A regularized variational principle

Consider the Sobolev space $\mathcal{H} = H^1(\mathbb{R}/_{T\mathbb{Z}}, \mathbb{C})$ composed by all T-periodic functions $x : \mathbb{R} \to \mathbb{C}$ that are absolutely continuous and such that the derivative [defined almost everywhere] satisfies $\dot{x} \in L^2(\mathbb{R}/_{T\mathbb{Z}}, \mathbb{C})$. Then $\mathcal{H}$ becomes a Hilbert space with the dot product

$$\langle x, y \rangle_{\mathcal{H}} = \int_0^T \langle x(t), y(t) \rangle dt + \int_0^T \langle \dot{x}(t), \dot{y}(t) \rangle dt .$$

The associated norm is stronger than the uniform norm $\| \cdot \|_\infty$ and so the subset of functions without collisions

$$\mathcal{D} = \{ x \in \mathcal{H} : x(t) \neq 0 \text{ for each } t \in \mathbb{R} \}$$

is open in $\mathcal{H}$.

Given a smooth and T-periodic force function $U = U(t,x)$ we define the action functional

$$(*) \quad \mathcal{A} : \mathcal{D} \to \mathbb{R}, \quad \mathcal{A}[x] = \int_0^T \left\{ \frac{1}{2} |\dot{x}|^2 + \frac{1}{|x|} + U(t,x) \right\} dt .$$

It is well known that $\mathcal{A}$ is C^1 and the critical points $\mathcal{A}'[x] = 0$ are precisely the T-periodic solutions of the system

$$(**) \quad \ddot{x} = -\frac{x}{|x|^3} + \nabla_x U(t,x), \quad x \in \mathbb{R}^2 \setminus \{0\} .$$

The change of variables

$$x = z^2, \quad ds = \frac{dt}{|x(t)|}$$

has been useful to study (**). Now we try to introduce it in (*).

We know that if $x(t)$ is a T-periodic solution of (**) then $z(s)$ is a periodic / anti-periodic function with period $S = \int_0^T \frac{dt}{|x(t)|}$. Then $S = S_x$ depends upon the solution $x(t)$. Our first task will be to rescale

$$ds = \frac{dt}{|x(t)|}$$ so that all z's have period 1

Assume now that H_1 is the Sobolev space of functions $z = z(\tau)$, $z : \mathbb{R} \to \mathbb{C}$ of period 1. Analogously we can work in the Sobolev space H_{-1} of anti-periodic functions of period 1. The definitions are similar to those for $\mathcal{H}$.

Given $z \in H_{11} \setminus \{0\}$ we want to find $\lambda_z \in \,]0,\infty[$ such that the change

$$x = z^2, \quad d\tau = \lambda_z \, ds$$

will send $z(\tau)$ 1-periodic into $x(t)$ T-periodic.

Therefore

$$t_z(\tau) = \frac{1}{\lambda_z} \int_0^\tau |z(\xi)|^2 d\xi$$

and we want

$$t_z(\tau + 1) = t_z(\tau) + T.$$

To this end we impose

$$\frac{1}{\lambda_z} \int_\tau^{\tau+1} |z(\xi)|^2 d\xi = \frac{1}{\lambda_z} \int_0^1 |z|^2 = T$$

We arrive at a first definition:

For each $z \in H_{\pm 1} \setminus \{0\}$,

$$t_z(\tau) = \frac{T}{\|z\|^2} \int_0^\tau |z(\xi)|^2 d\xi$$

where $\|z\| = \left(\int_0^1 |z|^2 \right)^{1/2}$. [L^2-norm]

Note that if $z \in \mathcal{D}$ then $t_z : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism [of class H^2_{loc}] satisfying

$$t_z(\tau + 1) = t_z(\tau) + T.$$

Next we introduce the normalized LC change

$$x \circ t_z = z^2.$$

~~Since~~ Differentiating with respect to τ,

$$(\dot{x} \circ t_z)\, t_z' = 2 z z' \Rightarrow \dot{x} \circ t_z = \frac{\|z\|^2}{T} \frac{1}{|z|^2} 2 z z',$$

$$\dot{x} \circ t_z = \frac{2\|z\|^2}{T} \frac{z'}{\bar{z}}.$$

Changing variables in the action, $t = t_z(\tau)$,

$$\mathcal{B}(z) = \mathcal{A}[x] = \int_0^1 \left\{ \frac{1}{2} |\dot{x} \circ t_z|^2 + \frac{1}{|x \circ t_z|} + U(t_z, x \circ t_z) \right\} \cdot \frac{T}{\|z\|^2} |z|^2 d\tau$$

$$= \frac{T}{\|z\|^2} \int_0^1 \left\{ \frac{2\|z\|^4}{T^2} \frac{|z'|^2}{|z|^2} + \frac{1}{|z|^2} + U(t_z, z^2) \right\} |z|^2 d\tau$$

and we obtain the new functional

$$\mathcal{B}(z) = \frac{2}{T}\|z\|^2\|z'\|^2 + \frac{T}{\|z\|^2} + \frac{T}{\|z\|^2}\int_0^1 |z|^2\, U(t_z, z^2)\,d\tau .$$

We observe that $\mathcal{B}$ is not a standard functional in Calculus of Variations of the type

$$\Phi[z] = \int_0^1 L(\tau, z(\tau), z'(\tau))\,d\tau$$

where the Lagrangian L is a function in three variables.

In our case

$$L = \frac{2}{T}\|z\|^2|z'|^2 + \frac{T}{\|z\|^2}\left(1 + \oint_0 |z|^2\, U(t_z(\tau), z^2)\right)$$

non-local terms

The Lagrangian now is a functional

$$L = L(\tau, z, z', \|z\|^2, P_z)$$

with $P_z(\tau) = \int_0^\tau |z(\xi)|^2 d\xi$. Note that $\|z\|^2 = P_z(1)$.

Due to the change of variables we have employed, the functional $\mathcal{B}$ will be defined on the domain

$$\hat{\mathcal{D}} = \{z \in H_{\pm 1} : z(\tau) \neq 0 \text{ for each } \tau \in \mathbb{R}\}.$$

The key observation is that $\mathcal{B}$ can be extended to $H_{\pm 1} \setminus \{0\}$. Now collisions $[z(\tau_*) = 0]$ do not produce singularities in the functional. The non-local term $\frac{1}{\|z\|^2}$ has replaced the potential term $\frac{1}{|z(\tau)|^2}$.

The LC change of variables can be thought as a transformation

$$\begin{tikzcd}
H_{\pm 1}\setminus\{0\} \supset \hat{\mathcal{D}} \arrow[r, "LC"] & \mathcal{D} \arrow[r, "\mathcal{A}"] & \mathbb{R} \\
H_{\pm 1}\setminus\{0\} \arrow[urr, bend right, "\mathcal{B}"'] & &
\end{tikzcd}$$

with $LC(z) = z^2 \circ t_z^{-1}$.

This nice diagram is taken from the paper by Frauenfelder and Weber.

<u>Lemma</u> The functional $\mathcal{B} : H_{\pm 1}\setminus\{0\} \to \mathbb{R}$ is C^1 if $U \in C^1(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}^2)$.

<u>Proof</u> We prove that the functional

$$\mathcal{R} : H_{\pm 1} \to \mathbb{R}, \quad \mathcal{R}(z) = \int_0^1 |z(\tau)|^2\, U\left(t_z(\tau), z(\tau)^2\right) d\tau$$

is C^1 and observe that

$$\mathcal{B}(z) = \frac{2}{T}\|z\|^2 \|z'\|^2 + \frac{T}{\|z\|^2}\left(1 + \mathcal{R}(z)\right).$$

First we observe that $\mathcal{R}$ is Gateaux differentiable

$$\frac{d}{d\varepsilon}\mathcal{R}(z + \varepsilon w)\Big|_{\varepsilon = 0} = \int_0^1 \left\{ \frac{\partial P}{\partial t}\,\delta + \langle \nabla_z P, w \rangle \right\} d\tau$$

where $P(t, z) = |z|^2\, U(t, z^2)$ and

$$\delta(\tau) = \frac{\partial}{\partial \varepsilon} t_{z+\varepsilon w}(\tau)\Big|_{\varepsilon = 0} = \frac{2T}{\|z\|^4}\Big[\Big(\int_0^1 |z|^2\Big)\Big(\int_0^\tau \langle z, w\rangle\Big) - \Big(\int_0^\tau |z|^2\Big)\Big(\int_0^1 \langle z, w\rangle\Big) \Big]$$

for each $z \in H_{\pm 1}\setminus\{0\}$, $w \in H_{\pm 1}$.

The differential of $\mathcal{R}$ at z is

$$\mathcal{R}'(z) : w \mapsto \int_0^1 \left\{ \frac{\partial P}{\partial t} \delta + \langle \nabla_z P, w \rangle \right\} d\tau .$$

This is a bounded linear form, $\mathcal{R}'(z) \in H_{\pm 1}^*$, because

$$|\delta(\tau)| \leq \frac{4T}{\|z\|} \|w\|$$

and the partial derivatives $\frac{\partial P}{\partial t}$ and $\nabla_z P$ are evaluated along $(t_z(\tau), z(\tau))$, $\tau \in \mathbb{R}$, a bounded set in $\mathbb{R}|_{T\mathbb{Z}} \times \mathbb{R}^2$.

Finally we observe that $\mathcal{R}' : H_{\pm 1} \to H_{\pm 1}^*$ is continuous.

Here we use $V \in C^1$ and the definition of δ. ∎

Therefore $\mathcal{R}$ is Fréchet differentiable and indeed C^1.

Once we know that $\mathcal{B}$ is C^1, we can look for critical points. These critical points are consistent with our problem. More precisely,

$z \in \hat{\mathcal{D}}$, $\quad \mathcal{B}'(z) = 0 \Leftrightarrow x = LC(z)$ is a T-periodic solution of $(**)$ in the classical sense

$z \in H_{\pm 1} \setminus \{0\}$, $\quad \mathcal{B}'(z) = 0 \Leftrightarrow x = LC(z)$ is a T-periodic solution of $(**)$ in the generalized sense

The proof of these equivalences is delicate.

We will do something easier: to describe the geometry of $\mathcal{B}$ when $V \equiv 0$. Then

$$\mathcal{B}(z) = \frac{2}{T} \|z\|^2 \|z'\|^2 + \frac{T}{\|z\|^2}$$

We observe that $\mathcal{B}(z) > 0 \quad \forall z \in H_{\pm 1} \setminus \{0\}$. Moreover,

$$\inf_{H_1} \mathcal{B} = 0$$

because if we take a sequence of constant functions z_n with $|z_n| \to 0$, then $\mathcal{B}(z_n) = \frac{T}{|z_n|^2} \to 0$.

In consequence $\mathcal{B}$ does not reach its infimum on H_1.
The situation is different on H_{-1}. Now $\inf_{H_{-1}} \mathcal{B} > 0$ is reached through a set a minimizers homeomorphic to $\mathbb{S}^1$.
To prove this we recall the inequality

$$\|z'\|^2 \geq \pi^2 \|z\|^2 \quad \text{for each } z \in H_{-1}$$

<u>Exercise</u> Prove this inequality using Fourier series

~~Then~~ $\cancel{\mathcal{B}(z) \geq \pi^2 \|z\|^4 + \frac{T}{\|z\|^2}}$

Assume that $(z_n) \subset H_{-1}$ is a minimizing sequence, $\mathcal{B}(z_n) \longrightarrow \inf_{H_{-1}} \mathcal{B} = c_{-1} > 0$. Then, for large n,

$$c_{-1} + 1 \geq \mathcal{B}(z_n) \geq \frac{2\pi^2}{T} \|z_n\|^4 + \frac{T}{\|z_n\|^2}.$$

From here we can find positive numbers $a < A$ independent of n such that

$$0 < a \leq \|z_n\| \leq A.$$

Finally,

$$c_{-1} + 1 \geq \mathcal{B}(z_n) \geq \frac{2}{T} a^2 \|z_n'\|^2.$$

Once we know that the sequence (z_n) is bounded in H_{-1}, we can extract a subsequence such that $z_n \to z_*$ in L^2, $z_n \rightharpoonup z_*$ in H_{-1}, where $z_* \in H_{-1}$ and $\rightharpoonup$ is the weak convergence. We conclude that z_* is a minimizer because $\|z_*\| \leq \liminf_{n\to\infty} \|z_n\|$ and

$$\mathcal{B}(z_*) \leq \liminf \mathcal{B}(z_n) = c_{-1}.$$

Note that $\|z_*\| \geq a$, in particular, $z_* \neq 0$

In this simple case $(U \equiv 0)$ it is possible to compute all critical points.
Given $z \in H_{\pm 1} \setminus \{0\}$ and $w \in H_{\pm 1}$,

$$\delta\mathcal{B}'(z)w = \frac{2}{T}\, \|z\|^2 \int_0^1 \langle z', w'\rangle + \left(\frac{2}{T}\|z'\|^2 - \frac{2T}{\|z\|^4}\right)\int_0^1 \langle z, w\rangle .$$

From the standard theory of boundary value problems,

$$\delta\mathcal{B}'(z)w = 0 \quad \text{for each } w \in H_{\pm 1}$$

is equivalent to

$$z \in (H_{\pm 1} \setminus \{0\}) \cap H^2_{loc}(\mathbb{R}), \quad z'' = \left(\frac{\|z'\|^2}{\|z\|^2} - \frac{T^2}{\|z\|^6}\right) z .$$

Then $z_n = z$ with

$$z_n(\tau) = A_n e^{i n\pi\tau} + B_n e^{-i n \pi \tau}, \quad \text{where } A_n, B_n \in \mathbb{C} \text{ must}$$

be determined and n is an odd integer if $z_n \in H_{-1}$ and an even integer if $z_n \in H_{+1}$. After some computations

$\|z_n\|^2 = |A_n|^2 + |B_n|^2, \quad \|z_n'\|^2 = n^2\pi^2 (|A_n| + |B_n|^2)$

$|A_n|^2 + |B_n|^2 = \left(\frac{T}{2n^2\pi^2}\right)^{1/3}.$

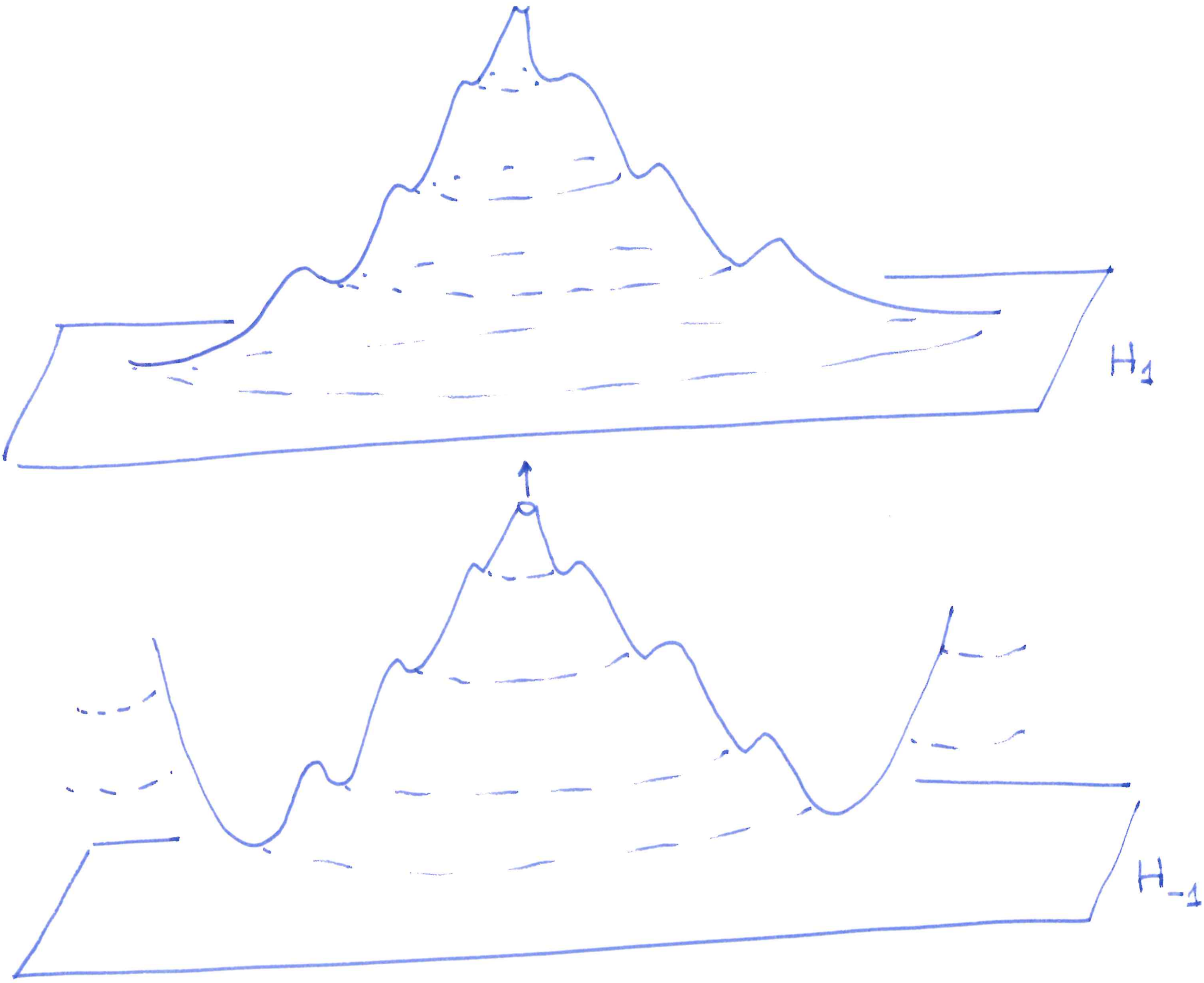
In consequence

$$\|z_n\|^2 = \left(\frac{T}{2n^2\pi^2}\right)^{1/3} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\|z_n'\|^2 = n^2\pi^2 \left(\frac{T}{2n^2\pi^2}\right)^{1/3} \longrightarrow +\infty$$

$$\delta\mathcal{B}(z_n) = 2^{1/3}\pi^{2/3} n^{2/3}\left(\frac{1}{T^{1/3}} + T^{2/3}\right) \longrightarrow +\infty$$

We draw two revolution surfaces that partially resemble the graph of $\delta\mathcal{B}$ on H_1 and H_{-1}



An important property of the functional $\mathcal{B}$ is that it is even,

$$\mathcal{B}(z) = \mathcal{B}(-z) \quad \forall z \in H_{\pm 1} \setminus \{0\}.$$

This is the case even when the function $U(t,z)$ is not even. A consequence of this symmetry is that it is possible to prove the existence of infinitely many critical points of $\mathcal{B}$ on H_{-1} under certain conditions on U.

We will prove a simpler result using only a standard minimitation technique. From now on assume that

$$U(t,x) \leq \alpha |x|^2 + \beta \,, \quad x \in \mathbb{R}^2$$

with $\alpha, \beta > 0$ and $\alpha < \frac{2}{T^2}$.

The first step will be to prove that $\mathcal{B}$ has a ~~positive~~ lower bound,

$$\inf_{H_{-1} \setminus \{0\}} \mathcal{B} > -\beta T$$

To this end we employ the inequality

$$\|z\|_\infty^2 \leq \|z\| \, \|z'\| \qquad \forall z \in H_{-1}.$$

Then

$$R(z) \leq \alpha \int_0^1 |z|^6 + \beta \int_0^1 |z|^2$$

$$\leq \alpha \|z\|_\infty^4 \|z\|^2 + \beta \|z\|^2 \leq \alpha \|z'\|^2 \|z\|^4 + \beta \|z\|^2$$

$$\mathcal{B}(z) = \frac{2}{T} \|z\|^2 \|z'\|^2 + \frac{T}{\|z\|^2} + \frac{T}{\|z\|^2} R(z) >$$

$$\frac{2}{T} \|z\|^2 \|z'\|^2 - T\alpha \|z'\|^2 \|z\|^2 - \beta T$$

$$= \left(\frac{2}{T} - T\alpha \right) \|z\|^2 \|z'\|^2 - \beta T$$

Let us now take a minimizing sequence $\{z_n\}$,

$$z_n \in H_{-1}, \quad \mathcal{B}(z_n) \to \inf_{H_{-1} \setminus \{0\}} \mathcal{B} =: \mu$$

For large n,

$$\mu + 1 \geq \mathcal{B}(z_n) \geq \frac{T}{\|z_n\|^2} + \left(\frac{2}{T} - T\alpha \right) \|z_n\|^2 \|z_n'\|^2 - \beta T$$

$$> \frac{T}{\|z_n\|^2} - \beta T.$$

In particular, there exists $m > 0$ such that

$$\|z_n\| \geq m.$$

The minimizing sequence is far from the origin.

Next we prove that $\{z_n\}$ is bounded in H_{-1},

$$\mu+1 \geq \mathcal{B}(z_n) \geq \left(\frac{2}{T} - T\alpha\right) \|z_n\|^2 \|z_n'\|^2 - \beta T$$

$$\geq \left(\frac{2}{T} - T\alpha\right) m^2 \|z_n'\|^2 - \beta T.$$

Now we can extract a subsequence $\{z_k\}$ satisfying

$z_k \rightharpoonup z_*$ in H_{-1} weak

$z_k \to z_*$ uniformly

Then $\|z_k\| \to \|z_*\|$, ~~lim~~ $\|z_*'\| \leq \liminf \|z_k'\|$

$\mathcal{R}(z_k) \to \mathcal{R}(z_*)$. In consequence

$$\mathcal{B}(z_*) \leq \liminf \mathcal{B}(z_k) = \mu$$

and we conclude that z_* is a minimizer.

Remarks. 1. It could be of interest to find the optimal value for α

2. Connection with the result by Boscaggin, Dambrosio and Papini

Appendix: proof of $\|z\|_\infty \leq \|z\| \, \|z'\|$, $z \in H_{-1}$

$$2\,|z(\tau)|^2 = \big|\, |z(\tau+1)|\,z(\tau+1) - |z(\tau)|\,z(\tau) \big| =$$

$$\left| \int_\tau^{\tau+1} \frac{d}{ds}\big(|z(s)|\,z(s)\big)\,ds \right| \leq 2 \int_\tau^{\tau+1} |z(s)|\,|z'(s)|\,ds$$

$$\leq 2\,\|z\|\,\|z'\|$$