Levi-Civita change of variables

$\mathbb{R}^2 \equiv \mathbb{C} \qquad (x_1, x_2) \longleftrightarrow x = x_1 + i x_2$

$$\ddot{x} = -\frac{x}{|x|^3}, \quad x \in \mathbb{C} \setminus \{0\}$$

LC : $\quad x = z^2, \quad ds = \dfrac{dt}{|x|}$

$x : I \subset \mathbb{R} \to \mathbb{C} \setminus \{0\}, \; x = x(t) \quad \rightsquigarrow \quad z : J \subset \mathbb{R} \to \mathbb{C} \setminus \{0\}$

$z = z(s)$

From $x(t)$ to $z(s)$ (i) $x = z^2$

$\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$, $z \longmapsto z^2$ covering map

$x : I \to \mathbb{C} \setminus \{0\}$, C^∞ $\quad x(t) = r(t)\, e^{i\theta(t)}$, $r > 0$, $r, \theta \in C^\infty$

Square roots:

$$+\sqrt{x}\,(t) = r(t)^{1/2} e^{i\theta(t)/2}, \quad -\sqrt{x}\,(t) = -r(t)^{1/2} e^{i\theta(t)/2}$$

$$x(t) \begin{array}{l} \nearrow \; +\sqrt{x}\,(t) \\ \\ \searrow \; -\sqrt{x}\,(t) \end{array}$$

From $x(t)$ to $z(s)$

(ii) $ds = \dfrac{dt}{|x|}$

$t_0 \in I$ fixed, $\quad \mathcal{S}(t) = \displaystyle\int_{t_0}^{t} \frac{d\tau}{|x(\tau)|}$ Sundman's integral

$\mathcal{S} : I \to J$ diffeomorphism, $\quad J = \mathcal{S}(I)$

$\mathcal{T} = \mathcal{S}^{-1} : J \to I$, $\quad s = \mathcal{S}(t) \Longleftrightarrow t = \mathcal{T}(s)$

$$z = +\sqrt{x} \circ \mathcal{T}$$

From $z(s)$ to $x(t)$

$$x = z^2, \quad ds = \frac{dt}{|x|}$$

$$dt = |x|\,ds = |z|^2 ds$$

$s_0 \in J$ fixed, $\quad \Pi(s) = \int_{s_0}^{s} |z(\sigma)|^2 d\sigma$

$\Pi : J \to I$ diffeomorphism, $\quad S = \Pi^{-1}$

$$x = (z \circ S)^2$$

Linearization of Kepler problem

$x = z^2, \quad ds = \frac{dt}{|x|}$

$$\ddot{x} = -\frac{x}{|x|^3} \longrightarrow z'' = \left(\frac{|z'|^2}{|z|^2} - \frac{1}{2|z|^2} \right) z$$

$$\frac{1}{2}|\dot{x}|^2 - \frac{1}{|x|} = E \longrightarrow 2\left|\frac{z'}{z}\right|^2 - \frac{1}{|z|^2} = E$$

$$z'' = \frac{1}{2} E z, \quad z \in \mathbb{C} \setminus \{0\}, \ E \in \mathbb{R}$$

$+$

$$\frac{1}{2}|z'|^2 - \frac{1}{4} E |z|^2 = \frac{1}{4}$$ (level of energy fixed)

Change of variables (Goursat)

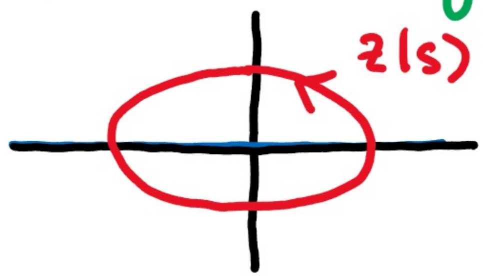
Regularization ($z \in \mathbb{C}$) (Levi-Civita)

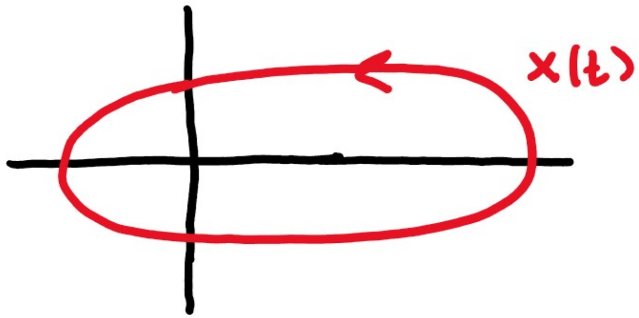
Examples

$E < 0$, $z(s) = a \cos \omega s + b \sin \omega s$

$\omega^2 \left(|a|^2 + |b|^2 \right) = \frac{1}{2}$, $a, b \in \mathbb{C}$

- a, b $\mathbb{R}$-linearly independent $\quad z(s) \neq 0 \;\; \forall s \in \mathbb{R}$





- a, b $\mathbb{R}$-linearly dependent