Generalized T- periodic solutions $\quad \ddot{x} = -\frac{x}{|x|^3} + \nabla_x V$

$\updownarrow \qquad$ 1 to 2

Closed orbits of regularized Hamiltonian system at zero energy $\quad \dot{\xi} = J\nabla H(\xi),\ H = 0$

Preliminaries 1

$D \subset \mathbb{R}^d$ open, $X : D \to \mathbb{R}^d$ continuous

$x : \mathbb{R} \to D$, $x = x(t)$, continuous

x is C^1 on $\mathbb{R} \setminus Z$ (Z discrete) $\quad \dot{x}(t) = X(x(t))$, $t \in \mathbb{R} \setminus Z$

$$\Longrightarrow$$

$x : \mathbb{R} \to D$ is C^1 and $\quad \dot{x}(t) = X(x(t))$, $t \in \mathbb{R}$

Proof. Volterra's integral equation

Preliminaries 2 Sundman's estimates

$x(t) \to 0$ as $t \to \omega^+ \implies x(t) \sim c\,(t-\omega)^{2/3}$ as $t \to \omega^+$

(Sperling) $\ddot{x} = -\dfrac{x}{|x|^3} + \underbrace{P(t, x, \dot{x})}_{\text{bounded in a nghd of } x=0 \text{ + continuous}}, \quad x \in \mathbb{R}^d \setminus \{0\}$

$x(t)$ solution on $[t_0, \omega[$, $\omega < \infty$

$$\liminf_{t \uparrow \omega} |x(t)| = 0 \implies \exists \xi \in \mathbb{R}^d \setminus \{0\} : \lim_{t \uparrow \omega} \frac{x(t)}{(\omega - t)^{2/3}} = \xi$$

$x(t)$ generalized T-periodic solution $\rightsquigarrow$

$(z(s), w(s), \bar{t}(s), E(s))$ periodic solution of $\xi' = J\nabla H(\xi)$, $H=0$

Step 1 Sundman's integral

$\frac{1}{|x(t)|} \sim \frac{1}{(t-\omega)^{2/3}}$ at collisions $\Rightarrow \frac{1}{|x|} \in L^1_{loc}(\mathbb{R})$

$S(t) = \int_0^t \frac{d\tau}{|x(\tau)|}$, $S: \mathbb{R} \to \mathbb{R}$ (absolutely) continuous + increasing

$S(t+T) = S(t) + \sigma$, $\sigma := \int_0^T \frac{d\tau}{|x(\tau)|} > 0$, S homeomorphism C^∞ on $\mathbb{R} \setminus Z$

$\mathcal{T} = S^{-1}: \mathbb{R} \to \mathbb{R}$ homeomorphism C^∞ on $\mathbb{R} \setminus Z^*$, $Z^* = S(Z)$

$\mathcal{T}(s+\sigma) = \mathcal{T}(s) + T$

Step 2 The square root $z = \sqrt{x}$

$t \in \mathbb{R} \setminus \mathbb{Z} \longmapsto \dfrac{x(t)}{|x(t)|} \in S^1$ continuous extension to $\mathbb{R} \to S^1$

Lifting property $\exists \theta : \mathbb{R} \to \mathbb{R}$ continuous

$$x(t) = |x(t)| e^{i\theta(t)}, \quad t \in \mathbb{R} \setminus \mathbb{Z} \quad (\Rightarrow t \in \mathbb{R})$$

$x(t)$ T-periodic $\Longrightarrow \exists k \in \mathbb{Z} : \theta(t+T) = \theta(t) + 2\pi k$

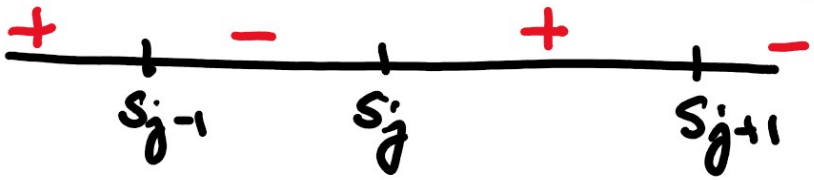
$$z(s) = \pm |x(\pi(s))|^{1/2} e^{i\theta(\pi(s))/2}, \quad s \in \mathbb{R}$$

+ or − ?

Step 2 bis   Definition of $z(s)$

$$Z_1 = \{ t_j : j \in \mathbb{Z} \}, \quad Z_1^* = \{ s_j : j \in \mathbb{Z} \} \qquad t_{j+J} = t_j + T, \quad s_{j+J} = s_j + \sigma$$

$$z(s) = (-1)^j \, |x(\tau(s))|^{1/2} e^{i\theta(\tau(s))/2}, \quad s \in I_j = [s_j, s_{j+1}]$$



$z : \mathbb{R} \to \mathbb{C}$ continuous, C^∞ on $\mathbb{R} \setminus Z_1^*$,

$$z(s+\sigma) = (-1)^{J+k} z(s)$$ σ-periodic or σ-antiperiodic

$$\lim_{s \to s_j^+} \frac{z(s)}{|z(s)|} = - \lim_{s \to s_j^-} \frac{z(s)}{|z(s)|}$$

Step 3 Definition of $\overline{t(s)}$ and $E(s)$

$\overline{t(s)} = \pi(s) + T\mathbb{Z}$, $\quad \bar{t} : \mathbb{R} \to \mathbb{R}/_{T\mathbb{Z}}$ continuous

σ-periodic, C^∞ on $\mathbb{R} \setminus \mathbb{Z}^*$

$E(s) = \mathcal{E}(\pi(s))$, $s \in \mathbb{R} \setminus \mathbb{Z}^*$,

$$\mathcal{E}(t) = \frac{1}{2} |\dot{x}(t)|^2 - \frac{1}{|x(t)|} - U(t, x(t))$$

E admits a continuous extension $E : \mathbb{R} \to \mathbb{R}$

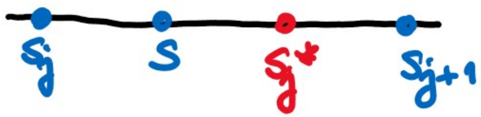
E is C^∞ on $\mathbb{R} \setminus \mathbb{Z}^*$, σ-periodic

Step 4 Definition of $w(s)$

$$w(s) = 4z'(s), \quad s \in \mathbb{R} \setminus Z^*$$

Claim 1 $\exists \lim\limits_{s \to s_j^\pm} w(s) = w(s_j^\pm)$

Proof: $$w(s) = w(s_j^*) + \underbrace{\int_{s_j^*}^{s} \left[\frac{1}{2} E(\xi) z(\xi) + \frac{1}{2} (\partial_{\bar z} P)(T(\xi), z(\xi), \overline{z(\xi)}) \right]}_{\text{continuous}} d\xi$$



Claim 2 $|w(s_j^\pm)|^2 = 8$ Proof: $H = 0$

Claim 3 $w(s_j^+) = w(s_j^-)$

Proof: $$\frac{z(s)}{|z(s)|} \to \frac{1}{2\sqrt{2}} w(s_j^+) \quad \text{if } s \to s_j,\ s > s_j$$

$$\frac{z(s)}{|z(s)|} \to -\frac{1}{2\sqrt{2}} w(s_j^-) \quad \text{if } s \to s_j,\ s < s_j$$

Choice of $\pm$ in $\sqrt{\ }$

$$z(s) = \underbrace{z(s_j)}_{=0} + \frac{1}{4} w(s_j^+)(s - s_j) + \underbrace{R(s)}_{\text{bounded}} (s - s_j)^2, \quad s > s_j$$

Conclusion

$\bar{t}(s), E(s), z(s), w(s)$ continuous, C^∞ on $\mathbb{R}\setminus\mathbb{Z}^*$

solution of $\dot{\xi} = J\nabla H(\xi), H=0$ if $s\in\mathbb{R}\setminus\mathbb{Z}^*$

σ-periodic or σ-antiperiodic ($\Rightarrow 2\sigma$-periodic)

$\Longrightarrow$ solution of $\dot{\xi}=J\nabla H(\xi)$, $H=0$ $\forall s\in\mathbb{R}$

It is a closed orbit because the system does not have equilibria, $t' = |z|^2 \Rightarrow t$ not constant