A brief introduction to Kepler problem

Consider the system

$$\ddot{x} = -\frac{x}{|x|^3}, \quad x \in \mathbb{R}^d \setminus \{0\}$$

with $d = 1, 2, 3$. The aim is to describe the set of T-periodic solutions for a fixed $T > 0$.

Assume that $x(t; x_0, v_0)$ is the solution with initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$. We want to describe the set

$$\mathcal{P}_T = \left\{ (x_0, v_0) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d : \begin{array}{l} x(t; x_0, v_0) \text{ is} \\ T\text{-periodic} \end{array} \right\}$$

Let us recall the most popular first integrals of Kepler problem, the energy

$$h = \frac{1}{2} |\dot{x}|^2 - \frac{1}{|x|}$$

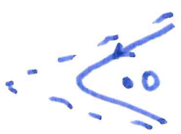
and the angular moment

$$\vec{c} = x \wedge \dot{x}.$$

The orbits can be classified in terms of h and c.

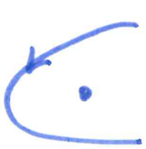
$\vec{c} \neq 0$ $\vec{c} = 0$

$\left[\begin{array}{l} h > 0 \\ h = 0 \end{array}\right.$



hyperbolic

ejection

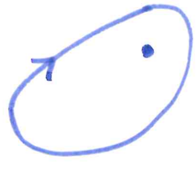


parabolic

collision

$\vec{c} \neq 0$ $\qquad\qquad$ $\vec{c} = 0$

$h<0$



elliptic

ejection-collision

In the case $h \geq 0$ the solution $x(t)$ is unbounded.

For $h<0$, $c=0$, $x(t)$ remains in a bounded set but $\dot{x}(t)$ is unbounded. This follows from the conservation of energy,

$$\frac{1}{2}|\dot{x}(t)|^2 - \frac{1}{|x(t)|} = h, \qquad x(t) \to 0 \iff |\dot{x}(t)| \to \infty .$$

The only candidates to periodic motions are the elliptic orbits.

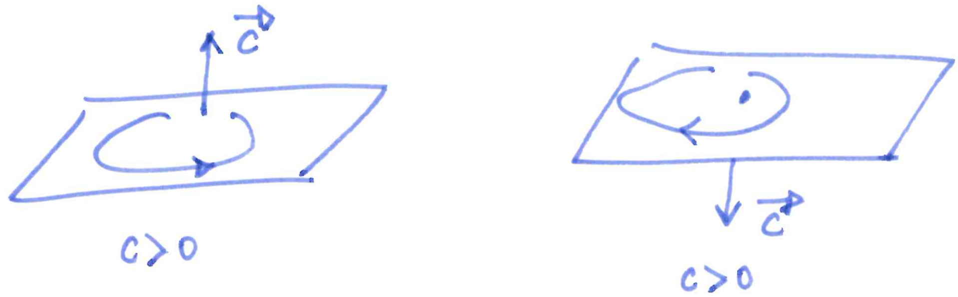
From now on we will concentrate on dimension $d=2$. For $d=1$, $\mathcal{P}_T = \phi$. For $d=3$ we observe that the group $SO(3)$ leaves invariant the equation (if $R \in SO(3)$ and $x(t)$ is a solution, then $Rx(t) \in SO(3)$). In consequence, for $d=3$, $\mathcal{P}_T$ is invariant under the action of $SO(3)$, $(x_0, v_0) \in \mathcal{P}_T \iff (Rx_0, Rv_0) \in \mathcal{P}_T$.

Then we can rotate the motion and assume that it lies on the plane $\mathbb{R}^2 \times \{0\}$.

Assume $d=2$. Take $\vec{e}_3 = (0,0,1)^*$ and

$$\vec{c} = c\,\vec{e}_3, \quad c \in \mathbb{R}.$$

Then $c>0$ says that the motion is counter-clockwise



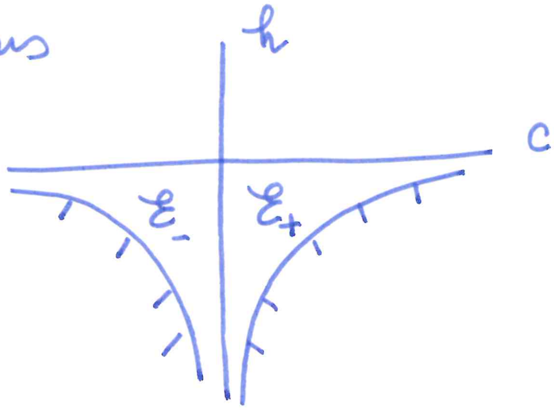
Define $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$ where

$$\mathcal{E}_+ = \{ (x_0, v_0) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 : h(x_0, v_0) < 0,\ c(x_0, v_0) > 0 \}$$

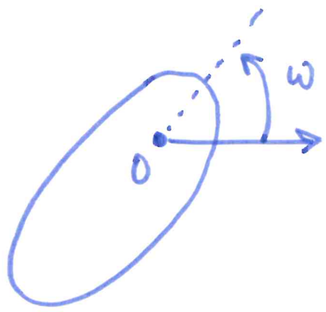
Since Kepler problem is reversible ($x(t)$ solution $\Longleftrightarrow$ $x(-t)$ solution), $S(\mathcal{E}_+) = \mathcal{E}_-$ where S is the symmetry $S(x, v) = (x, -v)$.

We observe that $\mathcal{E}_+$ and $\mathcal{E}_-$ are invariant open subsets of $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$

<u>Exercise</u> Prove that $h \geq -\frac{1}{2c^2}$ with identity on circular motions



Our next step is to describe all the solutions with initial condition lying in $\mathcal{E}_+$. To this end we will employ three geometric quantities in an ellipse



$a > 0$ major semi-axis

$e \in [0,1[$ eccentricity

$\omega \in \mathbb{R}/_{2\pi\mathbb{Z}}$

The solutions in $\mathcal{E}_+$ are

$$x(t) = R[\omega]\begin{pmatrix} a(\cos u - e) \\ a\sqrt{1-e^2}\,\sin u \end{pmatrix}, \quad u - e\sin u = \frac{1}{a^{3/2}}(t - T)$$

where $R[\omega] = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$

T is the time of passage by perihelion

It is possible to determine the geometric quantities from the initial conditions via the important first integral (Runge-Lenz, Laplace, ...)

$$\vec{e} = \dot{x} \wedge \vec{c} - \frac{x}{|x|} = e(\cos\omega, \sin\omega)$$

and the formula

$$h = -\frac{1}{2a}$$

(these identities are valid on $\mathcal{E}_+$).

Conversely, given a, e, ω and τ we can compute the initial conditions x_0, v_0.

We also observe that τ and $\cancel{a^{-3/2}\,\tau + 2\pi}$ $\tau + a^{3/2}2\pi$ will produce the same solution. The following map is a diffeomorphism

$$(x_0, v_0) \in \mathcal{E}_+ \longmapsto (a, \vec{e}, \underset{a^{-3/2}\tau}{\lambda}) \in \,]0,\infty[\, \times D \times \mathbb{R}/_{2\pi\mathbb{Z}},$$

where $D = \{ z \in \mathbb{R}^2 : |z| < 1 \}$.

In consequence $\mathcal{E}_+$ is diffeomorphic to $\mathbb{R}^3 \times \mathbb{S}^1$

We are ready for the description of $\mathcal{P}_T$. Looking at Kepler equation $u - e \sin u = \frac{1}{a^{3/2}}(t - \tau)$, we observe that the eccentric anomaly $u = u(t)$ has minimal period

$$p = 2\pi a^{3/2}.$$

This is also the minimal period of the solution $x(t)$ (3rd Kepler law). The solution will have period T if and only if $T = np$ for some $n = 1, 2, \dots$ In consequence $\mathcal{P}_T^+$ is described as the union of families

$$2\pi a_n^{3/2} = \frac{T}{n}, \quad n = 1, 2, \dots$$

We can now say that $\mathcal{P}_T^+$ has infinitely many connected components

$$\mathcal{P}_T^+ = \bigcup_{n=1}^{\infty} \mathcal{P}_{T,n}^+$$

and each of them is diffeomorphic to $D \times \mathbb{R}/_{2\pi\mathbb{Z}}$.

To visualize $\mathcal{P}_{T,n}^+$ we can think in the couples

$(E, \bar\theta)$ where E is an ellipse with focus at the origin, defined by a and $\vec{e}$, and θ is linked to the time of passage by perihel through the equation $a^{-3/2}\tau = \theta$. When θ runs in $\mathbb{R}/2\pi\mathbb{Z}$, τ runs in $\mathbb{R}$.

Topologically, each $\mathcal{P}^{\pm}_{T,n}$ is homeomorphic to $\mathbb{R}^2 \times S^1$ (an open solid torus).

The projection of $\mathcal{P}^{\pm}_{T,n}$ on the configuration space

$$\pi_1(\mathcal{P}^{\pm}_{T,n}) = \{ x_0 : (x_0, v_0) \in \mathcal{P}^{\pm}_{T,n} \}$$

is bounded because $|x_0|$ cannot go beyond $2a_n$.

However $\mathcal{P}^{\pm}_{T,n}$ is unbounded because the ellipse approaches the origin as $e \nearrow 1$. As $x_0 \to 0$, by conservation of energy, $|v_0| \to \infty$.

For $d=2$, $\mathcal{P}_T$ is a 3-dimensional manifold with infinitely many connected components, all of them homeomorphic to $\mathbb{R}^2 \times S^1$,

$$\mathcal{P}_T = \bigcup_{n=1}^{\infty} (\mathcal{P}^{+}_{T,n} \cup \mathcal{P}^{-}_{T,n}).$$

<u>The flow of Kepler problem is not complete.</u> The maximal interva of solutions with $\vec{c}=0$ is of the type

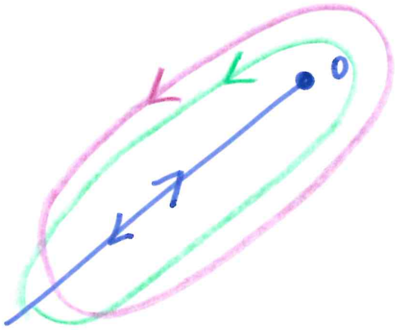
$]-\infty, \omega[$ collision, $\omega < +\infty$

$]\alpha, +\infty[$ ejection, $\alpha > -\infty$,

$]\alpha, \omega[$ ejection-collision, $-\infty < \alpha < \omega < +\infty$

At α or ω there is a collision $\left(x(t) \to 0, \ |\dot{x}(t)| \to \infty \right)$

Intuitively speaking, elliptic solutions with fixed a, ω and τ, will converge to an ejection-collision solution if $e \nearrow 1$



The degenerate ellipse $(e=1)$ is interpreted as a double segment. More analytically, assume for instance

$$a = 1, \quad \omega = 0, \quad \tau = 0$$

and $e_n \nearrow 1$. Then

$$x_n(t) = \begin{pmatrix} \cos u_n(t) - e_n \\ \sqrt{1-e_n^2}\, \sin u_n(t) \end{pmatrix}$$

where $u_n(t)$ satisfies $u_n - e_n \sin u_n = t$. This sequence ~~converges~~ of 2π-periodic solutions converges uniformly to the function

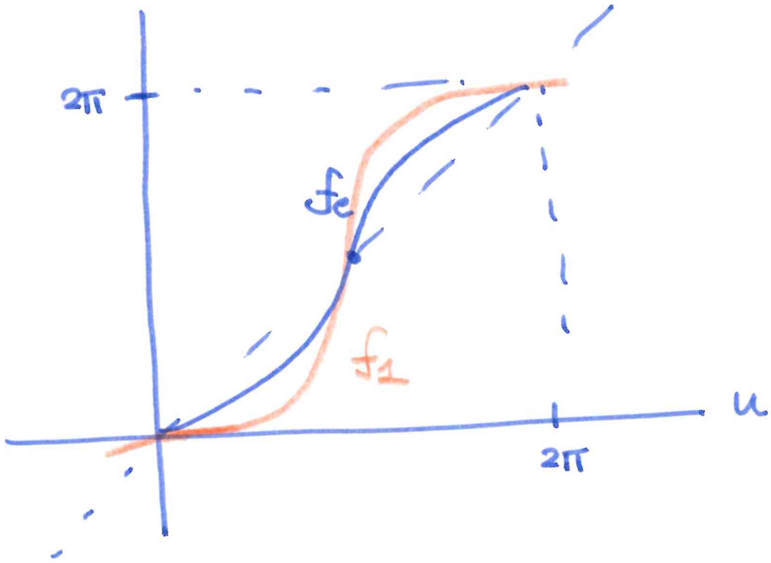
$$x_*(t) = \begin{pmatrix} \cos u_*(t) - 1 \\ 0 \end{pmatrix}$$

with $u_* - \sin u_* = t$.

Now we observe that the function $u_*(t)$ is continuous but not differentiable. The reason behind is in the behavior of the function

$$f_e : \mathbb{R} \to \mathbb{R}, \quad f_e(u) = u - e\sin u.$$

For $e \in]0,1[$ it is an analytic diffeomorphism but for $e=1$ it is only a homeomorphism with $f_1'(2n\pi)=0$

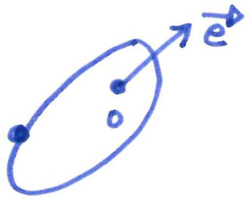


The function $x_*(t)$ is an ejection - collision solution in each interval of the type $]2n\pi, 2(n+1)\pi[$, $n \in \mathbb{Z}$. It seems natural to interpret $x_*(t)$ as a generalized 2π-periodic solution: the particle has succesive collision with the Sun but the energy and the direction are preserved. This seems a sort of completion of the flow in the region $h<0$. The idea goes back to T. Levi Civita.

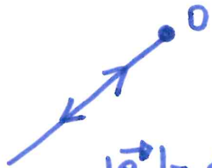
The set of T-periodic solutions has been enlarged, by adding collision solutions,

$$\tilde{\mathcal{P}}_T = \mathcal{P}_T \cup \{T\text{-periodic collision solutions}\}.$$

To introduce a topology in $\tilde{\mathcal{P}}_T$, we recall the diffeomorphisms from $\mathcal{P}^{\pm}_{T,n}$ onto $D \times \mathbb{R}/_{2\pi\mathbb{Z}}$, assigning to each solution the couple $(\vec{e}, g^{1/2}_{a}\tau)$



$|\vec{e}| < 1$



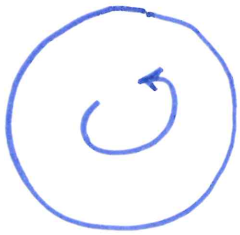
$|\vec{e}| = 1$

We have a bijection between $\tilde{\mathcal{P}}^{\pm}_{T,n}$ and $\overline{D} \times (\mathbb{R}/_{2\pi\mathbb{Z}})$,

where $\overline{D} = \{\vec{e} \in \mathbb{R}^2 : |\vec{e}| \leq 1\}$.

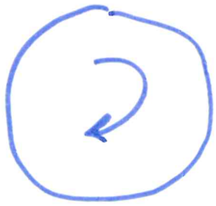
The boundary of $\overline{D}$ corresponds to the collision solutions with minimal period $\frac{T}{n}$

$\tilde{\mathcal{P}}^{+}_{T,n}$



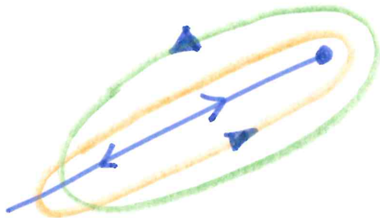
$\overline{D}$ $\times \mathbb{R}/_{2\pi\mathbb{Z}}$

$\tilde{\mathcal{P}}^{-}_{T,n}$

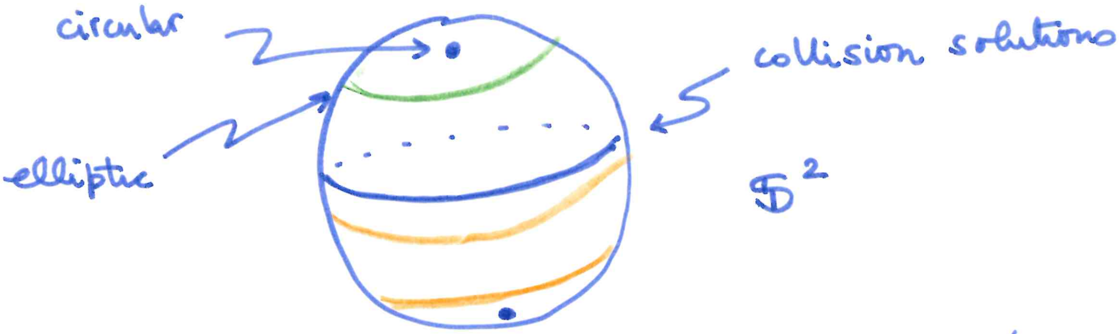


$\overline{D}$ $\times \mathbb{R}/_{2\pi\mathbb{Z}}$

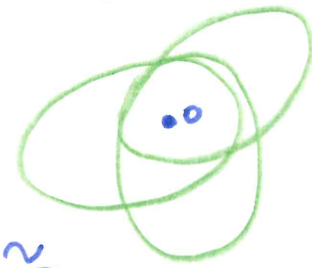
The ejection-collision solutions do not have an orientation an can be approximated by positive or negative ellipses



For this reason we can identify the boundaries of the two disks, to obtain a sphere



The poles correspond to circular solutions $(e=0)$ $N \circlearrowleft$, $S \circlearrowright$, the equator correspond to collision solutions $(e=1)$ and each parallel is a family of ellipses with fixed eccentricity $0<e<1$



In this way $\tilde{\mathcal{P}}_T$ becomes a 3-dimensional manifold with infinitely many connected components

$$\tilde{\mathcal{P}}_T = \bigcup_{n=1}^{\infty} \tilde{\mathcal{P}}_T$$

and each component is compact, ~~homeomorphic to~~ ~~$\mathbb{S}^2 \times \mathbb{S}^1$~~.

Levi-Civita regularization

a) The autonomous case

We identify $\mathbb{R}^2$ and $\mathbb{C}$, $x=(x_1,x_2) \mapsto x = x_1 + i x_2$

For the system

$$(\{\text{kepler}\}) \quad \ddot{x} = -\frac{x}{|x|^3}, \quad x \in \mathbb{C}\setminus\{0\}$$

we use the change of variables

$$x = z^2, \qquad ds = \frac{dt}{|x|}.$$

The first step is to understand the meaning of this change, since $z \in \mathbb{C}\setminus\{0\} \mapsto x = z^2 \in \mathbb{C}\setminus\{0\}$ is not a diffeomorphism. However it is a covering map and, given an analytic function $t \in I \mapsto x(t) \in \mathbb{C}\setminus\{0\}$, we can express it in analytic polar coordinates

$$x(t) = r(t)\, e^{i\theta(t)}$$

where $r(t) > 0$ and $\theta = \theta(t)$ are real analytic. Then we can define the square root

$$\xi(t) = r(t)^{1/2} e^{i\theta(t)/2}.$$

We observe that $\xi(t)$ and $-\xi(t)$ are solutions of $\xi^2 = x(t)$.

Assume that $x(t)$ is a solution of $(\{\text{kepler}\})$ defined on a maximal interval $I \subset \mathbb{R}$. We define the Sundman integral,

$$\mathcal{S}(t)=\int_{t_0}^{t}\frac{d\tau}{|x(\tau)|}\quad,\ t_0\in I \text{ fixed}$$

Then $\mathcal{S}$ is a diffeomorphism from I onto some open interval $J\subset\mathbb{R}$. We define

$$z(s)=\xi(\Pi(s))$$

where $\Pi: J\to I$ is the inverse of $\mathcal{S}$.

Once we have understood the meaning of the change of variables, let us compute

$$x=z^2,\qquad ds=\frac{dt}{|x|}=\frac{dt}{|z|^2}$$

$$\dot{x}=2zz'\frac{ds}{dt}=\frac{2z}{|z|^2}z'=2\frac{z'}{\bar{z}}$$

$$\ddot{x}=2\frac{\bar{z}z''-|z'|^2}{\bar{z}^2}\frac{ds}{dt}=2\frac{\bar{z}z''-|z'|^2}{\bar{z}^2|z|^2}$$

$$\ddot{x}=-\frac{x}{|x|^3}\Rightarrow 2\frac{\bar{z}z''-|z'|^2}{\bar{z}^2|z|^2}=-\frac{z^2}{|z|^6},$$

$$\left.\begin{array}{l}\bar{z}z''-|z'|^2=-\dfrac{1}{2}\\[2ex] \dfrac{1}{2}|\dot{x}|^2-\dfrac{1}{|x|}=E\Rightarrow 2\left|\dfrac{z'}{z}\right|^2-\dfrac{1}{|z|^2}=E\end{array}\right\}\longrightarrow$$

$$(\{\text{oscill}\})\qquad z''=\frac{1}{2}Ez,\quad E\in\mathbb{R}.$$

From Kepler problem we have arrived at a family of linear equations (note that for $E<0$ we have a harmonic oscillator) but not all solutions of the family ({oscill}) do correspond to Kepler motions. From conservation of energy they must stay on a concrete level energy, namely

({energy}) $\quad \frac{1}{2}|z'|^2 - \frac{1}{4}E|z|^2 = \frac{1}{4}.$

The solutions of ({oscill}) + ({energy}) are obtained from Kepler motions. Conversely, given $z(s)$ a solution of ({oscill}) + ({energy}) defined on $s \in J$ with $z(s) \neq 0$, we define

$$\Pi(s) = \int_{s_0}^{s} |z(\sigma)|^2 d\sigma, \quad s_0 \in J.$$

Then Π is a diffeomorphism from J to some open interval $I \subset \mathbb{R}$. We define

$$x(t) = z(S(t))^2, \quad t \in I$$

where $S: I \to J$ is the inverse of Π. After some computations as before we conclude that $x(t)$ is a motion of Kepler problem.

<u>Some examples</u>

For $E < 0$, $z(s) = a\cos\omega s + b\sin\omega s$, $\omega^2 = \frac{1}{2}|E|$

$\omega^2(|a|^2 + |b|^2) = \frac{1}{2}$, $a, b \in \mathbb{C}$

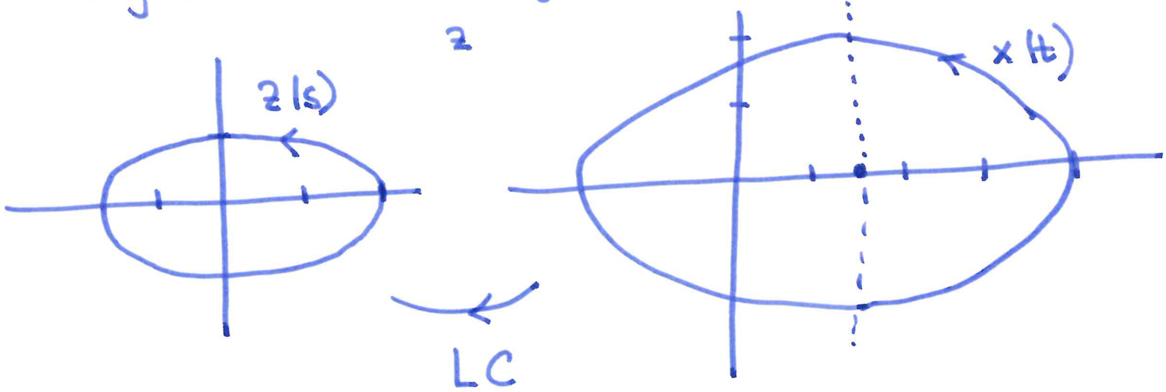
- If a and b are $\mathbb{R}$-linearly independent,

$z(s) \neq 0 \quad \forall s \in \mathbb{R}$

For instance, $z(s) = 2\cos\omega s + i\sin\omega s$ where ω can be adjusted is an ellipse centered at the origin. Then

$$x(t) = \frac{3}{2} + \frac{5}{2}\cos 2\omega s + 2i\sin 2\omega s$$ is an ellipse

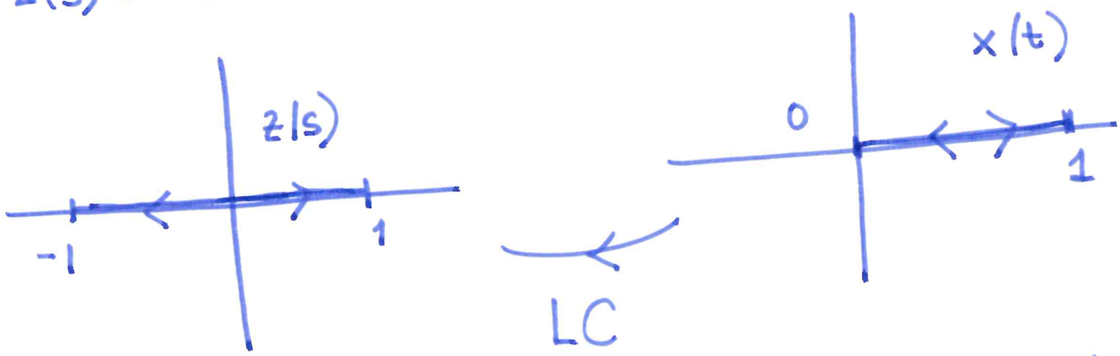
with a focus at the origin



Exercise (proposed by Antonio Ureña) The map $z \mapsto z^2$ transforms ellipses with center at the origin into ellipses with a focus at the origin

- If a and b are $\mathbb{R}$-linearly dependent, we can select an interval J of length $\frac{\pi}{\omega}$ where $z(s)$ does not vanish. It corresponds to an ejection-collision motion. For instance,

$z(s) = \cos\omega s$ $\qquad x(t) = \cos^2\omega s = \frac{1}{2} + \frac{1}{2}\cos 2\omega s$



Collisions in x correspond to a passage through the origin in z

The Levi-Civita transformation has two remarkable properties:

(i) Kepler problem is linearized (in the elliptic regions & it is transformed to a harmonic oscillator)
$\pm$

(ii) The problem $(\{\text{oscill}\}) + (\{\text{energy}\})$ makes sense when $z = 0$. This provides an analytic criterion to continue solutions of Kepler problem after collisions

Historical remark Goursat found this change of variables and noticed (i). The role of the change for regularization of collisions, property (ii), is the idea of Levi-Civita. Nowdays there are other methods to regularize, they satisfy (ii) but not (i). In contrast they are one-to-one instead of two-to-one

b) The time dependent case

Let us now consider the perturbed Kepler problem

$$(\{pKepler\})\quad \ddot{x} = -\frac{x}{|x|^3} + (\nabla_x U)(t,x), \quad x \in \mathcal{U} \setminus \{0\}$$

where $U : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is a C^∞ function and $\mathcal{U}$ is an open neighborhood of $x=0$ in $\mathbb{R}^2$. We also assume that U is T-periodic in t,

$$U(t+T,x) = U(t,x), \quad (t,x) \in \mathbb{R} \times \mathcal{U}.$$

By analogy with the unperturbed case we define

$$E = \frac{1}{2}|\dot{x}|^2 - \frac{1}{|x|} - U(t,x).$$

This quantity will not be constant along solutions because

$$\dot{E} = -\frac{\partial U}{\partial t}.$$

We are going to introduce the Levi-Civita change of variables in ($\{pKepler\}$),

$$x = z^2, \qquad ds = \frac{dt}{|x|}$$

After the identification $\mathbb{R}^2 \equiv \mathbb{C}$, $x=(x_1,x_2) \leftrightarrow x = x_1 + i x_2$, given a function $F: \mathbb{R}^2 \to \mathbb{R}$, $F = F(x_1,x_2)$, we employ the notation $F: \mathbb{C} \to \mathbb{R}$, $F = F(x,\bar{x})$, so that

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) \leftrightarrow \frac{\partial F}{\partial x_1} + i \frac{\partial F}{\partial x_2} = 2 \frac{\partial F}{\partial \bar{x}} .$$

In $\mathbb{C}$ the system ({pKepler}) becomes

$$\ddot{x} = -\frac{x}{|x|^3} + 2 \partial_{\bar{x}} U(t,x,\bar{x}) .$$

After introducing the change of variables

$$z'' = \left(\left|\frac{z'}{z}\right|^2 - \frac{1}{2|z|^2} \right) z + \bar{z} |z|^2 \partial_{\bar{z}} U(t,z,\bar{z}),$$

where $t = t(s) = t_0 + \int_{s_0}^{s} |z(\sigma)|^2 d\sigma$.

As it is written this is not a standard differential equation because it has the integral term $t(s)$. This is easily solved by considering $t = t(s)$ as a new unknown,

$$\begin{cases} z'' = \left(\left|\frac{z'}{z}\right|^2 - \frac{1}{2|z|^2} \right) z + \bar{z}|z|^2 (\partial_{\bar{z}} U)(t, z^2, \bar{z}^2) \\ t' = |z|^2 \end{cases}$$

From the definition of E,

$$E = 2 \left|\frac{z'}{z}\right|^2 - \frac{1}{|z|^2} - U(t, z^2, \bar{z}^2),$$

leading to

$$z'' = \frac{1}{2} E z + \frac{1}{2} z U(t, z^2, \bar{z}^2) + \bar{z}|z|^2 (\partial_{\bar{z}} U)(t, z^2, \bar{z}^2).$$

We define the function

$$\mathcal{P}(t,z,\bar z) = |z|^2\, U(t, z^2, \bar z^2),$$

smooth in some neighborhood $\mathcal{U}\subset\mathbb{C}$ of $z=0$. Then

$$(\{key\})\quad (\partial_{\bar z}\mathcal{P})(t,z,\bar z) = z\,U(t,z^2,\bar z^2) + 2|z|^2\bar z\,(\partial_{\bar x}U)(t,z^2,\bar z^2).$$

In consequence,

$$(\{sys\})\quad z'' = \frac{1}{2}Ez + \frac{1}{2}(\partial_{\bar z}\mathcal{P})(t,z,\bar z),\quad t' = |z|^2$$

and

$$(\{E2\})\qquad E|z|^2 = 2|z'|^2 - 1 - \mathcal{P}(t,z,\bar z).$$

Conversely, assume that $(z(s), t(s))$ is a solution of $(\{sys\})$, $(\{E2\})$ defined on some interval $J\subset\mathbb{R}$ where $z(s)\neq 0$. Note that we are not assuming that E is constant. Define $x(t) = z(\mathcal{S}(t))^2$, where $\mathcal{S}: I\to J$ is the inverse of $t(s)$. After computations,

$$\ddot x + \frac{x}{|x|^3} = \left[2\,\frac{\bar z z'' - |z'|^2}{\bar z^2|z|^2} + \frac{z^2}{|z|^6}\right]\circ\mathcal{S} \overset{(\{sys\})}{=}$$

$$= \frac{2}{\bar z^2|z|^2}\left[\frac{1}{2}E|z|^2 + \frac{1}{2}\bar z(\partial_{\bar z}\mathcal{P})(t,z,\bar z) - |z'|^2 + \frac{1}{2}\right]\circ\mathcal{S} \overset{(\{E2\})}{=}$$

$$= \frac{2}{\bar z^2|z|^2}\left[-\frac{1}{2}\mathcal{P}(t,z,\bar z) + \frac{1}{2}\bar z(\partial_{\bar z}\mathcal{P})(t,z,\bar z)\right]\circ\mathcal{S} \overset{(\{key\})}{=}$$

$$= 2(\partial_{\bar x}U)(t,x,\bar x)$$

Suming up, every solution of ({pKepler}) will produce ~~of~~ two solutions $(z(s), t(s))$ of ({sys}) + ({E2}) and the process can be reversed as long as $z(s) \neq 0$.

At this point the regularitation is still unclear. If we plug the value of $\mathcal{E}$ given by ({E2}) in ({sys}), we obtain a system in (z, t) having a singularity at $z = 0$. To overcome this difficulty we enlarge again the dimension of the system by assuming that also $E = E(s)$ is an unknown.

From $\mathring{E} = -\frac{\partial U}{\partial t}$ we deduce that

$$E' = \mathring{E}\,\frac{dt}{ds} = -|z|^2 (\partial_t U)(t, z^2, \bar{z}^2) = -(\partial_t P)(t, z, \bar{z}).$$

Then we obtain the system in six dimensions

$$(\{\text{reg}\}) \quad z'' = \frac{1}{2} E z + \frac{1}{2} (\partial_{\bar{z}} P)(t, z, \bar{z}), \quad t' = |z|^2, \quad E' = -(\partial_t P)(t, z, \bar{z})$$

Now there are no singularities at $z = 0$ but this system is too big, most solutions do not correspond to solutions of the perturbed Kepler problem. To understand this we consider the function $\mathcal{Y} = \mathcal{Y}(z, z', t, E)$

$$\mathcal{Y} = 2|z'|^2 - E|z|^2 - 1 - P(t, z, \bar{z})$$

and observe that it is a first integral of ({reg}),

$$\frac{d\mathcal{Y}}{ds} = 4\langle z', z''\rangle - 2E\langle z, z'\rangle - E'|z|^2 - (\partial_t P)t'$$
$$-(\partial_z P) z' - (\partial_{\bar{z}} P)\bar{z}'$$

$$= \langle z', \underbrace{4z'' - 2Ez - 2\partial_{\bar z} P}_{=0} \rangle - \underbrace{E'|z|^2 - (\partial_t P)t'}_{=0}$$

[Note that $\partial_z P\, z' + \partial_{\bar z} P\, \bar z' = \langle \nabla P, z' \rangle = 2\langle \partial_{\bar z} P, z' \rangle$]

Only the solutions at the level $\mathcal{J}=0$ will satisfy ({E2}). Therefore, the perturbed Kepler problem is in correspondence with the orbits of ({reg}) at level $\mathcal{J}=0$. This is an smooth system at $z=0$.

The Hamiltonian structure

Let us recall that a system defined on $\Omega \subset \mathbb{R}^N \times \mathbb{R}^N$ has a Hamiltonian structure if it can be expressed in the form

$$\dot z = J \nabla H(z)$$

where $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ and $H : \Omega \to \mathbb{R}$ is C^∞.

In coordinates $z = (q, p) \in \Omega$

$$\dot q = \frac{\partial H}{\partial p}(q,p), \quad \dot p = -\frac{\partial H}{\partial q}(q,p).$$

For system ({reg}) we define $w = 4z'$ and use the first integral to define

$$H(z, E, w, t) = \frac{1}{8}|w|^2 - E|z|^2 - 1 - P(t, z, \bar z)$$

The Hamiltonian structure

Consider an autonomous Hamiltonian system in $\mathbb{R}^N \times \mathbb{R}^N$,

$$\dot{\xi} = J \nabla H(\xi)$$

where $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$, $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is C^∞.

In coordinates $\xi = (q, p)$,

$$\dot{q} = \frac{\partial H}{\partial p}(q,p), \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p).$$

In our case $N = 3$ and $\mathbb{R}^3$ is identified to $\mathbb{C} \times \mathbb{R}$.

With $w = 4z'$ we consider

$$q = (z, E), \quad p = (w, t)$$

and
$$H = \frac{1}{8}|w|^2 - E|z|^2 - 1 - \mathcal{P}(t, z, \bar{z}).$$

Then $(\{reg\}) + \{y=0\}$ can be formulated as

$$(\{hamh\}) \quad \dot{\xi} = J\nabla H(\xi), \quad H = 0.$$

From the definition of $\mathcal{P}$ we observe that

$$H(-z, -w, E, t) = H(z, w, E, t).$$

This implies that the system $(\{hamh\})$ is invariant under the involution $(z, w, E, t) \mapsto (-z, -w, E, t)$. The two solutions $(z(s), w(s), E(s), t(s))$ and $(-z(s), -w(s), E(s), t(s))$ will produce the same solution of $(\{pKepler\})$

Also,

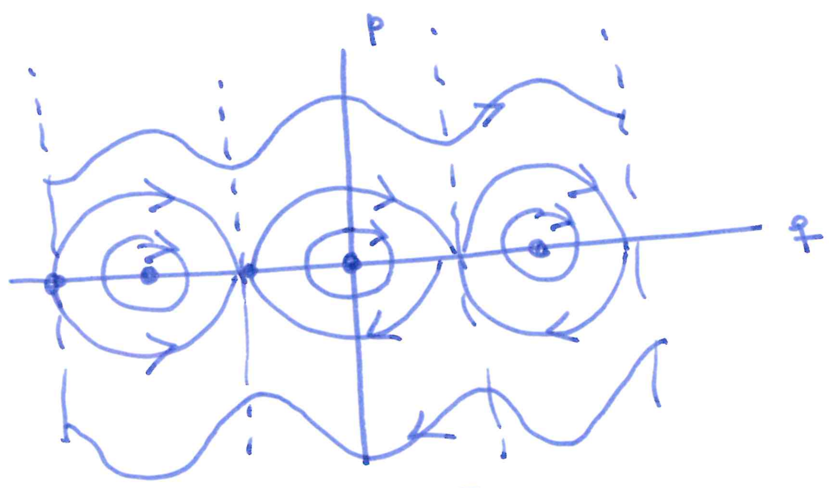
$$H(z, w, E, t+T) = H(z, w, E, t).$$

In general, given a Hamiltonian function which is periodic in one variable, say

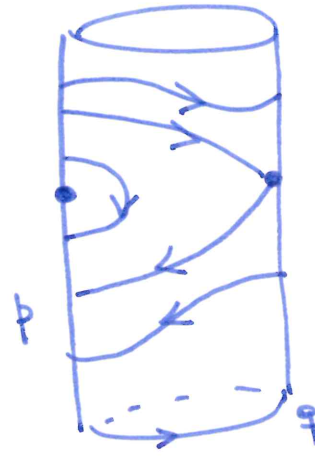
$$H(q_1 + Q, q_2 \cdots q_N, p_1 \cdots p_N) = H(q_1, q_2 \cdots q_N, p_1 \cdots p_N),$$

the phase space $\mathbb{R}^N \times \mathbb{R}^N$ can be changed to $(\mathbb{R}/Q\mathbb{Z}) \times \mathbb{R}^{N-1} \times \mathbb{R}^N$, with coordinates $\bar{q}_1, q_2 \cdots q_N, p_1, p_2 \cdots p_N$, $\bar{q}_1 = q_1 + Q\mathbb{Z}$. The typical example is the pendulum equation. Here $N = 1$ and $Q = 2\pi$, $\ddot{\theta} + a \sin\theta = 0$, $q = \theta$, $p = \dot{\theta}$,

$$H(q, p) = \frac{1}{2} p^2 - a \cos q$$



phase space $\mathbb{R} \times \mathbb{R}$



phase space $\mathbb{R}/_{2\pi\mathbb{Z}} \times \mathbb{R}$

In our case the periodicity is in t and we will work on $\mathbb{C} \times \mathbb{R} \times \mathbb{C} \times (\mathbb{R}/T\mathbb{Z})$

Experts in Hamiltonian dynamics will think on the symplectic manifold (M, ω) with $M = \mathbb{C} \times \mathbb{R} \times \mathbb{C} \times (\mathbb{R}/T\mathbb{Z})$ with the form

$$\omega = \sum_{j=1}^{2} dz_j \wedge dw_j + dE \wedge dt$$

Equivalence between generalited periodic solutions and closed orbits of the regularized Hamiltonian system

We will prove that there is a one-to-two correspondence between the generalized periodic solution of ({pKepler}) of period NT, $N = 1, 2, \dots$ and the closed orbits of ({hamR}).

The following elementary result will be very useful.

Lemma ({easy}) Assume that $X : D \subset \mathbb{R}^d \to \mathbb{R}^d$ is continuous vector field and $x : \mathbb{R} \to D$ is a continuous function such that there exists a discrete set $Z \subset \mathbb{R}$ with $x \in C^1(\mathbb{R} \setminus Z, \mathbb{R}^d)$ and $\dot{x}(t) = X(x(t))$, $t \in \mathbb{R} \setminus Z$.

Then x is C^1 on $\mathbb{R}$ and it is a global solution of the system $\dot{x} = X(x)$.

Proof

$$\underset{t \in I}{\dot{x}(t) = X(x(t))} \iff \underset{t, t_0 \in I}{x(t) = x(t_0) + \int_{t_0}^{t} X(x(\tau))\,d\tau}$$

i) From generalited periodic solutions to closed orbits

Assume that $x : \mathbb{R} \to \mathbb{R}^2$ is a generalized periodic solution with period NT, $N \geq 1$. The function

$$t \notin Z \mapsto \frac{x(t)}{|x(t)|} \in \mathbb{S}^1$$ admits a continuous extensi

to $\mathbb{R}$. Lifting the path we find a continuous function $\theta : \mathbb{R} \to \mathbb{R}$ such that $x(t) = |x(t)| e^{i\theta(t)}$ and $\theta(t+NT) = \theta(t) + 2k\pi$ for some $k \in \mathbb{Z}$.

Next we recall a property of the singularities of the perturbed Kepler problem (Sperling): Assume that $x(t)$ is a solution of $\ddot{x} = -\frac{x}{|x|^3} + P(t, x, \dot{x})$ where P is continuous and bounded in a neighborhood of $x = 0$. Assume that $x(t)$ is a solution defined on $[t_0, \omega[$ with $\omega < \infty$, $\liminf_{t \to \omega^-} |x(t)| = 0$. Then the following limit exists

$$\lim_{t \to \omega^-} \frac{x(t)}{|t-\omega|^{2/3}} = \xi \neq 0.$$

In consequence $\frac{1}{|x(t)|}$ behaves like $\frac{1}{|t-\omega|^{2/3}}$ near ω and $\frac{1}{|x|} \in L^1(t_0, \omega)$. We can define the Sundman integral

$$S : \mathbb{R} \to \mathbb{R}, \quad S(t) = \int_0^t \frac{d\tau}{|x(\tau)|}.$$

Since Z is discrete, S is continuous, increasing and

$$S(t+NT) = S(t) + \sigma,$$

where $\sigma = \int_0^{NT} \frac{d\tau}{|x(\tau)|}$.

In consequence S is a homeomorphism and S can be differentiated on $\mathbb{R} \setminus Z$ with $S' = \frac{1}{|x|}$.

The inverse homeomorphism $\mathcal{T}:\mathbb{R}\to\mathbb{R}$, $\mathcal{T}=\mathcal{T}(s)$, will satisfy $\mathcal{T}(s+\sigma)=\mathcal{T}(s)+NT$ and will be smooth on $\mathbb{R}\setminus Z^*$, where $Z^*=S(Z)$ is a discrete set.

Let $\{t_j\}_{j\in\mathbb{Z}}$ be the sequence of instants of collision of $x(t)$. Since $x(t)$ is NT-periodic, $t_{j+J}=t_j+NT$, where J is the number of zeros lying in the interval $[0,NT[$. The sequence $\{s_j\}_{j\in\mathbb{Z}}$ with $s_j=S(t_j)$ satisfies $s_{j+J}=s_j+\sigma$. In each interval $I_j=[s_j,s_{j+1}]$ we define

$$z(s)=(-1)^j\,|x(\mathcal{T}(s))|^{1/2}e^{i\theta(\mathcal{T}(s))/2}\quad \text{if } s\in I_j.$$

We observe that $z(s)$ is continuous and

$$z(s+\sigma)=(-1)^{J+k}z(s),\quad s\in\mathbb{R}.$$

Thus, either $z(s)$ is σ-periodic or σ-anti periodic. We are alternating the branches of the square root so that

$$\lim_{s\to s_j^+}\frac{z(s)}{|z(s)|}=-\lim_{s\to s_j^-}\frac{z(s)}{|z(s)|}=(-1)^{j+1}e^{i\theta(t_j)/2}$$

<u>Example</u> $z(s)=\cos s$, $x(\mathcal{T}(s))=\cos^2 s$

$x(t)$ is π-periodic, $\theta(t)\equiv 0$, $s_0=-\frac{\pi}{2}$, $s_1=\frac{\pi}{2}$,

$J=1$,

$$z(s)=\left\{\begin{array}{ll}|\cos s|, & s\in[-\frac{\pi}{2},\frac{\pi}{2}]\\ -|\cos s|, & s\in[\frac{\pi}{2},\frac{3\pi}{2}]\end{array}\right\}=\cos s,\quad \sigma=\pi$$

$$z(s+\pi)=-z(s)$$

From the definition of generalited solution we know that $E(s) = E(\Pi(s))$, $s \in \mathbb{R} \setminus Z^*$ admits a continuous extension which is σ-periodic. Also, $t(s) = \Pi(s)$ is continuous and $s \in \mathbb{R} \mapsto \overline{t(s)} \in \mathbb{R}/_{T\mathbb{Z}}$ is σ-periodic.

Finally, let us analyze $w(s) = 4z'(s)$ if $s \in \mathbb{R} \setminus Z^*$. In each interval I_j we select $s_j^* \in \mathring{I}_j$, then from the differential equations,

$$W(s) = W(s_j^*) + 4\int_{s_j^*}^{s} \left[\tfrac{1}{2} E(s_1) z(s_1) + \tfrac{1}{2}(\partial_{\bar z} P)\right]\underbrace{(t(s_1), z(s_1), \bar z(s_1))}_{ds_1}$$

for each $s \in \mathring{I}_j$. Since the integrand is continuous in $\mathbb{R}$ we conclude that $w(s_j^+)$ and $w(s_{j+1}^-)$ exist. We are going to prove that $w(s_j^-) = w(s_j^+)$ is also continuous. Then we can apply Lemma {easy} to conclude that $(z(s), w(s), E(s), t(s))$ is a solution of the system ({hamh}) on the whole line. This solution will be periodic with period σ or 2σ.

From the first integral we observe that $|w(s_j^{\pm})| = 2\sqrt{2}$.

Then

$$\left.\begin{aligned} \lim_{s \to s_j^+} \frac{z(s)}{|z(s)|} &= \frac{1}{2\sqrt{2}} w(s_j^+) \\ \lim_{s \to s_j^-} \frac{z(s)}{|z(s)|} &= -\frac{1}{2\sqrt{2}} w(s_j^-) \end{aligned}\right\} \Rightarrow w(s_j^+) = +w(s_j^-)$$

We also observe that the system $\dot{\xi} = J\nabla H(\xi)$, has no equilibria at $H=0$. Note that $|z(s)|$ is positive almost everywhere and $t' = |z|^2$, so that $\overline{t(s)}$ is not constant.

ii) From closed orbits to generalized periodic solutions

Assume that $(z(s), w(s), E(s), \overline{t(s)})$ is a solution of

$$\dot{\xi} = J\nabla H(\xi), \quad H(\xi) = 0$$

satisfying $z(s+\sigma) = \pm z(s)$, $w(s+\sigma) = \pm w(s)$, $E(s+\sigma) = E(s)$, $\overline{t(s+\sigma)} = \overline{t(s)}$. Let $t = t(s)$ be a lift of $\overline{t(s)}$. Then there exists $N \in \mathbb{Z}$ such that $t(s+\sigma) = t(s) + NT$.

From $H(\xi(t)) = 0$ we deduce that $z(s) = 0$ implies $z'(s) \neq 0$. In consequence

$$Z^* = \{s \in \mathbb{R} : z(s) = 0\}$$

is a discrete set. From the equation $t' = |z(s)|^2$ we deduce that $t(s)$ is a smooth homeomorphism. The inverse $\mathcal{S} : \mathbb{R} \to \mathbb{R}$, $\mathcal{S} = \mathcal{S}(t)$ is a homeomorphism, smooth on $\mathbb{R} \setminus Z$ where $Z = t(Z^*)$. Note that Z is also a closed and discrete set. Moreover,

$$\mathcal{S}(t + NT) = \mathcal{S}(t) + \sigma.$$

Define

$$x(t) = z(\mathcal{S}(t))^2, \quad t \in \mathbb{R}.$$

Then $x(t)$ is continuous over $\mathbb{R}$ and it satisfies

$$\ddot{x}(t) = -\frac{x(t)}{|x(t)|^3} + \nabla_x U(t, x(t)), \quad t \in \mathbb{R} \setminus Z.$$

The function $E(t) = E(\mathcal{S}(t))$ is continuous and this implies that $\frac{1}{2}|\mathring{x}(t)|^2 - \frac{1}{|x(t)|} = E(t) + U(t, x(t))$ has a limit if $t \to t_0$ with $t_0 \in Z$.

Let $s_0 \in Z^*$ be such that $\mathcal{S}(t_0) = s_0$. Then $z(s_0) = 0$, $z'(s_0) \neq$ implies that $z(s) = \alpha(s)(s - s_0)$ where $\alpha(s) \to z'(s_0)$ as $s \to s_0$

Therefore,

$$\frac{x(t)}{|x(t)|} = \left(\frac{z(\mathcal{S}(t))}{|z(\mathcal{S}(t))|}\right)^2 = \left(\frac{\alpha(\mathcal{S}(t))}{|\alpha(\mathcal{S}(t))|}\right)^2 \longrightarrow \left(\frac{z'(s_0)}{|z'(s_0)|}\right)^2 \text{ if } t \to$$