Kepler problem $\ddot{x} = -\dfrac{x}{|x|^3}, \; x \in \mathbb{R}^d \setminus \{0\}$

$d = 1, 2, 3,$ $\quad |\cdot|$ Euclidean norm

<u>Goal</u> : Given $T > 0$, to find all T-periodic solutions

$x(t; x_0, v_0)$ solution $x(0) = x_0$, $\dot{x}(0) = v_0$

$$\mathcal{P}_T = \left\{ (x_0, v_0) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d \;/\; \begin{array}{l} x(\cdot\,; x_0, v_0) \text{ is} \\ T\text{-periodic} \end{array} \right\}$$

First integrals

$\vec{c} = x \wedge \dot{x}$ angular momentum

$h = \frac{1}{2}|\dot{x}|^2 - \frac{1}{|x|}$ energy

Classification of orbits

$\vec{c} \neq 0$ $h > 0$

hyperbolic

$h = 0$

parabolic

$h < 0$

elliptic

$\vec{c} = 0$

ejection

$h \geq 0$

ejection/collision

$h < 0$

$d=1$ $\quad \nexists$ periodic solutions $\quad (\Rightarrow \mathcal{P}_T = \emptyset)$

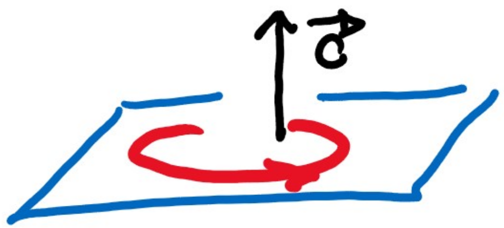
$\sup |x(t)| < \infty, \ \sup |\dot{x}(t)| = +\infty$

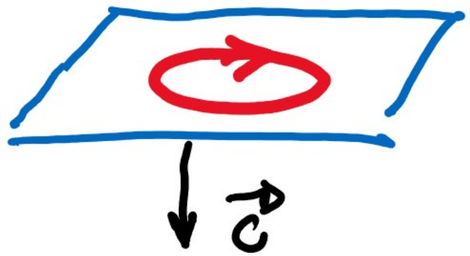
$x(t) \to 0$ si $t \nearrow \omega \implies |\dot{x}(t)| \to \infty$ si $t \nearrow \omega$

$$\left(\frac{1}{2} |\dot{x}|^2 = h + \frac{1}{|x(t)|} \right)$$

$d=2$ $\quad x(t)$ periodic $\iff \vec{c} \neq 0, \ h < 0$

$\vec{c} = c\,\vec{e}_3$



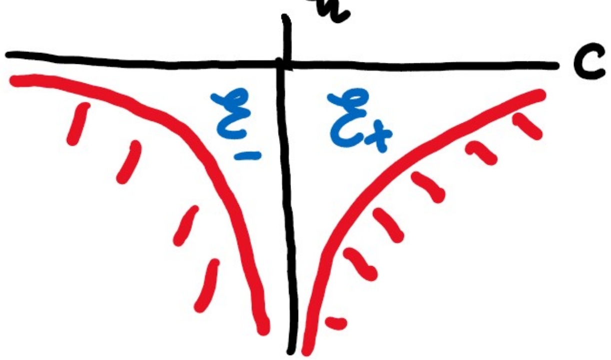


Elliptic region $\quad \mathcal{E} = \left\{ (x_0, v_0) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \,/\, \begin{array}{l} c(x_0,v_0) \neq 0 \\ h(x_0,v_0) < 0 \end{array} \right\}$

$\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$ $\qquad \mathcal{E}_+ : c > 0,\ h < 0, \quad \mathcal{E}_- : c < 0,\ h < 0$

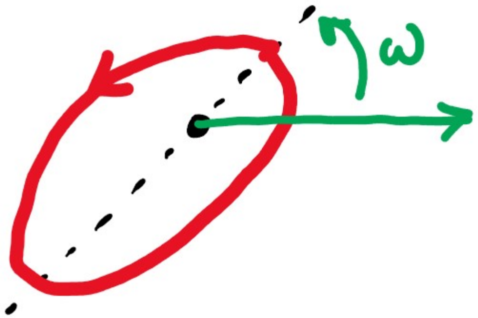
Kepler problem is reversible $[\, x(t) \text{ solution} \Rightarrow x(-t) \text{ solution} \,]$

$\mathcal{E}_- = S(\mathcal{E}_+), \quad S(x_0, v_0) = (x_0, -v_0)$



Exercise $\quad c^2 h \geq -\frac{1}{2}$

Elliptic motions



$2a$ major axis $\qquad a > 0$

e eccentricity $\qquad e \in [0,1[$

ω argument of eccentricity vector $\qquad \tilde{\omega} \in \mathbb{R}/_{2\pi\mathbb{Z}}$

$$x(t) = R[\omega] \begin{pmatrix} a(\cos u - e) \\ a\sqrt{1-e^2}\,\sin u \end{pmatrix}$$

$$u - e\sin u = \frac{1}{a^{3/2}}(t-\tau)$$

τ time of passage through perihelion

$R[\omega]$ rotation of angle ω ↺

Topology of $\mathcal{E}_+$

Runge-Lenz, Laplace...

$$h = -\frac{1}{2a}, \quad \vec{e} = e(\cos\omega, \sin\omega) = \dot{x} \wedge \vec{c} - \frac{x}{|x|}$$

$$u - e\sin u = \frac{1}{a^{3/2}}(t - \tau) \qquad \tau \equiv \tau + \frac{2\pi}{a^{3/2}}$$

$a^{3/2}\tau$ is an angle

$$(x_0, v_0) \in \mathcal{E}_+ \longmapsto (a, \vec{e}, a^{3/2}\tau) \in \,]0,\infty[\, \times D \times \mathbb{R}/2\pi\mathbb{Z}$$

diffeomorphism

$$\mathcal{E}_+ \approx \,]0,\infty[\, \times D \times \mathbb{R}/2\pi\mathbb{Z} \approx \mathbb{R}^3 \times S^1$$

$$D = \{ \vec{e} \in \mathbb{R}^2 \,/\, |\vec{e}| < 1 \}$$

Finding $\mathcal{P}_T$ $(d=2)$ $\qquad u - e\sin u = \frac{1}{a^{3/2}}(t-\tau)$

p minimal period of $u = u(t)$ $\qquad \boxed{p = 2\pi a^{3/2}}$ 3rd Kepler Law

$$x(t) \text{ T-periodic} \iff p = \frac{1}{n} T, \quad n = 1, 2, 3, \dots$$

$$\mathcal{P}_T = \bigcup_{n=1}^{\infty} \left(\mathcal{P}_{T,n}^{+} \cup \mathcal{P}_{T,n}^{-} \right)$$

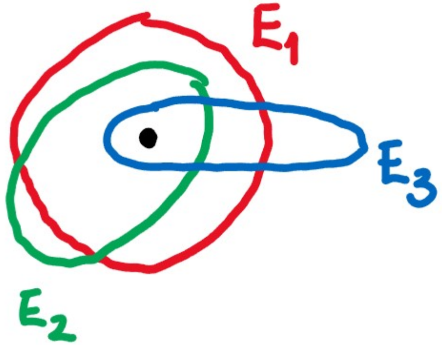
$$\mathcal{P}_{T,n}^{\pm} : \quad a = a_n := \left(\frac{T}{2\pi n} \right)^{2/3}$$

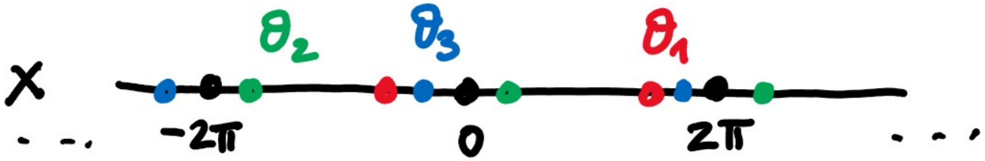
$$\mathcal{P}_{T,n}^{\pm} \approx D \times (\mathbb{R}/2\pi\mathbb{Z}) \approx \mathbb{R}^2 \times S^1$$ open solid torus

$$(E,\bar{\theta}) \equiv (x_0, v_0) \in \mathcal{P}^{+}_{T,n}$$

E ellipse, focus at O, $a = a_n$

$\tau = a^{2/3}\theta$ time of passage by perihelion, $\bar{\theta} \in \mathbb{R}/2\pi\mathbb{Z}$





$\mathcal{P}^{\pm}_{T,n}$ is unbounded $(x_0 \to 0 \iff |v_0| \to \infty)$

$\pi_1(\mathcal{P}^{\pm}_{T,n})$ bounded, $\pi_1(x_0, v_0) = x_0$

Finding φ_T $(d=3)$ Exercise

Hint: $SO(3)$ invariant action on the phase space