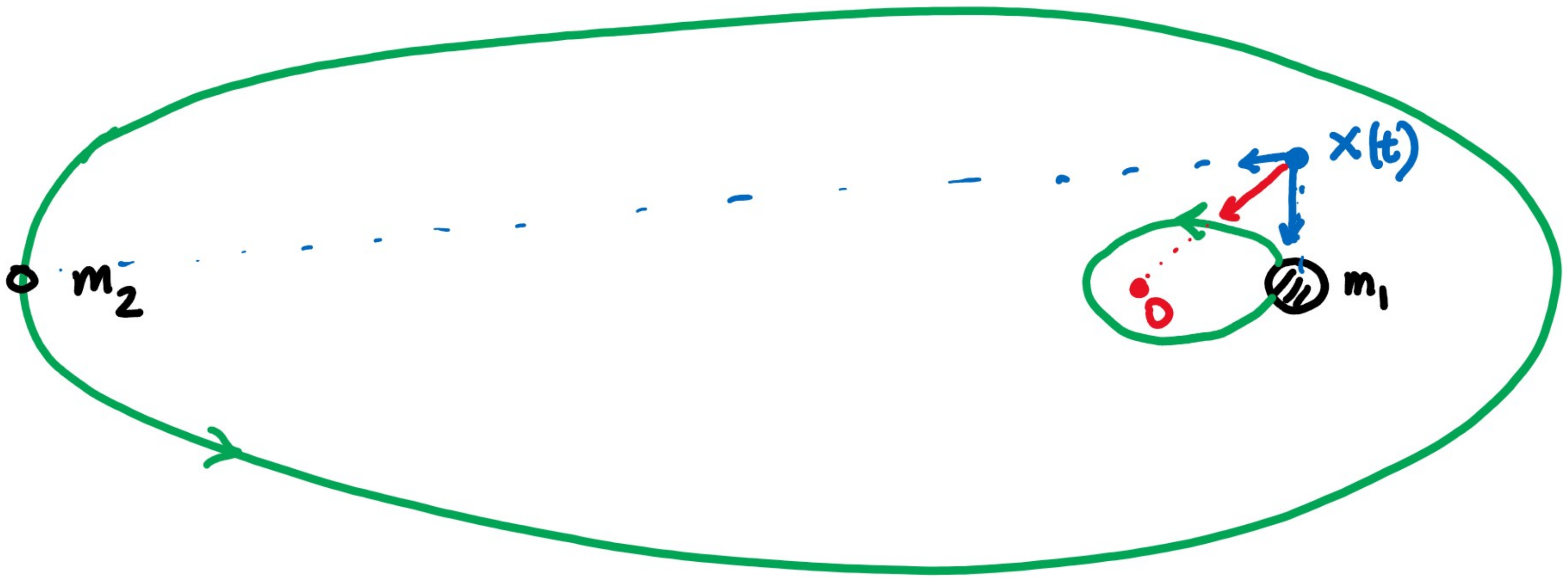
$$\ddot{x} = \underbrace{-\frac{x}{|x|^3}}_{\text{Kepler problem}} + \underbrace{\nabla_x U(t,x)}_{\text{perturbation}}, \quad x \in \mathbb{R}^d \setminus \{0\}, \quad d = 1, 2, 3$$

$m = 1, \quad G = 1$

$U : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \quad U(t+T, x) = U(t, x)$

T- periodic solutions

# Restricted elliptic 3 body problem



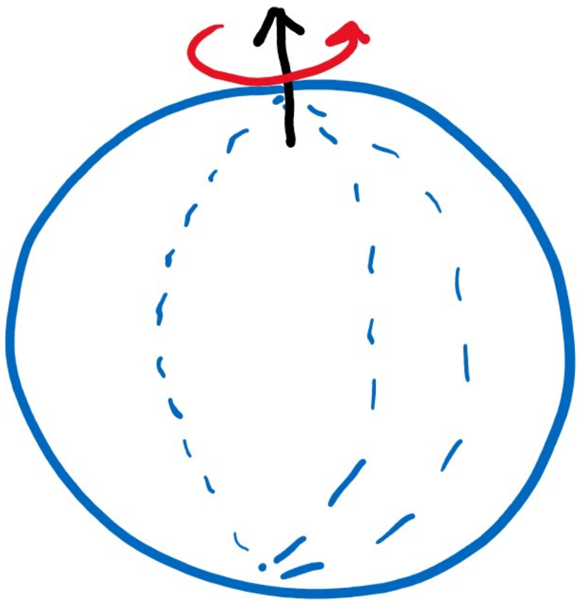

$m_1 \gg m_2$ $\quad T$ period of the primaries

Kepler problem at $x=0$ with $m_1 = 1$, $m_2$ small parameter

$U(t,x)$ smooth at $x=0$

$x(t)$ infinitesimal body close to $m_1$

# Satellite around the Earth (Fatou)



• $x(t)$ satellite

The Earth is not a perfect homogeneous sphere

$T = 24$ hours

$U$ has a singularity at $x=0$

## Celestial Mechanics

Perturbation theory, averaging, small parameters

Regularization (Levi-Civita, Kustaanheimo-Stiefel...)

## Nonlinear Analysis (80's)

Classical solutions: Lazer-Solimini ($d=1$)

Ambrosetti-Coti Zelati-Ekeland, Serra-Terracini

($d \geq 2$) symmetries on $V(t,x)$

Solutions with collisions (ad hoc definitions):

Rabinowitz, Tanaka, ...

# Generalized T-periodic solution

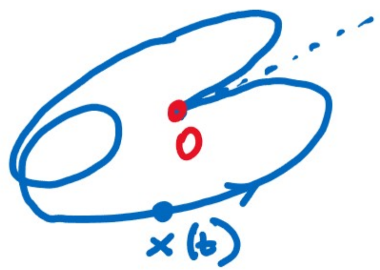
$x : \mathbb{R} \to \mathbb{R}^d$, $x = x(t)$ continuous, T-periodic

$Z_1 = \{ t \in \mathbb{R} : x(t) = 0 \}$ is a discrete set

$x \in C^\infty(\mathbb{R} \setminus Z_1, \mathbb{R}^d)$, $\ddot{x}(t) = -\dfrac{x(t)}{|x(t)|^3} + \nabla_x U(t, x(t))$, $t \notin Z_1$

$\forall t_* \in Z_1$

$$\exists \lim_{t \to t_*} \left\{ \frac{1}{2} |\dot{x}(t)|^2 - \frac{1}{|x(t)|} \right\}, \quad \exists \lim_{t \to t_*} \frac{x(t)}{|x(t)|}$$



## Recent literature

$d=1$ O. (2011), Zhao (2016), Rebelo-Simoes (2018)

$\exists\ \infty$ T-periodic generalized solutions

$d=2,3$ Boscaggin-O.-Zhao (2019)

small perturbation $\exists$ "many" T-p.g.s.

$d=2$ Boscaggin-Dambrosio-Papini (2020)

$\exists$ T-p.g.s.

$d=2,3$ Barutello-O-Verzini (2021)

$\exists\ \infty$ T-p.g.s. + Variational principles

$d\geqslant 2$ Zhao (2022)

$\exists\ \infty$ T-p g.s.

$d=1$ Frauenfelder-Weber

Variational principles

1. Kepler problem Preliminaries. Topology of the set of T-periodic solutions $(d=2)$

2. Levi-Civita regularization Classical and time-dependent cases. Hamiltonian structure

3. Generalized periodic solutions and regularization Equivalence

4. Regularized variational principles Periodic solutions $(d=2)$

5. Dimension $d=1$. The successor map is a twist map