Non-autonomous systems

$$\ddot{x} = -\frac{x}{|x|^3} + \nabla_x U(t,x), \quad x \in \mathbb{R}^2 \setminus \{0\}$$

$U : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ $\quad C^\infty, \quad U(t+T, x) = U(t,x), \ T>0$

$x = (x_1, x_2) \in \mathbb{R}^2, \quad x = x_1 + i x_2 \in \mathbb{C}, \quad F = F(x_1, x_2), \quad F = F(x, \bar{x})$

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right), \quad \frac{\partial F}{\partial x_1} + i \frac{\partial F}{\partial x_2} = 2\frac{\partial F}{\partial \bar{x}}$$

$$\boxed{\ddot{x} = -\frac{x}{|x|^3} + 2\partial_{\bar{x}} U(t, x, \bar{x}), \quad x \in \mathbb{C} \setminus \{0\}}$$

Towards a new phase space (I)

LC : $x = z^2$

$ds = \dfrac{dt}{|x|}$

(1) $z'' = \left(\left|\dfrac{z'}{z}\right|^2 - \dfrac{1}{2|z|^2} \right) z + \bar{z}\,|z|^2 \partial_{\bar{x}} V(t, z^2, \bar{z}^2)$

singularity

non-local term

$t(s) = t_0 + \int_{s_0}^{s} |z(\sigma)|^2 d\sigma$, (1) is not an O.D.E. !!

(2) $t' = |z|^2$

(1) + (2) O.D.E system, unknowns $z, w = z', t$

Towards a new phase space (II)

$$E = 2\left|\frac{z'}{z}\right|^2 - \frac{1}{|z|^2} - V(t, z^2, \bar{z}^2), \quad P(t, z, \bar{z}) = |z|^2 V(t, z^2, \bar{z}^2$$

$$\begin{cases} z'' = \frac{1}{2} E z + \frac{1}{2} (\partial_{\bar{z}} P)(t, z, \bar{z}) \\ t' = |z|^2 \\ E|z|^2 = 2|z'|^2 - 1 - P(t, z, \bar{z}) \end{cases}$$

Algebraic-Differential Equation. Unknowns:

$z, \ w = z', \ t, \ E$

Towards a new phase space (III)

$$E' = \dot{E}\,\frac{dt}{ds} = \dot{E}\,|z|^2 = -\frac{\partial U}{\partial t}\,|z|^2 = -\frac{\partial \mathcal{P}}{\partial t}$$

$$\begin{cases} z'' = \frac{1}{2} E z + \frac{1}{2} \left(\partial_{\bar{z}} \mathcal{P}\right)(t, z, \bar{z}) \\ t' = |z|^2 \\ E' = -\frac{\partial \mathcal{P}}{\partial t}(t, z, \bar{z}) \end{cases}$$

O. D. E. system in $z, w = z', t, E$

The phase space is too large dimension: $2+2+1+1 = 6 > 5$

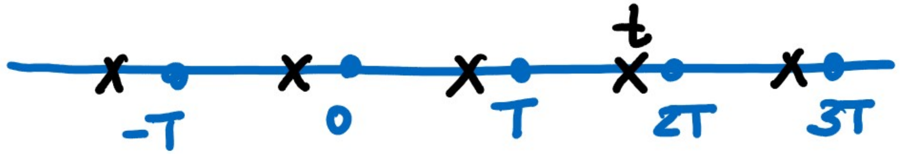
First integral $\quad \mathcal{J} = 2|w|^2 - E|z|^2 - 1 - \mathcal{P}(t, z, \bar{z})$

The phase space

$$M_{*} = \{ (z,w,\bar{t},E) \in \mathbb{C}\setminus\{0\} \times \mathbb{C}\setminus\{0\} \times \mathbb{R}/_{T\mathbb{Z}} \times \mathbb{R} \;/\; 2|w|^2 - E|z|^2 - 1 - \mathcal{P}(t,z,\bar{z}) = 0 \}$$

manifold of dimension 5

$\mathbb{R}/_{T\mathbb{Z}}$, $\bar{t} = t + T\mathbb{Z}$



The vector field on M is well defined

U T-periodic in $t \Longrightarrow \mathcal{P}$ T-periodic in t

$$\mathcal{P} : \mathbb{R}/_{T\mathbb{Z}} \times \mathbb{C} \longrightarrow \mathbb{R}$$