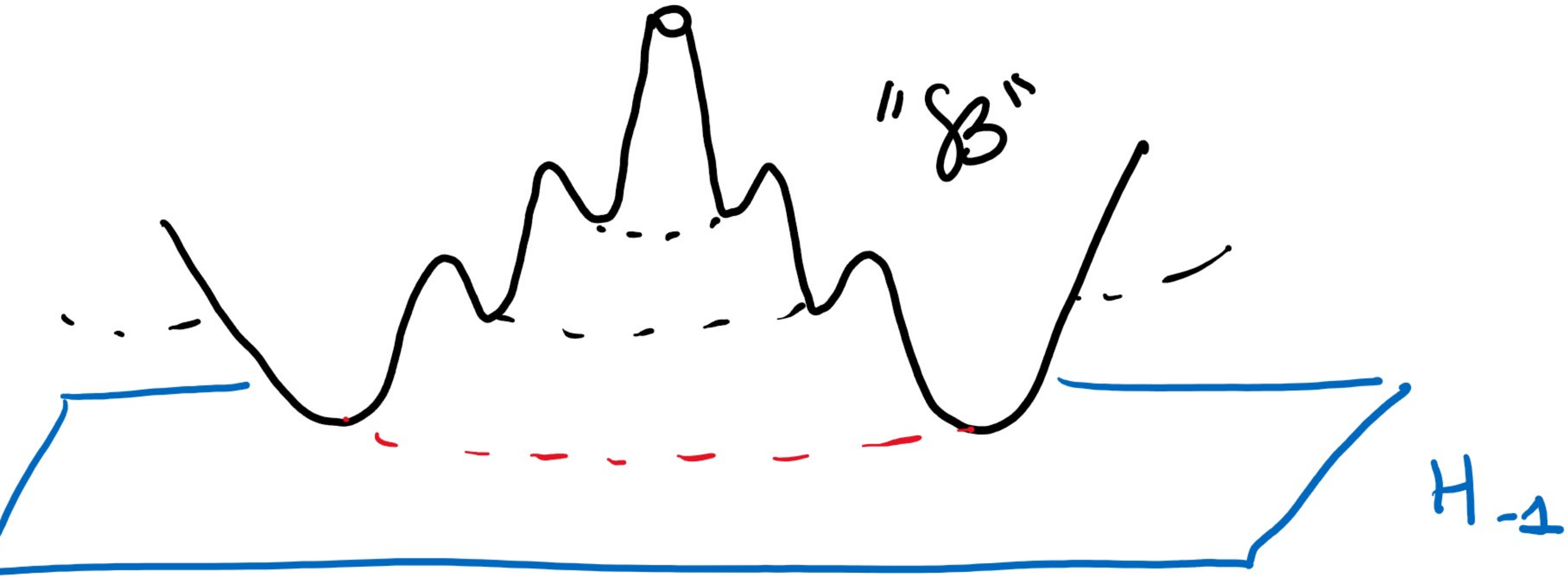
$\exists \{z_k\} : z_k \rightharpoonup z_* \text{ weak in } H_{-1}, \; z_k \to z_* \text{ strong in } L^2$

$\|z_k\| \to \|z_*\|, \; \|z_*'\| \leq \liminf \|z_k'\| \Longrightarrow$

$$\mathcal{B}(z_*) \leq \lim \mathcal{B}(z_k) = c_{-1}.$$



"δ_B"
H_1



"δ_B"
H_{-1}

A hidden symmetry

$$\mathcal{B}(z) = \frac{2}{T}\|z\|^2\|z'\|^2 + \frac{T}{\|z\|^2}\left(1+\int_0^1 |z(\tau)|^2\, U\big(t_z(\tau), z(\tau)^2\big)\,d\tau\right)$$

$t_z = t_{-z}$

$$\mathcal{B}(-z) = \mathcal{B}(z) \quad \forall z \in H_{-1}\setminus\{0\}$$

$\rightsquigarrow$

$\exists\, \infty$ many critical points

(slight assumptions on U)

A minimization result

$U(t,x) \le \alpha|x|^2+\beta \quad \forall x\in\mathbb{R}^2$, $\alpha < \frac{2}{T^2} \Rightarrow \mathcal{B}: H_{-1}\to\mathbb{R}$ has a minimum

1st Step $\mathcal{B}$ is bounded below $\left(\inf_{H_{-1}\setminus\{0\}} \mathcal{B} > -\beta T\right)$

$\|z\|_\infty^2 \le \|z\|\,\|z'\| \quad \forall z\in H_{-1}$ (known inequality)

$$\mathcal{B}(z) \ge \left(\frac{2}{T} - T\alpha\right)\|z\|^2\|z'\|^2 - \beta T + \frac{T}{\|z\|^2} \qquad (*)$$

2nd Step Minimizing sequences are far from the origin

$z_n\in H_{-1}$, $\mathcal{B}(z_n)\to \inf_{H_{-1}\setminus\{0\}}\mathcal{B} =: \mu$, For large n

$$\mu + 1 \ge \mathcal{B}(z_n) \overset{(*)}{\ge} -\beta T + \frac{T}{\|z_n\|^2}$$

$$\|z_n\| \geq m > 0 \quad \text{(Step 2)}$$

3rd Step $\{z_n\}$ bounden in H_{-1}

$$\mu + 7 \geq \mathcal{B}(z_n) \geq \left(\frac{2}{T} - T\alpha\right) m^2 \|z_n'\|^2 - \beta T$$

4th Step Passage to the limit

$z_k \longrightarrow z_*$ weak in H_{-1}, $z_k \longrightarrow z$ uniformly

$\|z_k\| \to \|z_*\|$, $\mathcal{R}(z_k) \longrightarrow \mathcal{R}(z_*)$, $\|z_*'\| \leq \liminf \|z_k'\|$

$$\Longrightarrow \quad \mathcal{B}(z_*) \leq \lim \mathcal{B}(z_k)$$