Perturbed Kepler problem in one dimension

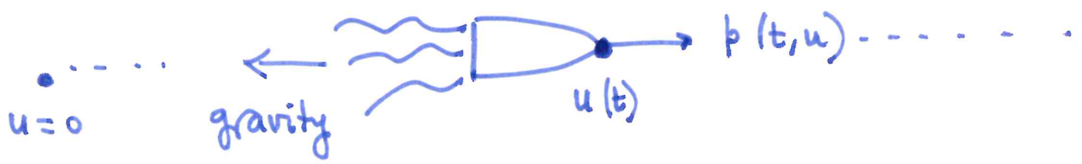
Consider the equation

$$\ddot{u} = -\frac{1}{u^2} + p(t,u), \quad u \in \,]0,\infty[$$

where $p : \mathbb{R} \times [0,\infty[\,\to \mathbb{R}$ is smooth (C^∞), T-periodic in t,

$p(t,\cdot)$ is monotone non-decreasing for each t

$p(t,u) \leq M < \infty$ for each $(t,u) \in \mathbb{R} \times [0,\infty[$



A monotonicity property

Let us write the equation as a first order system

$$\dot{u} = f(t,u,v), \quad \dot{v} = g(t,u,v)$$

where $f(t,u,v) = v$, $g(t,u,v) = -\frac{1}{u^2} + p(t,u)$.

The vector field satisfies Kamke's condition (cooperative system)

$$\frac{\partial f}{\partial v} \geq 0, \quad \frac{\partial g}{\partial u} \geq 0.$$

In consequence the flow is monotone and the method of upper and lower solutions works

$$\left.\begin{array}{l} u(t_0) \leq u^*(t_0) \\ \dot{u}(t_0) \leq \dot{u}^*(t_0) \end{array}\right] \Longrightarrow u(t) \leq u^*(t),\ \dot{u}(t) \leq \dot{u}^*(t),\ t \geq t_0$$

as long as both solutions are defined

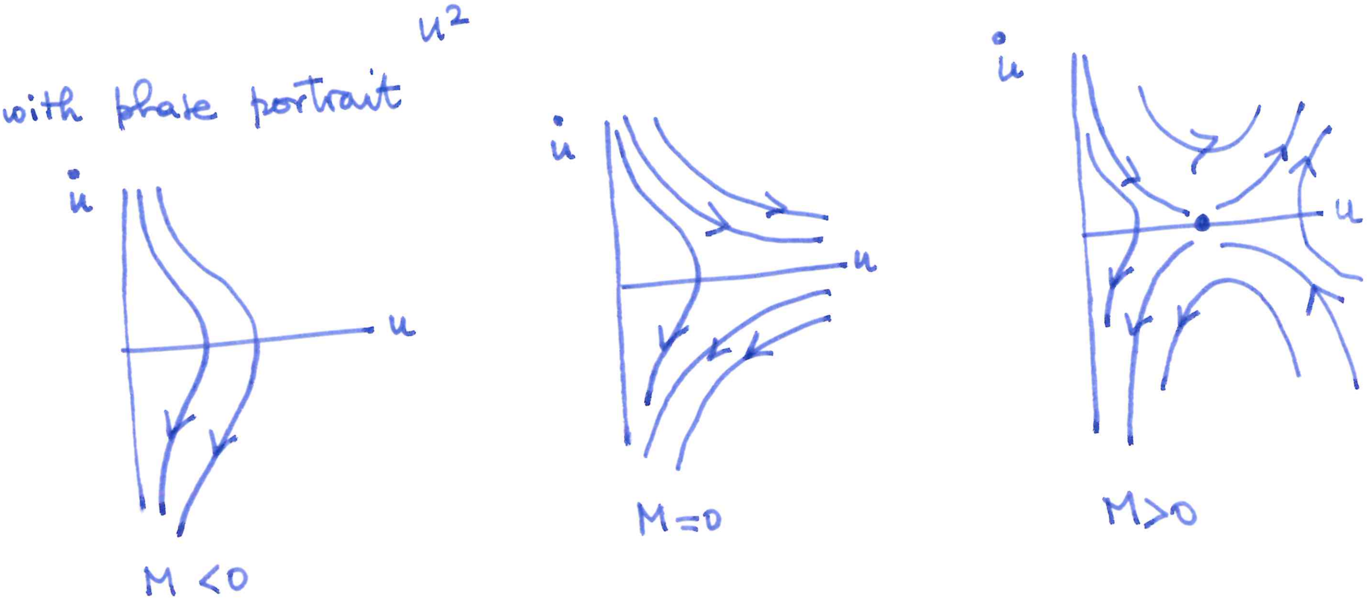
<u>Exercise</u> Assume that u and u^* have forward maximal intervals $[t_0, \omega[$ and $[t_0, \omega^*[$. Then $\omega \leq \omega^*$.

[Remark: $\omega < \infty \Rightarrow \liminf_{t \nearrow \omega} u(t) = 0$]

The natural upper and lower solutions will be provided by the autonomous equation

$$\ddot{u} = -\frac{1}{u^2} + M$$

with phase portrait



<u>Regularization</u> We take $x \in \mathbb{C}$ with $u = \frac{1}{2}(x + \bar{x})$ and consider the force function

$$U(t, x, \bar{x}) = \mathcal{U}\left(t, \frac{x+\bar{x}}{2}\right)$$

where $\mathcal{U}(t,u) = \int_0^u p(t,\xi)\, d\xi$

For the perturbed 2-d Kepler problem

$$\ddot{x} = -\frac{x}{|x|^3} + 2\partial_{\bar{x}} U(t, x, \bar{x})$$

the set

$$\{ (x, \dot{x}) : x, \dot{x} \in]0, \infty[\}$$

is invariant.

After regularization we obtain the Hamiltonian system

$$z' = \frac{1}{4} w, \quad w' = -2Ez - \partial_{\bar{z}} P, \quad E' = -\frac{\partial P}{\partial t}, \quad t' = |z|^2$$

on $\quad H := \frac{1}{8} |w|^2 - E|z|^2 - 1 - P(t, z, \bar{z}) = 0$

with $P(t, z, \bar{z}) = |z|^2 \, U(t, z^2, \bar{z}^2)$.

We observe that $\{ z = \bar{z}, \ w = \bar{w} \}$ is an invariant set. We will work on it.

Let us take the surfaces (contained in $H = 0$)

$$\Pi_+ = \{ z = 0, \ w = 2\sqrt{2} \}$$

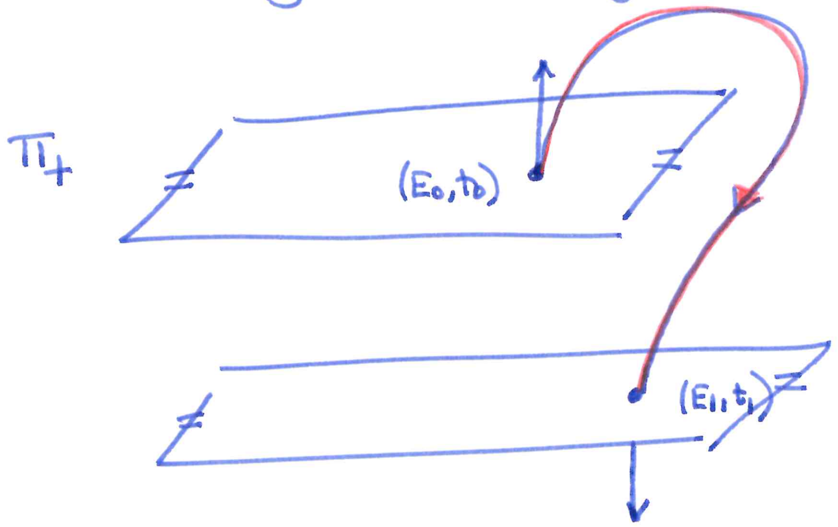
$$\Pi_- = \{ z = 0, \ w = -2\sqrt{2} \}$$

We observe that they are transversal to the flow ($z' = \frac{1}{4} w = \pm \frac{1}{2}\sqrt{2}$ on $\Pi_\pm$)

and we will consider the map

$$S : (E_0, t_0) \mapsto (E_1, t_1)$$

described by the following picture



We consider the domain of S,

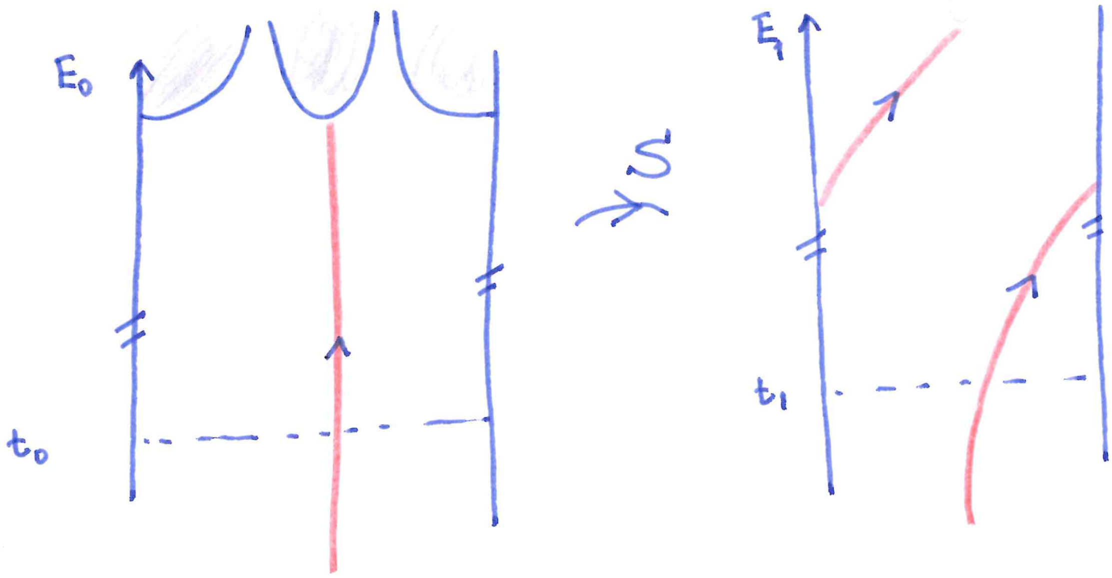
$$\mathcal{D} = \{ (E_0,t_0) : t_1 < +\infty \}$$

We need two properties of this map:

i) $\mathcal{D} = \{ (E_0,t_0) : E_0 < \psi(t_0) \}$ where $\psi : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}\cup\{\infty\}$

is lower semi-continuous

ii) $E_0 \mapsto t_1(E_0,t_0)$ is strictly increasing (twist property)



To prove these properties we use the monotonicity of the original flow. Let $z(s;E_0,t_0)$ be the first component of the regularized system such that $z(0;E_0,t_0)=0$, $w(0;E_0,t_0)=2\sqrt{2}$. Let $s_1>0$ be the first instant when Π_- is reached. We claim that $s_1 \leq s_1^*$

$$z(s;E_0,t_0) \leq z^*(s;E_0^*,t_0), \quad s\in[0,s_1]$$

if $E_0<E_0^*$.

[Proof: by an approximation argument we take $z_0=\varepsilon$, $\frac{1}{8}w_\varepsilon^2 - E_0\varepsilon^2 - 1 - \varepsilon^2 \mathcal{U}(t,\varepsilon^2)=0$, $w_\varepsilon>0$. We notice that w_ε is increasing with E_0. Then we can apply continuous dependence for the regularized system. By comparison with the autonomous system we observe that $s_1<\infty$ if E_0 is large and negative. Also, $z(s;E_0,t_0)$ remains uniform bounded. Then $t_1<\infty$. The rest is more or less automatic]

$t' = |z|^2$

Also,

iii) $t_1(E_0,t_0)-t_0 \to 0$ as $E_0 \searrow -\infty$ uniformly in t_0

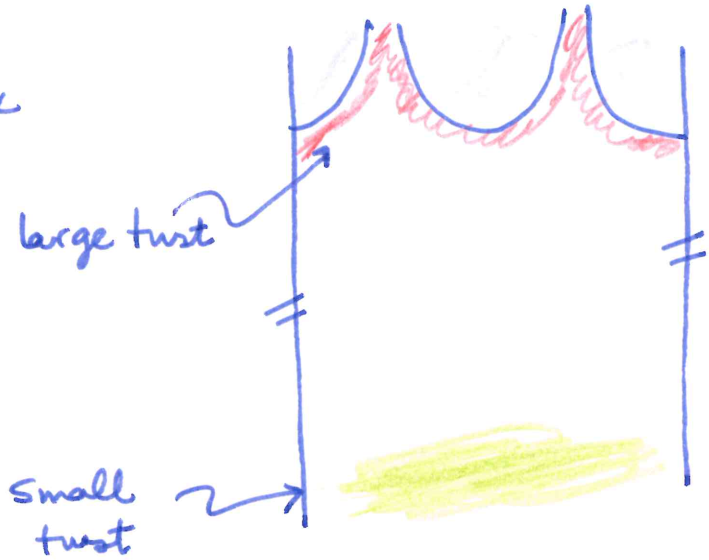
$t_1(E_0,t_0) \nearrow +\infty$ as $E_0 \nearrow \psi(t_0)$

[This follows from comparison with the autonomous system]

In order to apply the Poincaré-Birkhoff theorem (elementary version) we only need to check that

$\mathcal{S}$ is exact symplectic

We do this in two steps



1. $\mathcal{S}$ is symplectic : $dE_0 \wedge dt_0 = dE_1 \wedge dt_1$

The Hamiltonian flow is symplectic

$$\sum_{i=1}^{2} dz_{s,i} \wedge dw_{s,i} + dE_s \wedge dt_s = \sum_{1}^{2} dz_{0,i} \wedge dw_{0,i} + dE_0 \wedge dt_0$$

where (z_s, w_s, E_s, t_s) is the flow.

After considering the pull back of the inclusions $\Pi_\pm \longrightarrow \mathbb{C}^2 \times \mathbb{R} \times \mathbb{R}/T\mathbb{Z}$, we obtain the conclusion

(On $\Pi_\pm$, $z_s = 0$) Explain more or go to page 9

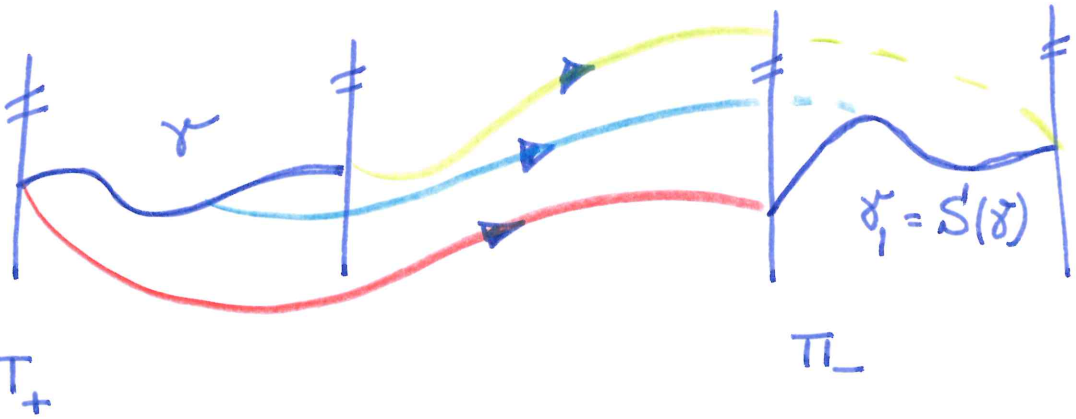
2. $\mathcal{S}$ is exact symplectic We must prove that

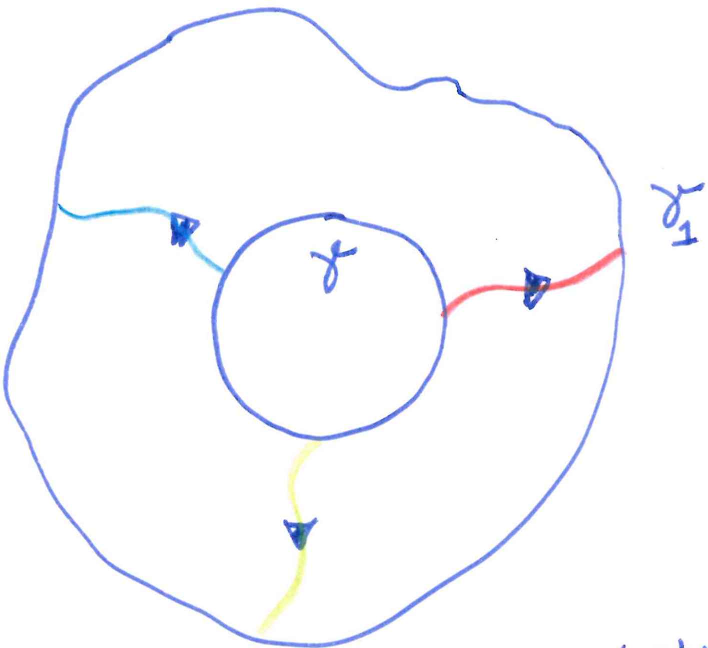
$$\int_\gamma (E_1\, dt_1 - E_0\, dt_0) = 0$$

for some closed smooth Jordan curve which is not contractible in $\mathbb{R} \times (\mathbb{R}/T\mathbb{Z})$

[see the Appendix]

We take such a curve γ and describe the situation with two intuitions





$$\gamma : \begin{cases} E = E(\lambda) \\ t = t(\lambda) \end{cases}, \lambda \in [0,1] \qquad \begin{gathered} (E'(\lambda), t'(\lambda)) \neq (0,0) \\ \forall \lambda \in [0,1] \\ t(1) = t(0) + T \end{gathered}$$

Let $\sigma(\lambda) > 0$ be the time of arrival at Π_-

$$z(s, \lambda) \neq 0 \text{ if } \lambda \in]0, \sigma(\lambda)[\,, \; z(\sigma(\lambda), \lambda) = 0$$

By transversality $\sigma(\lambda)$ is C^∞. Define the surface

$$\mathcal{S} = \left\{ \left(z(s,\lambda), w(s,\lambda), E(s,\lambda), \bar{t}(s,\lambda) \right) : s \in [0, \sigma(\lambda)], \; \lambda \in [0,1] \right\}$$

By uniqueness of the i.v.p.,

$$(s, \lambda) \in [0, \sigma(\lambda)] \times [0, 1[\; \mapsto (z, w, E, \bar{t}) \text{ is}$$

one-to-one.

Therefore $\mathcal{S}$ is homeomorphic to a closed annulus.

Let us now prove the $\mathcal{S}$ is a smooth surface with boundary: $\xi = \xi(s, \lambda)$ parameterization, $\xi = (z, w, E, \bar{t})$

We must prove that the vectors $\frac{\partial \xi}{\partial s}$ and $\frac{\partial \xi}{\partial \lambda}$ are linearly independent. Indeed they are solutions of the variational equation

$$\gamma' = J D^2 H(\xi(s; \lambda)) \gamma$$

and

$$\frac{\partial \xi}{\partial s}(0, \lambda) = J \nabla H(\xi(s, \lambda)) = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial \xi}{\partial \lambda}(0, \lambda) = \begin{pmatrix} 0 \\ 0 \\ E'(\lambda) \\ t'(\lambda) \end{pmatrix}$$

Since they are linearly independent at $s = 0$, the same can be said for each s

<u>Exercise</u> $x' = A(s) x$, $\phi_1(s), \phi_2(s)$ solutions

$\phi_1(0), \phi_2(0)$ linearly independent $\Rightarrow \phi_1(s), \phi_2(s)$ l. i. for each s

Once we know S is smooth, we can apply Stoke's theorem on S to the 1-form $\Sigma = i^*\alpha$ where

$$\alpha = ~~-\sum dw_i \wedge dz_i + E~~ \quad -\sum_1^2 w_i\, dz_i + E\, dt \quad \text{and}$$

$i : S \to \mathbb{C}^2 \times \mathbb{R} \times \mathbb{R}/_{T\mathbb{Z}}$. Then

$$\int_S d(i^*\alpha) = \int_{\partial S} i^*\alpha$$

Also, $i^*(d\alpha) = i^*\left(\sum_1^2 dz_i \wedge dw_i + dE \wedge dt\right)$

$$= \omega\left(\frac{\partial \xi}{\partial s}, \frac{\partial \xi}{\partial \lambda}\right) ds \wedge d\lambda$$

where ω is the symplectic form. This form is independent of s because $\frac{\partial \xi}{\partial s}$ and $\frac{\partial \xi}{\partial \lambda}$ are solutions of a linear Hamiltonian system.

From the initial conditions we know that

$\omega\left(\frac{\partial \xi}{\partial s}, \frac{\partial \xi}{\partial \lambda}\right)\Big|_{s=0} = 0$. In consequence, $d(i^*\alpha) = 0$

Therefore,

$$0 = \int_{\partial S} E\, dt = \int_{\gamma_1} E\, dt - \int_{\gamma} E\, dt$$

$$= \int_{\gamma} E_1\, dt_1 - \int_{\gamma} E\, dt$$

Appendix

Closed 1-forms on the cylinder

$(\bar{\theta}, z) \in \mathbb{T} \times \mathbb{R}, \quad \bar{\theta} = \theta + 2\pi \mathbb{Z}$

Let $\eta = A(\theta, z)\,d\theta + B(\theta, z)\,dz$ be a closed 1-form in the cylinder (A, B are 2π-periodic in θ, $d\eta = 0 \iff \frac{\partial A}{\partial z} = \frac{\partial B}{\partial \theta}$). Then the number

$$\lambda = \frac{1}{2\pi} \int_0^{2\pi} A(\theta, z)\,d\theta$$

is independent of z $\left(\frac{d\lambda}{dz} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial B}{\partial \theta}(\theta, z)\,d\theta = 0 \right)$

Let $W(\theta, z)$ be a primitive of η in $\mathbb{R}^2$ ($dW = \eta$ but W is not necessarily 2π-periodic).

$$W(\theta, z) = B(0, z) + \int_0^\theta A(\varphi, z)\,d\varphi,$$

$$W(\theta, z) = \underbrace{\tilde{W}(\theta, z)}_{2\pi\text{-p in } \theta} + \lambda\theta$$

η is exact in the cylinder $\iff \lambda = 0$

$\iff \int_\gamma \eta = 0$ for every non-contractible Jordan curve

$\iff$ " for some "

$$\int_\gamma \eta = \int_0^1 \frac{d}{dt} W(\theta(t), z(t))\,dt = W(\theta(0) + 2\pi, z(0)) - W(\theta(0), z(0))$$
$$= 2\pi\lambda$$

Symplectic and exact symplectic in the cylinder

$$M : (\theta, z) \mapsto (\theta_1, z_1)$$

Definitions
$$\begin{cases} M \text{ symplectic if } d\theta_1 \wedge dz_1 = d\theta \wedge dz \\ M \text{ exact symplectic if } z_1 d\theta_1 - z\, d\theta \text{ is exact} \end{cases}$$

M exact symplectic $\Longleftrightarrow$ M symplectic +

$$\int_\gamma (z_1 d\theta_1 - z\, d\theta) = 0$$

on some γ