Isoperimetric inequalities in unbounded convex bodies

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Abstract

We consider the problem of minimizing the relative perimeter under a volume constraint in an unbounded convex body $C \subset \mathbb{R}^n$, without assuming any further regularity on the boundary of C. Motivated by an example of an unbounded convex body with null isoperimetric profile, we introduce the concept of unbounded convex body with *uniform geometry*. We then provide a handy characterization of the uniform geometry property and, by exploiting the notion of *asymptotic cylinder* of C, we prove existence of isoperimetric regions in a generalized sense. By an approximation argument we show the strict concavity of the isoperimetric profile and, consequently, the connectedness of generalized isoperimetric regions. We also focus on the cases of small as well as of large volumes; in particular we show existence of isoperimetric regions with sufficiently large volumes, for special classes of unbounded convex bodies. We finally address some questions about isoperimetric rigidity and analyze the asymptotic behavior of the isoperimetric profile in connection with the notion of *isoperimetric dimension*.

List of symbols

$\mathbb{R}^n$ Euclidean n-space;

$|x|$ Euclidean norm of $x \in \mathbb{R}^n$;

$\langle x, y \rangle$ Euclidean inner product of $x, y \in \mathbb{R}^n$;

C convex body in Euclidean space;

$\operatorname{int} C$ interior points of C;

$B(x, r)$ open Euclidean ball of center $x \in \mathbb{R}^n$ and radius $r > 0$;

$\overline{B}(x, r)$ closed Euclidean ball of center $x \in \mathbb{R}^n$ and radius $r > 0$;

$B_C(x, r)$ relative open ball;

$\overline{B}_C(x, r)$ relative closed ball;

∂E topological boundary of E;

$\partial_C E$ relative boundary of E in the interior of C;

$\partial^* E$ reduced boundary of E;

$d(x, C)$ distance of x from C;

C_r tubular neighborhood of radius r of C

$\delta(C, C')$ Hausdorff distance of C and C';

$h_{p, \lambda}$ homothety of center p and ratio λ;

λC $h_{0, \lambda}(C)$;

C_p tangent cone to C at $p \in \partial C$;

C_∞ asymptotic cone of C;

$|E|$ Lebesgue measure, or volume, of E

ω_n volume of the unit ball in $\mathbb{R}^n$

$\mathcal{H}^s$ s-dimensional Hausdorff measure in $\mathbb{R}^n$

∂f subdifferential of f;

$P(E, A)$ perimeter of E in a Borel set A;

$P_C(E)$ same as $P(E, \operatorname{int} C)$ when C is a convex body;

I_C isoperimetric profile of C;

Y_C renormalized isoperimetric profile of C;

$\mathcal{K}(C)$ The set of asymptotic cylinders of C;

$\alpha(K)$ $\mathcal{H}^{n-1}(\partial B(x, 1) \cap K)$, measure of the solid angle of the cone $K \subset \mathbb{R}^n$ with vertex x;

$I_{C_{\min}}$ infimum of isoperimetric profiles of tangent cones to C;

$\mathcal{C}_0^n$ family of convex bodies in $\mathbb{R}^n$ that contain the origin and have non-degenerate asymptotic cone;

Γ_k^n family of convex bodies $C \in \mathcal{C}_0^n$ so that $C = \tilde{C} \times \mathbb{R}^{n-k}$, $0 \leqslant k \leqslant n$, up to an isometry, where $\tilde{C} \subset \mathbb{R}^k$ is a line-free convex body, or just the origin in the case $k = 0$;

$\mathcal{C}_{0,m}^n$ $\bigcup_{k=0}^m \Gamma_k^n$, $0 \leqslant m \leqslant n$;

CHAPTER 1

Introduction

Given a closed set $C \subset \mathbb{R}^n$ with non-empty interior, the *relative isoperimetric problem* on C looks for sets $E \subset C$ of given finite volume $|E|$ minimizing the *relative perimeter* $P_C(E)$ of E in the interior of C. When the boundaries of E and C are regular hypersurfaces, it is known that $\partial E \cap C$ is a constant mean curvature hypersurface and its closure meets ∂C orthogonally.

The *isoperimetric profile function* I_C assigns to each positive volume $0 < v < |C|$ the infimum of the relative perimeter of sets $F \subset C$ of volume v. An *isoperimetric region* is a set $E \subset C$ whose relative perimeter coincides with $I_C(|E|)$. The function I_C provides an optimal isoperimetric inequality on C since $P_C(F) \geqslant I_C(|F|)$ for any set $F \subset C$.

In this work, we consider the relative isoperimetric problem in *unbounded convex bodies*, i.e. unbounded closed convex sets with non-empty interior in Euclidean space, without assuming any further regularity of their boundaries. We focus on existence of isoperimetric regions, concavity properties of the isoperimetric profile, questions related to isoperimetric rigidity (i.e. given an isoperimetric inequality valid for a convex body C, does equality implies a geometric characterization of C?), as well as asymptotic isoperimetric inequalities in connection with the problem of determining the isoperimetric dimension of an unbounded convex body.

1.1. Historical background

Isoperimetric sets are at once a modern and classical topic: they arise in many fields, from physics of interfaces to optimal design of structures and shapes, and have fascinated scientists since antiquity. For instance, they appear in physical systems where surface tension is one of the main driving forces acting in the system. Surface tension was related to the mean curvature of a boundary interface by Young and Laplace in the equation $\Delta p = \sigma H$, where Δp is the difference of the internal and the external pressures, H is the mean curvature of the interface and σ is the surface tension.

The capillarity phenomenon is one of the most relevant examples where relative isoperimetric problems come into play. There one observes a liquid and a gas constrained by a solid support whose shape is determined by surface tension of the liquid-gas interface and by the wetting properties of the support, see Michael [46], Bostwick and Steen [13], and Finn [30]. Other examples related to the isoperimetric problem include: the Van der Waals-Cahn-Hilliard theory of phase transitions [37] (see in particular the Γ-convergence results by Modica [48] and Sternberg [81], as well as the construction of solutions of the Allen-Cahn equation by Pacard and Ritoré [58]); the shape of A/B block copolymers consisting of two macromolecules bonded together, which separate into distinct phases (Thomas et al. [85], Ohta-Kawasaki [57]).

Moreover, isoperimetric problems are relevant for their close, and deep, connection with functional inequalities of paramount importance in analysis, mathematical physics, and probability (like for instance the Sobolev-Poincaré inequality, the Faber-Krahn inequality and the Cheeger inequality, see [**18**], [**17**], [**84**], [**39**], [**19**]).

Central questions for the relative isoperimetric problem are the existence, regularity and geometric properties of isoperimetric regions, as well as the properties of the isoperimetric profile function.

For *bounded* convex bodies many results are known. When the boundary is *smooth*, the concavity of the isoperimetric profile and the connectedness of the reduced boundary of isoperimetric regions was proved by Sternberg and Zumbrun [**82**], while the concavity of the function $I_C^{n/(n-1)}$ were proved by Kuwert [**40**]. An essential ingredient in [**82**] and [**40**] is the use of the second variation formula to obtain estimates of the second derivative in a weak sense of the isoperimetric profile, an argument first used by Bavard and Pansu [**6**]. The behavior of the isoperimetric profile for small volumes was considered by Bérard and Meyer [**9**], and the behavior of isoperimetric regions for small volumes by Fall [**27**]. Connectedness of isoperimetric regions and of their complements was obtained by Ritoré and Vernadakis [**72**]. See also the works by Bayle [**7**], Bayle and Rosales [**8**] and Morgan and Johnson [**54**]. The results in all these papers make a strong use of the regularity of the boundary. In particular, as shown in [**82**] and [**40**], the $C^{2,\alpha}$ regularity of the boundary implies a strong regularity of the isoperimetric regions up to the boundary, except for a singular set of large Hausdorff codimension, that allows the authors to apply the classical first and second variation formulas for volume and perimeter. The convexity of the boundary then implies the concavity of the profile and the connectedness of the regular part of the free boundary.

Up to our knowledge, the only known results for non-smooth boundary are the ones by Bokowski and Sperner [**11**] on isoperimetric inequalities for the Minkowski content in Euclidean convex bodies; the isoperimetric inequality for convex cones by Lions and Pacella [**44**] using the Brunn-Minkowski inequality, with the characterization of isoperimetric regions by Figalli and Indrei [**29**]; the extension of Levy-Gromov inequality, [**35**, App. C], to arbitrary convex sets given by Morgan [**52**]; the extension of the concavity of the $\left(\frac{n}{n-1}\right)$ power of the isoperimetric profile to arbitrary convex bodies by E. Milman [**47**, § 6]. In his work on the isoperimetric profile for small volumes in the *boundary* of a polytope, Morgan mentions that his techniques can be adapted to handle the case of small volumes in a solid polytope, [**51**, Remark 3.11], without uniqueness, see Remark after Theorem 3.8 in [**51**]. Previous estimates on least perimeter in convex bodies have been obtained by Dyer and Frieze [**25**], Kannan, Lovász and Simonovits [**38**] and Bobkov [**10**]. Outside convex bodies in Euclidean space, isoperimetric inequalities have been established by Choe, Ghomi and Ritoré [**20**], [**21**], and, in the case of 3-dimensional Hadamard manifolds, by Choe and Ritoré [**22**].

In the case of unbounded convex bodies, several results on the isoperimetric profile of *cylindrically bounded* convex bodies have been obtained in [**70**] and for *conically bounded* ones in [**73**]. In convex cones, the results by Lions and Pacella [**44**] were recovered by Ritoré and Rosales [**69**] using stability techniques.

It is important to mention that existence of isoperimetric regions in non-compact spaces is not always guaranteed. For instance, complete planes of revolution with (negative) increasing Gauss curvature are known to have no isoperimetric regions

for any value of the two-dimensional volume as shown in [**64**, Theorem 2.16]. While general existence of solutions of variational problems in non-compact spaces is generally treated by means of concentration-compactness arguments ([**42**], [**43**]), the use of geometric arguments in the theory of isoperimetric inequalities to study the behavior of minimizing sequences is quite old and can be traced back to Fiala's paper [**28**], where it was shown that in a complete surface with non-negative Gauss curvature, a sequence of discs escaping to infinity have worse isoperimetric ratio that some compact regions of the same area. This argument was exploited in [**65**] to prove existence of isoperimetric regions in complete surfaces with non-negative Gauss curvature. An essential ingredient in this proof was the geometric description of the behavior of minimizing sequences given in [**64**, Lemma 1.8] and used in [**64**, Theorem 2.8] to show existence of isoperimetric regions in complete planes of revolution with non-decreasing Gauss curvature. Lemma 1.8 in [**64**] was extended to Riemannian manifolds in [**69**, Theorem 2.1] and used to prove existence of isoperimetric regions in convex cones with smooth boundary in the same paper. More modern existence results can be traced back to Almgren [**2**, Chapter VI], who proved existence of solutions of the isoperimetric problem in $\mathbb{R}^n$ for multiple volume constraints as a particular case of a more general theory for elliptic integrands. Morgan [**50**], based on Almgren's results, proved existence and regularity of clusters of prescribed volume in $\mathbb{R}^3$ minimizing area plus length of singular curves. The same author obtained in [**53**] existence of isoperimetric regions in a Riemannian manifold whose quotient by its isometry group is compact, see also [**50**, § 4.5]. Eichmair and Metzger showed in [**28**] that the leaves of the canonical foliation by stable constant mean curvature spheres in asymptotically flat manifolds asymptotic to a Schwarzschild space of positive mass are the only isoperimetric boundaries for the (large) volume they enclose, thus proving existence for large volumes. The same authors proved in [**28**] that an asymptotically flat Riemannian 3-manifold with non-negative scalar curvature contains a sequence of isoperimetric regions whose volumes diverge to infinity. Mondino and Nardulli [**49**] showed existence of isoperimetric regions of any volume in asymptotically flat manifolds with Ricci curvature uniformly bounded below.

As for the regularity of isoperimetric regions, the classical result on interior regularity was proved by Gonzalez, Massari and Tamanini [**34**], after the pioneering work by De Giorgi on regularity of perimeter-minimizing sets without a volume constraint. The boundary regularity for perimeter minimizing sets under a volume constraint inside a set with smooth boundary follows by the work of Grüter [**36**], see also [**82**].

As for the geometric characterization of isoperimetric regions in convex sets, isoperimetric solutions in a half-space are easily shown to be half-balls by reflecting an isoperimetric set with respect to the boundary hyperplane and applying the classical isoperimetric inequality in Euclidean space. The characterization of spherical caps as isoperimetric boundaries in balls was given by Bokowski and Sperner [**11**] (see also Burago and Zalgaller [**16**]) as an application of spherical symmetrization. Results on smooth second order minimizers of the perimeter in balls were given by Ros and Vergasta [**77**], see also [**76**]. In a slab, the non-empty intersection of two half-spaces determined by two parallel hyperplanes, isoperimetric solutions were classified by Vogel [**86**] and Athanassenas [**4**] in the 3-dimensional case. In both papers, the problem is reduced by symmetrization to axially symmetric sets. The only solutions

of the problem are half-balls and tubes around segments connecting orthogonally the two boundary hyperplanes. Both results were later extended by Pedrosa and Ritoré [**60**] to higher dimensional Euclidean spaces using stability techniques. The three-dimensional case also follows from the results in [**67**]. In Theorem 4.2 in [**62**] it was proved the existence of a constant $\varepsilon > 0$ such that isoperimetric regions in $[0,1] \times [0,\delta] \times \mathbb{R}$, with $\delta > \varepsilon$ are half-balls, tubes around closed segments connecting vertical walls and horizontal slabs, see also [**63**] and [**68**]. A similar result can be proved for cuboids. Ros proved estimates on the genus of isoperimetric surfaces in quotients of $\mathbb{R}^3$ by crystallographic groups [**75**]

Finally, we would like to remark that the relative isoperimetric problem is quite different from the minimization of the Euclidean perimeter under a volume constraint inside C, a problem considered by several authors, [**83**], [**78**], [**3**].

1.2. Outline of contents

This work has been organized into several chapters.

In Chapter 2 the notation used along the manuscript is fixed and basic definitions and facts about convex bodies and finite perimeter sets are presented. In particular, the notion of local convergence in Hausdorff distance is introduced at the beginning of § 2.1, followed by the proof of some useful properties of this notion of convergence. Given a convex body C, the concavity of the function

$$(x,r) \in C \times \mathbb{R}^+ \mapsto |B(x,r) \cap C|^{1/n}$$

is proved in Lemma 2.7 using the Brunn-Minkowski inequality.

In Chapter 3 we introduce and study some key concepts, in particular the notion of unbounded convex body of *uniform geometry* and its close relationship with the non-triviality of the isoperimetric profile of C. We first realized the importance of uniform geometry after the discovery of an unbounded convex body whose isoperimetric profile is identically zero, see Example 3.12. The definition is as follows: we say that an unbounded convex body C is of uniform geometry if the volume of a relative ball $B_C(x,r) = B(x,r) \cap C$ of a fixed radius $r > 0$ cannot be made arbitrarily small by letting $x \in C$ go off to infinity, see (3.3). We remark that this condition does not require any further regularity of ∂C. Proposition 3.14 shows a useful characterization of the uniform geometry assumption on an unbounded convex body and, in particular, its equivalence to the positivity of the isoperimetric profile I_C for any given volume. Moreover, uniform geometry is proved to be equivalent to the fact that any *asymptotic cylinder* of C is a convex body (with nonempty interior). The concept of asymptotic cylinder is introduced at the beginning of Section 3.1 as a local limit in Hausdorff distance of a sequence of translations $\{-x_j + C\}$, with $\{x_j\}_{j \in \mathbb{N}} \subset C$ being a divergent sequence of points. As mentioned above, this concept turns out to be crucial also in exploiting the connection between uniform geometry and non-degeneracy of the isoperimetric profile I_C (see again Proposition 3.14). Moreover it can be shown by a slight modification of Example 3.12 that the assumption of uniform geometry is stronger than simply requiring a uniform lower bound on the solid angles of the tangent cones to C. Various classes of unbounded convex bodies with uniform geometry are presented in Examples 3.28, 3.29, 3.30 and 3.31. A detailed account of asymptotic cylinders of a convex cone is done in Proposition 3.32. A consequence of this proposition is that whenever the boundary of a convex cone is C^1 out of a vertex, then all its asymptotic cylinders are either

half-spaces or the whole Euclidean space. Moreover in Proposition 3.33 we prove that unbounded convex bodies with non degenerate asymptotic cone having a C^1 boundary out of a given vertex present the same type of asymptotic cylinders. Then in Section 3.3 we exploit some further consequences of uniform geometry, that will be of use in Chapters 4 and 5 (in particular some uniform density and concentration estimates for sets of finite perimeter in C, see Lemma 3.22 and Corollaries 3.24 and 3.25, as well as the boundedness of isoperimetric regions, see Proposition 3.21).

In Chapter 4 we prove a generalized existence result, Theorem 4.6, in the spirit of Nardulli's existence theorem [55] for non-compact Riemannian manifolds (without boundary) satisfying a so-called *smoothly bounded geometry* property. Theorem 4.6 says that the isoperimetric profile $I_C(v)$ of an unbounded convex body C of uniform geometry is attained for any fixed $v > 0$ by a *generalized isoperimetric region* consisting of an array of sets $(E_0, \ldots, E_\ell)$, such that $E_0 \subset C = K_0$ and $E_i \subset K_i$ for $i = 1, \ldots, \ell$ and for suitable asymptotic cylinders $K_1, \ldots, K_\ell$, which satisfy $\sum_{i=0}^{\ell} |E_i| = v$ and $\sum_{i=0}^{\ell} P_{K_i}(E_i) = I_C(v)$. As Theorem 5.8 will later show, this result can be significantly improved as soon as the concavity of the isoperimetric profile is proved (which in turn requires a continuity result, Theorem 5.1, as an essential intermediate step). For the proof of Theorem 4.6 we closely follow the scheme adopted by Galli and Ritoré [31]. Essentially, we combine the uniform Poincaré inequality stated in Lemma 3.16, a doubling property on C proved in Lemma 2.6, an upper bound on $I_C(v)$ stated in Remark 3.10, and a well-known volume fixing deformation where the perimeter change is controlled by the volume change, up to a multiplicative constant (see Lemma 4.2).

In Chapter 5 we show the strict concavity of the isoperimetric profile of an unbounded convex body C of uniform geometry. To this aim we first approximate C by a sequence $\{C_i\}_{i \in \mathbb{N}}$ of unbounded convex bodies in the Hausdorff distance, so that all C_i and all their asymptotic cylinders are of class $C^{2,\alpha}$, see Lemma 5.3. Then we prove that the renormalized isoperimetric profiles $Y_{C_i} := I_{C_i}^{n/(n-1)}$ are concave, Lemma 5.5 (the function I_C is $(\frac{n}{n-1})$-concave in the terminology of Brascamp and Lieb [14]), and then we obtain that Y_C is also concave by passing to the limit. An implication of the concavity of Y_C is the strict subadditivity of the isoperimetric profile I_C, yielding a further refinement of the generalized existence Theorem 4.6, i.e., that $I_C(v)$ is attained by a single, connected isoperimetric set of volume v contained either in C or in some asymptotic cylinder of C (see Theorem 5.8). We stress that essential ingredients in the proof of Lemma 5.5 are the generalized existence of isoperimetric regions in Theorem 4.6 and the continuity of the isoperimetric profile of C given in Theorem 5.1. We also remark that such a continuity may fail in a non-compact space, as shown by the recent example by Nardulli and Pansu [56]. However, the existence of a Lipschitz continuous strictly convex exhaustion function on a manifold guarantees the continuity of the profile [66]. Conditions on the sectional curvature of a complete manifold, such as non-negativity or non-positivity, imply the existence of such an exhaustion function.

Chapter 6 contains several new isoperimetric inequalities and rigidity results for the equality cases. First it is proved in Theorem 6.3 that for a convex body C with non-degenerate asymptotic cone C_∞, the inequality $I_C \geqslant I_{C_\infty}$ holds and that the quotient $I_C(v)/I_{C_\infty}(v)$ tends to 1 as $v \to +\infty$. Existence of isoperimetric regions for large volumes, as well as convergence of rescalings to balls in C_∞, are also shown. Apart from its own interest, Theorem 6.3 is also used as a tool in Theorem 6.9 to

prove that

$$\lim_{v \to 0} \frac{I_C(v)}{I_{C_{\min}}(v)} = 1,$$

where $C_{\min}$ is a tangent cone to C or to an asymptotic cylinder of C with minimum solid angle. The existence of such a cone is established in Lemma 3.6. An interesting consequence of Theorem 6.9 is a new proof of the characterization of isoperimetric regions of small volume in polytopes or prisms given in Corollary 6.11: they are relative balls in C centered at vertices of C with the smallest solid angle. Finally, some rigidity results are given. In Theorem 6.14, given a convex body C with non-degenerate asymptotic cone C_∞, it is shown that, if the equality $I_C(v_0) = I_{C_\infty}(v_0)$ holds for some $v_0 > 0$ then C is isometric to C_∞. Then in Theorem 6.16 we prove that, whenever the equality holds for some $0 < v_0 < |C|$ in any of the inequalities

$$I_C \leqslant I_{C_{\min}}, \quad I_C \geqslant I_{\lambda C},$$

or for $v_0 < w$ in the inequality

$$\frac{I_C(v_0)}{v_0^{(n-1)/n}} \geqslant \frac{I_C(w)}{w^{(n-1)/n}},$$

then $I_C = I_{C_{\min}}$ in the interval $(0, v_0]$. Moreover, if K is either C or an asymptotic cylinder where the minimum of the solid angle is attained for some $p \in K$ then $K \cap B(p, r) = K_p \cap B(p, r)$ for any $r > 0$ such that $|B_K(p, r)| \leqslant v_0$, and in this case $B_K(p, r)$ is a generalized isoperimetric region. Then, Corollary 6.17 and Theorem 6.18 show that, if equality holds in any of the inequalities

$$I_C \leqslant I_H, \quad I_C \leqslant I_{\mathbb{R}^n \setminus C},$$

then C is a closed half-space or a slab. A consequence of Corollary 6.17 is a proof of existence of isoperimetric regions in convex bodies whose asymptotic cylinders are either closed half-spaces or the entire space $\mathbb{R}^n$, as it happens for convex bodies of revolution that are not cylindrically bounded, or for convex bodies with a non degenerate asymptotic cone that is of class C^1 outside a vertex. These results are proven in Theorem 6.21 and Corollary 6.22. We finally consider the case of cylindrically bounded convex bodies and we prove in Theorem 6.20 a generalization of the existence of isoperimetric regions of large volumes shown in [**70**] and a rigidity result in the same spirit of Theorems 6.16 and 6.18.

Finally, Chapter 7 is focused on the problem of estimating the isoperimetric dimension of C, which can be defined as the number $\alpha > 0$ such that there exist $0 < \lambda_1 < \lambda_2$ and $v_0 > 0$ with the property

$$(1.1) \qquad \lambda_1 v^{(\alpha-1)/\alpha} \leqslant I_C(v) \leqslant \lambda_2 v^{(\alpha-1)/\alpha} \qquad \forall\, v > v_0\,.$$

In general the isoperimetric dimension is not well-defined, moreover the crucial estimate in (1.1) is the first one, i.e., the lower bound on $I_C(v)$. Therefore it seems more convenient to formulate the problem in terms of an asymptotic isoperimetric inequality exploiting the growth rate of the volume of relative balls, as estimated by a non-decreasing function $V(r)$ depending only on the radius r. This is the approach followed by Coulhon and Saloff-Coste in [**24**], where isoperimetric-like inequalities are proved for graphs, groups, and manifolds in the large volume regime. Given a non-decreasing function $V(r)$, such that $V(r) \to +\infty$ as $v \to +\infty$ and $|B_C(x, r)| \geqslant V(r)$ for all $x \in C$, one introduces its *reciprocal function* ϕ_V as

$$\phi_V(v) = \inf\{r > 0 : V(r) \geqslant v\}\,.$$

In Theorem 7.4 the following result is proved: if C is an unbounded convex body of uniform geometry, then for the optimal choice $V(r) = b(r) = \inf_{x \in C} |B_C(x, r)|$, and denoting by $\phi(v)$ the reciprocal function of $b(r)$, one has

$$(1.2) \qquad n\frac{v}{\phi(v)} \geqslant I_C(v) \geqslant 24^{-n}\frac{v}{\phi(v)}\,.$$

Clearly, in order to derive from (1.2) an (asymptotic) estimate of $I_C(v)$ in terms of some explicit function of v one would need to compute $b(r)$ with sufficient precision, which is not an easy task for a generic convex body C. We thus focus on a special class of three-dimensional convex bodies of revolution: we let

$$C = \{(x, y) = (x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R} : \ y \geqslant f(|x|)\}\,,$$

where $f : [0, +\infty) \to [0, +\infty)$ is a convex function such that $f(0) = 0$ and $\lim_{s \to +\infty} s^{-1}f(s) = +\infty$. Any such C is an unbounded convex body whose asymptotic cone is a half-line, therefore one expects an isoperimetric dimension strictly smaller than 3. If one further assumes f strictly convex, of class $C^3(0, +\infty)$ and such that $f''' \leqslant 0$ on $(0, +\infty)$, then Theorem 7.10 proves that $b(r) = |B_C(0, r)|$, hence the explicit computation of $V(r) = b(r)$ becomes quite easy in this case. In particular, in Example 7.11 we compute the isoperimetric dimension of

$$C_a = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} : \ y \geqslant |x|^a\}$$

for $1 < a \leqslant 2$ and show that it is given by $\frac{a+2}{a}$, thus the isoperimetric dimension continuously changes from 3 to 2 as the parameter a goes from 1 to 2.

Several interesting problems remain open. A first one is the existence of isoperimetric regions in an unbounded convex body C. Although we have given conditions on particular classes of convex bodies ensuring existence for any volume, like the one in Theorem 6.21, it is not clear whether an example can be provided exhibiting an isoperimetric region in an asymptotic cylinder of C, but not in C. Another interesting open question is the range of possible isoperimetric dimensions for an unbounded convex body in $\mathbb{R}^n$. It would be reasonable to expect that the range is exactly $[1, n]$. However, the results in Chapter 7 only show that the range includes the set $[1, 2] \cup \{3\}$ in the three-dimensional case.

Convex bodies and finite perimeter sets

2.1. Convex bodies and local convergence in Hausdorff distance

In this paper, a *convex body* $C \subset \mathbb{R}^n$ is defined as a closed convex set with non-empty interior. The interior of C will be denoted by $\operatorname{int} C$. In the following we shall distinguish between *bounded* and *unbounded* convex bodies. Given $x \in C$ and $r > 0$, we define the *intrinsic ball* $B_C(x,r) = B(x,r) \cap C$, and the corresponding closed ball $\overline{B}_C(x,r) = C \cap \overline{B}(x,r)$. For $E \subset C$, the *relative boundary* of E in the interior of C is $\partial_C E = \partial E \cap \operatorname{int} C$.

Given a convex set C, and $r > 0$, we define $C_r = \{p \in \mathbb{R}^n : d(p,C) \leqslant r\}$. The set C_r is the tubular neighborhood of radius r of C and is a closed convex set. Given two convex sets C, C', we define their *Hausdorff distance* $\delta(C,C')$ by

$$\delta(C,C') = \inf\{r > 0 : C \subset (C')_r, C' \subset C_r\}.$$

We shall say that a sequence $\{C_i\}_{i \in \mathbb{N}}$ of convex sets converges to a convex set C in *Hausdorff distance* if $\lim_{i \to \infty} \delta(C_i, C) = 0$. Then we will say that a sequence $\{C_i\}_{i \in \mathbb{N}}$ of convex bodies *converges locally in Hausdorff distance* to a convex body C if, for every open ball B such that $C \cap B \neq \emptyset$, the sequence of bounded convex bodies $\{C_i \cap \overline{B}\}_{i \in \mathbb{N}}$ converges to $C \cap \overline{B}$ in Hausdorff distance. Whenever the convergence of C_i to C holds up to extracting a subsequence, we will say that C_i *subconverges* to C.

LEMMA 2.1. *Let $\{C_i\}_{i \in \mathbb{N}}$ be a sequence of convex bodies converging to a convex body C in Hausdorff distance. Then $\{C_i\}_{i \in \mathbb{N}}$ converges locally in Hausdorff distance to C.*

PROOF. Consider an open ball B such that $C \cap B \neq \emptyset$. To prove that $C_i \cap \overline{B}$ converges to $C \cap \overline{B}$ in Hausdorff distance, we shall use the Kuratowski criterion: every point in $C \cap \overline{B}$ is the limit of a sequence of points $x_i \in C_i \cap \overline{B}$, and the limit x of a subsequence $x_{i_j} \in C_{i_j} \cap \overline{B}$ belongs to $C \cap \overline{B}$, see [**79**, Theorem 1.8.7].

The second assertion is easy to prove: if $x \notin C$, then there exists some $\rho > 0$ such that $B(x,\rho) \cap (C + \rho \overline{B}) = \emptyset$ (just take $\rho > 0$ so that $\overline{B}(x,2\rho) \cap C = \emptyset$). This is a contradiction since $x_{i_j} \in B(x,\rho)$ and $C_{i_j} \subset C + \rho \overline{B}$ for j large enough.

To prove the first one, take $x \in C \cap \overline{B}$. Let x_i be the point in C_i at minimum distance from x. Such a point is unique by the convexity of C_i. It is clear that $\{x_i\}_{i \in \mathbb{N}}$ converges to x since $|x - x_i| \leqslant \delta(C, C_i) \to 0$. If $x \in B$, then $x_i \in C_i \cap B \subset C_i \cap \overline{B}$ for i large enough. If $x \in \partial \overline{B}$, then take a point $y \in C \cap B$ and a sequence $y_i \in C_i \cap \overline{B}$ converging to y. The segment $[x_i, y_i]$ is contained in C_i. If $x_i \in \overline{B}$ we take $z_i = x_i$. In case $x_i \notin \overline{B}$, we choose $z_i \in [x_i, y_i] \cap \partial B$ as in this case the intersection is nonempty (more precisely, the intersection contains exactly one point for i large enough). We claim that $z_i \to x$. Otherwise there is a subsequence $z_{i_j} \in \partial \overline{B}$ (we

may assume $z_{i_j} \neq x_{i_j}$) converging to some $q \in C \cap \partial \overline{B}$ different from x. But

$$\frac{q-y}{|q-y|} = \lim_{j \to \infty} \frac{z_{i_j} - y_{i_j}}{|z_{i_j} - y_{i_j}|} = \lim_{j \to \infty} \frac{x_{i_j} - y_{i_j}}{|x_{i_j} - y_{i_j}|} = \frac{x-y}{|x-y|}.$$

This implies that the points q and x lie in the same half-line leaving from y. Since $y \in B$ and $q, x \in \partial \overline{B}$, we get $q = x$, a contradiction. $\qquad \square$

REMARK 2.2. If condition $C \cap B \neq \emptyset$ is not imposed in the definition of local convergence in Hausdorff distance, Lemma 2.1 does not hold. Simply consider the sequence $C_i := \{x \in \mathbb{R}^n : x_n \geqslant 1 + 1/i\}$, converging in Hausdorff distance to $C := \{x \in \mathbb{R}^n : x_n \geqslant 1\}$. Then $C_i \cap \overline{B}(0,1) = \emptyset$ does not converge to $C \cap \overline{B}(0,1) = \{(0, \dots, 0, 1)\}$.

A condition guaranteeing local convergence in Hausdorff distance is the following

LEMMA 2.3. *Let* $\{C_i\}_{i \in \mathbb{N}}$ *be a sequence of convex bodies, and* C *a convex body. Assume that there exists* $p \in \mathbb{R}^n$ *and* $r_0 \geqslant 0$ *such that, for every* $r > r_0$, *the sequence* $C_i \cap \overline{B}(p,r)$ *converges to* $C \cap \overline{B}(p,r)$ *in Hausdorff distance. Then* C_i *converges locally in Hausdorff distance to* C.

PROOF. Let B be an open ball so that $C \cap B \neq \emptyset$. Choose $r > r_0$ so that $\overline{B} \subset \overline{B}(p,r)$. By hypothesis, $C_i \cap \overline{B}(p,r)$ converges to $C \cap \overline{B}(p,r)$ in Hausdorff distance. Lemma 2.1 implies that $C_i \cap \overline{B}(p,r) \cap \overline{B} = C_i \cap \overline{B}$ converges to $C \cap \overline{B}(p,r) \cap \overline{B} = C \cap \overline{B}$ in Hausdorff distance. $\qquad \square$

The following two lemmata will be used in Chapter 3.

LEMMA 2.4. *Let* F, G *be closed convex sets containing* 0, *then*

$$(2.1) \qquad \delta(F \cap \overline{B}(0,r), G \cap \overline{B}(0,r)) \leqslant \delta(F,G), \quad \text{for all } r > 0.$$

PROOF. The proof of (2.1) directly follows from the inequality

$$(2.2) \qquad \sup_{x \in F \cap \overline{B}(0,r)} d(x, G \cap \overline{B}(0,r)) \leqslant \delta(F,G),$$

up to interchanging the role of F and G. In order to prove (2.2), we fix $x \in F \cap \overline{B}(0,r)$ and denote by y the metric projection of x to G and by y_0 the metric projection of y to $\overline{B}(0,r)$. Then, we notice that $0 \in G$ implies $y_0 \in G \cap \overline{B}(0,r)$ by convexity. We thus find $|x - y_0| \leqslant |x - y| \leqslant \delta(F,G)$, which proves (2.2). $\qquad \square$

LEMMA 2.5. *Let* $A \subset \mathbb{R}^n$ *be a convex body with* $0 \in A$ *and let* $r > 0$ *and* $v \in \mathbb{R}^n$ *be such that* $|v| < r/2$. *Then*

$$(2.3) \qquad \delta\big(A \cap \overline{B}(0,r), (v+A) \cap \overline{B}(0,r)\big) \leqslant 2|v|.$$

PROOF. Let us start proving

$$(2.4) \qquad \sup_{x \in A \cap \overline{B}(0,r)} d(x, (v+A) \cap \overline{B}(0,r)) \leqslant 2|v|.$$

To this aim, given $x \in A \cap \overline{B}(0,r)$, denoting by x_v the metric projection of x to $\overline{B}(0,|v|)$, and setting $z = x - x_v$, we find that $z \in A \cap \overline{B}(0, r - |v|)$, hence $z + v \in A \cap \overline{B}(0,r)$, so that $|x - (z+v)| \leqslant |x_v| + |v| \leqslant 2|v|$, which proves (2.4). Then with a similar argument we show that

$$(2.5) \qquad \sup_{x \in (v+A) \cap \overline{B}(0,r)} d(x, A \cap \overline{B}(0,r)) \leqslant 2|v|.$$

Indeed, take $x \in (v + A) \cap \overline{B}(0, r)$, set $a = x - v$ and denote by a' the metric projection of a on $\overline{B}(0, r)$. It is then immediate to check that $a' \in A \cap \overline{B}(0, r)$ and that $|x - a'| \leqslant |x - a| + |a - a'| \leqslant 2|v|$, which gives (2.5) at once. Finally, by combining (2.4) and (2.5) we get (2.3) as wanted. $\square$

Let $p \in \mathbb{R}^n$ and denote by $h_{p,\lambda}$ the homothety of center p and ratio λ, defined as $h_{p,\lambda}(x) = p + \lambda(x - p)$. We define the *tangent cone C_p* of a convex body C at a given boundary point $p \in \partial C$ as the closure of the set

$$\bigcup_{\lambda > 0} h_{p,\lambda}(C).$$

Tangent cones of convex bodies have been widely considered in convex geometry under the name of supporting cones [**79**, § 2.2] or projection cones [**12**]. From the definition it follows easily that C_p is the smallest cone, with vertex p, that includes C.

We define the *asymptotic cone C_∞* of an unbounded convex body C by

$$(2.6) \qquad C_\infty = \bigcap_{\lambda > 0} \bigcup_{0 < \mu < \lambda} \mu C,$$

where $\mu C = \{\mu x : x \in C\}$ is the image of C under the homothety of center 0 and ratio μ. In other words, C_∞ is the "blow-down" of C. Moreover, it turns out that $\bigcap_{\lambda > 0} h_{p,\lambda}(C) = p + C_\infty$ for all $p \in C$ (we recall that $h_{p,\lambda}$ denotes the homothety of center p and ratio λ). Hence the shape of the asymptotic cone is independent of the chosen origin. When C is bounded the set C_∞ defined by (2.6) is $\{0\}$. Observe that λC converges, locally in Hausdorff sense, to the asymptotic cone C_∞ as $\lambda \to 0$ [**15**]. We shall say that the asymptotic cone is *non-degenerate* if $\dim C_\infty = \dim C$ (note that in general one has $\dim C_\infty \leqslant \dim C$). From the definition it follows easily that $p + C_\infty \subset C$ whenever $p \in C$, and that $p + C_\infty$ is the largest cone, with vertex p, included in C.

The *volume* of a measurable set $E \subset \mathbb{R}^n$ is defined as the Lebesgue measure of E and will be denoted by $|E|$. The r-dimensional Hausdorff measure in $\mathbb{R}^n$ will be denoted by $\mathcal{H}^r$. We recall the well-known identity $|E| = \mathcal{H}^n(E)$ for all measurable $E \subset \mathbb{R}^n$.

LEMMA 2.6. *Let $C \subset \mathbb{R}^n$ be a convex body, not necessarily unbounded. Given $r > 0$, $\lambda > 1$, we have*

$$(2.7) \qquad |B_C(x, \lambda r)| \leqslant \lambda^n |B_C(x, r)|,$$

for any $x \in C$. In particular, C is a doubling metric space with constant 2^{-n}.

PROOF. Since $\lambda > 1$, the convexity of C implies $B_C(x, \lambda r) \subset h_{x,\lambda}(B_C(x, r))$, where $h_{x,\lambda}$ is the homothety of center x and ratio λ. $\square$

Using Brunn-Minkowski Theorem we can prove the following concavity property for the power $1/n$ of the volume of relative balls in C.

LEMMA 2.7. *Let $C \subset \mathbb{R}^n$ be a convex body. Then the function $F : C \times \mathbb{R}^+ \to \mathbb{R}$ defined by $F(x, r) := |B_C(x, r)|^{1/n}$ is concave.*

PROOF. Take $(x, r), (y, s) \in C \times \mathbb{R}^+$, and $\lambda \in [0, 1]$. Assume $z \in \lambda \overline{B}_C(x, r) + (1 - \lambda)\overline{B}_C(y, s)$. Then there exist $z_1 \in \overline{B}_C(x, r)$, $z_2 \in \overline{B}_C(y, s)$ such that $z = \lambda z_1 + (1 - \lambda)z_2$. The point z belongs to C by the convexity of C. Moreover

$$|z - (\lambda x + (1 - \lambda)y| \leqslant \lambda|z_1 - x| + (1 - \lambda)|z_2 - y| \leqslant \lambda r + (1 - \lambda)s.$$

This implies $z \in \overline{B}_C(\lambda x + (1-\lambda)y, \lambda r + (1-\lambda)s)$ and so

$$(2.8) \qquad \lambda \overline{B}_C(x,r) + (1-\lambda)\overline{B}_C(y,s) \subset \overline{B}_C(\lambda x + (1-\lambda)y, \lambda r + (1-\lambda)s).$$

From (2.8) and the Brunn-Minkowski inequality [**79**, Thm. 6.11] we obtain

$$\begin{aligned}
F(\lambda x + (1-\lambda)y, \lambda r + (1-\lambda)s) &= |B_C(\lambda x + (1-\lambda)y, \lambda r + (1-\lambda)s)|^{1/n} \\
&\geqslant |\lambda B_C(x,r) + (1-\lambda)B_C(y,s)|^{1/n} \\
&\geqslant \lambda |B_C(x,r)|^{1/n} + (1-\lambda)|B_C(y,s)|^{1/n} \\
&= \lambda F(x,r) + (1-\lambda)F(y,s),
\end{aligned}$$

which proves the concavity of F. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Given a convex set $C \subset \mathbb{R}^n$, and $x \in \partial C$, we shall say that $u \in \mathbb{R}^n \setminus \{0\}$ is *an outer normal vector* to C at x if C is contained in the closed half-space $H_{x,u}^- := \{y \in \mathbb{R}^n : \langle y - x, u \rangle \leqslant 0\}$. The set $H_{x,u}^-$ is a supporting half-space of C at x and the set $\{y \in \mathbb{R}^n : \langle y - x, u \rangle = 0\}$ is a supporting hyperplane of C at x, see [**79**, § 1.3]. The *normal cone* of C at x, denoted by $N(C,x)$, is the union of $\{0\}$ and all outer normal vectors of C at x, see [**79**, § 2.2].

Given a convex function $f : \Omega \to \mathbb{R}$ defined on a convex domain $\Omega \subset \mathbb{R}^n$, and a point $x \in \Omega$, the *subdifferential* of f at x is the set

$$\partial f(x) = \{u \in \mathbb{R}^n : f(y) \geqslant f(x) + \langle u, y - x \rangle \text{ for all } y \in \Omega\},$$

see [**79**, p. 30] and also [**74**] and [**23**]. Given a convex function, its epigraph $\{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \geqslant f(x)\}$ is a convex set. A vector u belongs to $\partial f(x)$ if and only if the vector $(u, -1)$ is an outer normal vector to the epigraph at the point $(x, f(x))$.

For future reference we shall need a technical lemma about the Painlevé-Kuratowski convergence of the graphs of the subdifferentials of convex functions that locally converge to a convex function. This lemma, that we state and prove for the reader's convenience, is well-known for convex functions defined on Banach spaces (see [**5**]).

LEMMA 2.8. *Let $\{f_i\}_i$ be a sequence of convex functions defined on some fixed ball $B_R \subset \mathbb{R}^n$, that uniformly converge to a convex function f. Then for any $x \in B_{R/2}$ and $u \in \partial f(x)$ there exist sequences $\{x_i\}_i \subset B_R$ and $\{u_i\}_i$ such that $u_i \in \partial f_i(x_i)$ and $(x_i, u_i) \to (x, u)$ as $i \to \infty$.*

PROOF. We split the proof in two steps.

Step one. We show that for any $r, \varepsilon > 0$ there exists $i_{r,\varepsilon}$ such that for all $i \geqslant i_{r,\varepsilon}$ we can find $y_i \in B_r(x)$ and $u_i \in \partial f_i(y_i)$ such that $|u - u_i| \leqslant 2r\varepsilon$.

Up to a translation we may assume that $x = 0$. Let us fix $\varepsilon > 0$ and define $f^\varepsilon(y) = f(y) + \varepsilon|y|^2$ and, similarly, $f_i^\varepsilon(y) = f_i(y) + \varepsilon|y|^2$. By assumption we have $f(y) \geqslant f(0) + \langle u, y \rangle$, hence for all $y \neq 0$ we have

$$(2.9) \qquad\qquad f^\varepsilon(y) \geqslant f(0) + \langle u, y \rangle + \varepsilon|y|^2 > f^\varepsilon(0) + \langle u, y \rangle,$$

which means in particular that $u \in \partial f^\varepsilon(0)$. We now define the set

$$N_{i,\varepsilon} = \left\{y \in B_r : f_i^\varepsilon(y) \leqslant f^\varepsilon(0) + \langle u, y \rangle + \frac{r^2\varepsilon}{16}\right\}.$$

We notice that, since $f_i^\varepsilon \to f^\varepsilon$ uniformly on B_r as $i \to \infty$, the set $N_{i,\varepsilon}$ is not empty and contained in $B_{r/2}$ for i large enough. Indeed, up to taking i large enough we can assume that $\sup_{y \in B_r} |f_i(y) - f(y)| \leqslant \frac{\varepsilon r^2}{16}$, so that we obtain for any $y \in N_{i,\varepsilon}$

$$
\begin{aligned}
f(0) + \langle u, y \rangle + \varepsilon|y|^2 - \frac{\varepsilon r^2}{16} &\leqslant f(y) + \varepsilon|y|^2 - \frac{\varepsilon r^2}{16} \\
&\leqslant f_i(y) + \varepsilon|y|^2 = f_i^\varepsilon(y) \\
&\leqslant f(0) + \langle u, y \rangle + \frac{\varepsilon r^2}{16},
\end{aligned}
$$

whence the inequality $|y|^2 \leqslant \frac{r^2}{8} \leqslant \frac{r^2}{4}$. Now set

$$
t_{i,\varepsilon} = \sup\{ t \in \mathbb{R} : f_i^\varepsilon(y) \geqslant f^\varepsilon(0) + \langle u, y \rangle + t, \ \forall y \in B_r \}.
$$

We remark that $-\infty < t_{i,\varepsilon} < \frac{r^2 \varepsilon}{16}$, as $N_{i,\varepsilon}$ is nonempty and contained in $B_{r/2}$ for i large enough, moreover its value is obtained by minimizing the function $f_i^\varepsilon(y) - f^\varepsilon(0) - \langle u, y \rangle$ on B_r, so that there exists $y_i \in \overline{B}_{r/2}$ such that for all $y \in B_r$ we have

$$
f_i^\varepsilon(y) - f^\varepsilon(0) - \langle u, y \rangle \geqslant t_{i,\varepsilon} = f_i^\varepsilon(y_i) - f^\varepsilon(0) - \langle u, y_i \rangle,
$$

that is,

$$
f_i^\varepsilon(y) \geqslant f_i^\varepsilon(y_i) + \langle u, y - y_i \rangle,
$$

which shows that $u \in \partial f_i^\varepsilon(y_i)$. Now let us define $u_i = u - 2\varepsilon y_i$ and notice that

$$
\begin{aligned}
f_i(y) &\geqslant f_i(y_i) + 2\varepsilon \langle y, y_i - y \rangle + \langle u, y - y_i \rangle \\
&= f_i(y_i) + \langle u_i, y - y_i \rangle - 2\varepsilon|y - y_i|^2 \\
&= f_i(y_i) + \langle u_i, y - y_i \rangle + o(|y - y_i|) \qquad \text{as } y \to y_i.
\end{aligned}
$$

Replacing y by $y_i + t(y - y_i)$, for $t \in [0,1]$, dividing both sides of the inequality by t and taking limits when $t \downarrow 0$ we get

$$
(2.10) \qquad\qquad f_i'(y_i; y - y_i) \geqslant \langle u_i, y - y_i \rangle,
$$

where $f'(y_i; y - y_i)$ is the directional derivative of f at the point y_i in the direction of $y - y_i$. By the convexity of f_i, see Theorem 24.1 in [**74**], we have

$$
f_i(y) - f_i(y_i) \geqslant f_i'(y_i; y - y_i),
$$

which together with (2.10) implies that u_i is a subgradient of f_i at the point y_i. See also Proposition 2.2.7 in [**23**] for a variant of this argument. Finally, $|u_i - u| = 2\varepsilon|y_i| \leqslant 2r\varepsilon$, as wanted.

Step two. In order to complete the proof of the lemma, we argue by contradiction assuming the existence of $r_0, \varepsilon_0 > 0$ and of a subsequence $\{f_{i_k}\}_k$, such that for all $y \in B_{r_0}(x)$ and all $v \in \partial f_{i_k}(y)$ we have $|v - u| \geqslant \varepsilon_0$. By applying Step one to the subsequence, with parameters $r = r_0$ and $\varepsilon = \varepsilon_0/(4r_0)$, we immediately find a contradiction. $\qquad \square$

2.2. Finite perimeter sets and isoperimetric profile

Given a convex body C and a Borel set $E \subset C$, we define the *relative perimeter* of E in $\mathrm{int}(C)$ by

$$
P_C(E) = \sup \left\{ \int_E \mathrm{div}\, \xi \, d\mathcal{H}^n, \xi \in \Gamma_0(C), |\xi| \leqslant 1 \right\},
$$

where $\Gamma_0(C)$ is the set of smooth vector fields with compact support in $\text{int}(C)$. We shall say that E has *finite perimeter* in C if $P_C(E) < \infty$. When $C = \mathbb{R}^n$ we simply write $P(E)$ instead of $P_{\mathbb{R}^n}(E)$. Then we denote by either $P_A(E)$ or $P(E, A)$ the relative perimeter of a Borel set E inside an open (or, more generally, Borel) set A. When E is a set of locally finite perimeter, the set function $P(E, \cdot)$ is a Radon measure. The perimeter $P(E)$ agrees with the standard, $(n-1)$-dimensional Hausdorff measure of ∂E in the smooth or Lipschitz case. Moreover, the perimeter measure satisfies lower-semicontinuity and compactness properties, which are essential in proving existence of solutions to isoperimetric problems. Before stating these well-known results, we introduce the notion of L^1-convergence of Borel sets. We say that a sequence of Borel sets $\{E_h\}$ converges to a Borel set E_∞ in L^1 (or in L^1_{loc}) if the corresponding sequence of characteristic functions χ_{E_h} converges to χ_{E_∞} in L^1 (L^1_{loc}). Equivalently, we can define the symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ and say that E_h converges to E_∞ in L^1 whenever $|E_h \triangle E_\infty| \to 0$ as $h \to \infty$.

PROPOSITION 2.9. *Let A be a fixed open set and let $\{E_h\}_{h \in \mathbb{N}}$ be a sequence of Borel sets. Then*

  (i) *if $\sup_{h \in \mathbb{N}} P(E_h, A) < +\infty$ then there exists a subsequence E_{h_k} converging to a Borel set E_∞ in $L^1_{loc}(A)$, as $k \to \infty$;*
  (ii) *if E_h converges to E_∞ in $L^1_{loc}(A)$, then $\liminf_h P(E_h, A) \geqslant P(E_\infty, A)$.*

A deeper link between the perimeter measure and the $(n-1)$-dimensional Hausdorff measure is exploited through the notion of *reduced boundary* $\partial^* E$, which is a subset of the topological boundary ∂E where a weak inner normal and an approximate tangent plane are defined. A fundamental result, due to De Giorgi, states that $\partial^* E$ is countably $(n-1)$-rectifiable and one has $P(E, A) = \mathcal{H}^{n-1}(\partial^* E \cap A)$ for all A Borel. We refer the reader to Maggi's book [**45**] for an up-to-date reference on sets of finite perimeter.

For future reference we denote by ω_n the volume of the unit ball in $\mathbb{R}^n$, and notice that its perimeter (i.e., the $(n-1)$-dimensional Hausdorff measure of its boundary) is $n\omega_n$.

We define the *isoperimetric profile* of C by

$$I_C(v) = \inf \left\{ P_C(E) : E \subset C, |E| = v \right\}.$$

We shall say that $E \subset C$ is an *isoperimetric region* if $P_C(E) = I_C(|E|)$. We also recall in the next lemma a scaling property of the isoperimetric profile, which directly follows from the homogeneity of the perimeter with respect to homotheties.

LEMMA 2.10. *Let C be a convex body, and $\lambda > 0$. Then, for all $v > 0$,*

$$I_{\lambda C}(\lambda^n v) = \lambda^{n-1} I_C(v), \tag{2.11}$$

For future reference, we define the *renormalized isoperimetric profile* of $C \subset \mathbb{R}^n$ by

$$Y_C := I_C^{n/(n-1)}.$$

The known results on the regularity of isoperimetric regions are summarized in the following Theorem.

THEOREM 2.11 ([**34**], [**36**], [**82**, Thm. 2.1]). *Let $A \subset \mathbb{R}^n$ be an open set and $E \subset A$ an isoperimetric region. Then $\partial E \cap A = S_0 \cup S$, where $S_0 \cap S = \emptyset$ and*

(i) S is an embedded C^∞ hypersurface of constant mean curvature.

(ii) S_0 is closed and $\mathcal{H}^s(S_0) = 0$ for any $s > n - 7$.

Moreover, if the boundary of A is of class $C^{2,\alpha}$ then $\mathrm{cl}(\partial E \cap A) = S \cup S_0$, where

(iii) S is an embedded $C^{2,\alpha}$ hypersurface of constant mean curvature.

(iv) S_0 is closed and $\mathcal{H}^s(S_0) = 0$ for any $s > n - 7$.

(v) At points of $S \cap \partial A$, S meets ∂A orthogonally.

We remark that the existence of isoperimetric regions in a compact convex body C is a straightforward consequence of the lower semicontinuity of the relative perimeter and of the compactness properties of sequences of sets with equibounded (relative) perimeter. On the other hand, the existence of isoperimetric regions in an unbounded convex body C is a quite delicate issue, that will be discussed later on.

We now state a result that will be widely used throughout this work

THEOREM 2.12 (Volume adjustment). E_0 be a set with locally finite perimeter in an open set $\Omega \subset \mathbb{R}^n$, and let $B \subset \Omega$ be a bounded open set such that $P(E_0, B) > 0$. Then, there exist three constants $M, \overline{m}, d > 0$, possibly depending on E_0 and B, with the following property: for every set E with finite volume and locally finite perimeter in Ω, such that $|(E \triangle E_0) \cap B| < d$, and for every $-\overline{m} < t < \overline{m}$, there exists a set $F \subset \Omega$ such that $F = E$ outside B, satisfying

(i) $|F| = |E| + t$,

(ii) $|P(F, B) - P(E, B)| \leqslant M|t|$.

PROOF. As $P(E_0, B) > 0$, there exists a vector field X with compact support in B such that $\int_{E_0} \mathrm{div}\, X \, d\mathcal{H}^n > 0$. The one-parameter group of diffeomorphisms $\{\varphi_t\}_{t \in \mathbb{R}}$ associated with X then satisfies $\frac{d}{dt}\big|_{t=0} |\varphi_t(E_0)| = \int_{E_0} \mathrm{div}\, X \, d\mathcal{H}^n > 0$. This implies the existence of $\overline{m}, d > 0$ and of an open interval I around 0 such that the function $t \in I \mapsto (|\varphi_t(E)| - |E|) \in (-\overline{m}, \overline{m})$ is a C^1 diffeomorphism for all E such that $|(E \triangle E_0) \cap B| < d$. This proves (i).

To prove (ii), we consider the reduced boundary $\partial^* E$ of E. Take $m \in (-\overline{m}, \overline{m})$ and $F = \varphi_{t_m}(E)$ so that $|F| = m$. Then $P(F, \Omega) - P(E, \Omega) = P(F, B) - P(E, B)$. By the area formula

$$|P(F, B) - P(E, B)| = \left| \int_{\partial^* E \cap B} \left(\mathrm{Jac}(\varphi_{t_m}) - 1 \right) d\mathcal{H}^{n-1} \right| \leqslant C'|t_m|\, P(E, B),$$

where C' is a constant depending only on the vector field X. As $|t_m| \leqslant C''|m|$, where $C'' > 0$ is a constant only depending on X and E_0, we obtain (ii). $\qquad\square$

For future reference we recall in the following proposition a useful property related to minimizing sequences for the relative perimeter

PROPOSITION 2.13. Let $A \subset \mathbb{R}^n$ be an open set and let $v \in (0, |A|)$. Then there exists a minimizing sequence for the relative perimeter in A with volume v, consisting of bounded sets.

PROOF. The proof is accomplished as soon as we prove that, for any Borel set $E \subset A$, such that $|E| = v$ and $0 < P(E, A) < \infty$, and for any $\epsilon > 0$, we can find a bounded Borel set $G \subset A$ with $|G| = v$ and $P(G, A) \leqslant P(E, A) + \epsilon$. Since $P(E, A) > 0$ there exists an open ball $B \subset A$ such that $P(E, B) > 0$ and $P(E, \partial B) = 0$. By coarea formula we have

$$\int_0^{+\infty} \mathcal{H}^{n-1}(E \cap \partial B_R \cap A) \, dR = v < +\infty \,,$$

hence we can find R_ϵ large enough so that

$$B \subset\subset B_{R_\epsilon}, \quad m = |E \setminus B_{R_\epsilon}| < \min(\bar{m}, \epsilon/(2C))$$

and

$$\mathcal{H}^{n-1}(E \cap \partial B_{R_\epsilon} \cap A) < \epsilon/2 \,,$$

where $\bar{m}$ and C are as in Theorem 2.12. We can thus apply Theorem 2.12 with $E = E_0$ and find F such that $F = E$ outside B, $|F| = |E| + m$, and $P(F, B) \leqslant P(E, B) + \epsilon/2$. By choosing $G = F \cap B_{R_\epsilon}$ one has

$$P(G, A) = P(F, B) + P(E, B_{R_\epsilon} \setminus B) + \mathcal{H}^{n-1}(E \cap \partial B_R \cap A) \leqslant P(E, A) + \epsilon \,,$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We end this chapter with some properties of the isoperimetric profile of a convex cone and the definition of isoperimetric dimension.

Let now $K \subset \mathbb{R}^n$ be a closed solid cone with vertex p. Let

$$\alpha(K) = \mathcal{H}^{n-1}(\partial B(p, 1) \cap \operatorname{int}(K))$$

be the *solid angle* of K. If K is also convex then it is known that the intrinsic balls centered at the vertex are isoperimetric regions in K, [44], [69], and that they are the only ones [29] for general convex cones, without any regularity assumption on the boundary. The invariance of K by dilations centered at the vertex p yields

$$(2.12) \qquad P_K(B_K(p, r)) = \alpha(K)^{1/n} \, n^{(n-1)/n} \, |B_K(p, r)|^{(n-1)/n}$$

And if K is also convex, then by the above equality and the fact that intrinsic balls centered at p are isoperimetric, we obtain,

$$(2.13) \qquad I_K(v) = \alpha(K)^{1/n} \, n^{(n-1)/n} v^{(n-1)/n} = I_K(1) \, v^{(n-1)/n}.$$

Consequently the isoperimetric profile of a convex cone is completely determined by its solid angle.

Finally we define the isoperimetric dimension of a set in $\mathbb{R}^n$. We say that C satisfies an *m-dimensional isoperimetric inequality* if there are positive constants λ, v_0 such that

$$I_C(v) \geqslant \lambda v^{(m-1)/m} \quad \text{for all } v \geqslant v_0.$$

The *isoperimetric dimension of C* is the supremum of the real $m > 0$ such that C satisfies an m-dimensional isoperimetric inequality.

These definitions have sense in any metric measure space, see Gromov [35, Chap. 6.B, p. 322], Coulhon and Saloff-Coste [24], and Chavel [17]. If C satisfies an m-dimensional isoperimetric inequality, then, for any intrinsic ball $B_C(x, r)$, and recalling that $\mathcal{H}^{n-1}(\partial B(0, 1)) = n\omega_n$, we have

$$n\omega_n r^{n-1} \geqslant P(B_C(x, r)) \geqslant \lambda |B_C(x, r)|^{(m-1)/m} \,.$$

Hence the growth of the volume of intrinsic balls is uniformly controlled (i.e., it does not depend on the center of the ball) in terms of $r^{m(n-1)/(m-1)}$.

CHAPTER 3

Unbounded convex bodies of uniform geometry

In this chapter we collect various key definitions and results concerning the asymptotic properties of unbounded convex bodies. In particular we will show the crucial role played by property (3.3) in order to ensure the non-triviality of the isoperimetric profile function I_C of an unbounded convex body C.

3.1. Asymptotic cylinders

We start by introducing the notion of *asymptotic cylinder* of C, which requires some preliminaries. Following Schneider [**79**, § 1.4], we shall say that a subset $A \subset \mathbb{R}^n$ is *line-free* if it does not contain a line. According to Lemma 1.4.2 in [**79**], every closed convex set $A \subset \mathbb{R}^n$ can be written as the direct sum $B \oplus V$, where V is a linear subspace of $\mathbb{R}^n$ and B is a line-free closed convex set contained in a linear subspace orthogonal to V. Throughout this work, a *convex cylinder* will be a convex set containing a line, that is, a set of the form $B \oplus V$, with $V \neq \{0\}$.

The following two lemmas will play an important role in the sequel. Given $v \in \mathbb{R}^n \setminus \{0\}$, we shall denote by $L(v)$ the vector space generated by v. Thus $L(v) := \{\lambda v : \lambda \in \mathbb{R}\}$.

LEMMA 3.1. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body. Consider a bounded sequence $\{\lambda_i\}_{i \in \mathbb{N}}$ of positive real numbers and an unbounded sequence of points $x_i \in \lambda_i C$. Further assume that $-x_i + \lambda_i C \to K$ locally in Hausdorff distance. Then K is a convex cylinder.*

PROOF. Since $\{\lambda_i\}_{i \in \mathbb{N}}$ is bounded and $\{x_i\}_{i \in \mathbb{N}}$ is unbounded, the sequence $\{\lambda_i^{-1} x_i\}_{i \in \mathbb{N}}$ of points in C is unbounded. For any point $x_0 \in C$ we have

$$v_i := \frac{\lambda_i^{-1} x_i - x_0}{|\lambda_i^{-1} x_i - x_0|} \to v,$$

where the convergence is up to extraction of a subsequence, and v is a unit vector. The convexity of C then implies that the half-line $x_0 + \{\lambda v : \lambda \geqslant 0\}$ is contained in C for any $x_0 \in C$. Scaling by λ_i we get that $z_i + \{\lambda v : \lambda \geqslant 0\}$ is contained in $\lambda_i C$ for any $z_i \in \lambda_i C$. Taking $z_i = x_i$, the local convergence in Hausdorff distance of $-x_i + \lambda_i C$ to K implies that $\{\lambda v : \lambda \geqslant 0\}$ is contained in K.

On the other hand, the convexity of C implies that $[x_0, \lambda_i^{-1} x_i] \subset C$ and so $[\lambda_i x_0 - x_i, 0] \subset -x_i + \lambda_i C$. The segment $[\lambda_i x_0 - x_i, 0]$ is contained in the half-line $\{-\lambda v_i : \lambda \geqslant 0\}$. The lengths of these segments diverge, while the directions v_i converge to v, as $i \to \infty$. Hence the half-line $\{-\lambda v : \lambda \geqslant 0\}$ is contained in K because of the local convergence in Hausdorff distance of $-x_i + \lambda_i C$ to K.

Summing up, we conclude that $L(v) \subset K$. By [**79**, Lemma 1.4.2], K is a convex cylinder. □

LEMMA 3.2. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body, and let $\{x_i\}_{i\in\mathbb{N}} \subset C$ be a divergent sequence. Then $\{-x_i + C\}_{i\in\mathbb{N}}$ subconverges locally in Hausdorff distance to an unbounded closed convex set K. Moreover, K is a convex cylinder.*

PROOF. First we set $C_i = -x_i + C$. Then we observe that, for every $j \in \mathbb{N}$, the sequence of convex bodies $\{C_i \cap \overline{B}(0,j)\}_{i\in\mathbb{N}}$ is bounded in Hausdorff distance. By Blaschke's Selection Theorem [**79**, Thm. 1.8.4], there exists a convergent subsequence. By a diagonal argument and Lemma 2.3, we obtain that C_i subconverges locally in Hausdorff distance to a limit convex set K.

Now we apply Lemma 3.1 (taking $\lambda_i = 1$ for all $i \in \mathbb{N}$) to conclude that K is a convex cylinder. $\square$

Given an unbounded convex body C, we shall denote by $\mathcal{K}(C)$ the set of convex cylinders that can be obtained as local Hausdorff limits of sequences $\{C_i\}_{i\in\mathbb{N}}$, where $C_i = -x_i + C$, and $\{x_i\}_{i\in\mathbb{N}} \subset C$ is a divergent sequence. Any element of $\mathcal{K}(C)$ will be called an *asymptotic cylinder* of C.

REMARK 3.3. In a certain sense, it is enough to consider diverging sequences contained in the boundary ∂C to obtain the relevant asymptotic cylinders. Let $\{x_i\}_{i\in\mathbb{N}}$ be a diverging sequence of points in C, and assume that $-x_i + C$ locally converges in Hausdorff distance to an asymptotic cylinder K of C. Take a sequence of points $y_i \in \partial C$ such that $r_i := d(x_i, \partial C) = |x_i - y_i|$.

If $\limsup_{i\to\infty} r_i = +\infty$ then, after passing to a subsequence, we may assume that r_i is increasing and $\lim_{i\to\infty} r_i = +\infty$. As $\overline{B}(x_i, r_i) \subset C$ we have $\overline{B}(0, r_i) \subset -x_i + C$ and, since r_i has been taken increasing we have $\overline{B}(0, r_i) \subset -x_j + C$ for all $j \geqslant i$. Taking limits in j we get $\overline{B}(0, r_i) \subset K$ for all $i \in \mathbb{N}$. Since $\lim_{i\to\infty} r_i = +\infty$ we have $K = \mathbb{R}^n$.

If $\limsup_{i\to\infty} r_i < +\infty$ then the sequence $x_i - y_i$ is bounded. Passing to a subsequence we may assume that $y_i - x_i$ converges to z, and we get $K = z + K'$, where K' is the local Hausdorff limit of a subsequence of $-y_i + C$.

The following lemma is a refinement of Lemma 3.2 above.

LEMMA 3.4. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of unbounded convex bodies converging in Hausdorff distance to an unbounded convex body C.*

(i) *Let $\{x_i\}_{i\in\mathbb{N}}$ be a divergent sequence with $x_i \in C_i$ for $i \in \mathbb{N}$. Then there exists an asymptotic cylinder $K \in \mathcal{K}(C)$ such that $\{-x_i + C_i\}_{i\in\mathbb{N}}$ subconverges locally in Hausdorff distance to K.*

(ii) *Let $K_i \in \mathcal{K}(C_i)$ for $i \in \mathbb{N}$. Then there exists $K \in \mathcal{K}(C)$ such that $\{K_i\}_{i\in\mathbb{N}}$ subconverges locally in Hausdorff distance to K.*

PROOF. For each $i \in \mathbb{N}$, let z_i be the metric projection of x_i to C. Since $|x_i - z_i| \leqslant \delta(C_i, C) \to 0$, the sequence $\{z_i\}_{i\in\mathbb{N}}$ is also divergent. By Lemma 3.2, the sequence $\{-z_i + C\}_{i\in\mathbb{N}}$ subconverges locally in Hausdorff distance to some $K \in \mathcal{K}(C)$. Let us prove the first claim, i.e., that $\{-x_i + C_i\}_{i\in\mathbb{N}}$ subconverges locally in Hausdorff distance to K. By Lemma 2.3 it is enough to show that, for any $r > 0$, the sequence $\{(-x_i + C_i) \cap \overline{B}(0,r)\}_{i\in\mathbb{N}}$ subconverges in Hausdorff distance to $K \cap \overline{B}(0,r)$.

First, one can observe that the sequence $\{(-z_i + C) \cap \overline{B}(0,r)\}_{i\in\mathbb{N}}$ subconverges in Hausdorff distance to $K \cap \overline{B}(0,r)$, since for some natural $j > r$ we know that $\{(-z_i + C) \cap \overline{B}(0,j)\}_{i\in\mathbb{N}}$ subconverges in Hausdorff distance to $K \cap \overline{B}(0,j)$), and we can apply Lemma 2.1.

Second, the Hausdorff distance between $(-x_i + C_i) \cap \overline{B}(0, r)$ and $(-z_i + C) \cap \overline{B}(0, r)$ converges to 0 by Lemma 2.4 since

$$\delta((-x_i + C_i) \cap \overline{B}(0, r), (-z_i + C) \cap \overline{B}(0, r)) \leqslant \delta(-x_i + C_i, -z_i + C),$$

and the term on the right converges to 0 when $i \to \infty$ because of the Hausdorff convergence of C_i to C and the convergence of $x_i - z_i$ to 0. From these two observations, (i) follows.

Let us now prove (ii). For every $i \in \mathbb{N}$, Lemma 3.2 implies the existence of a divergent sequence $\{x_j^i\}_{j \in \mathbb{N}} \subset C_i$ so that $\{-x_j^i + C_i\}_{j \in \mathbb{N}}$ converges locally in Hausdorff distance to K_i. For every $i \in \mathbb{N}$, we choose increasing $j(i)$ so that the Hausdorff distance between $K_i \cap \overline{B}(0, i)$ and $(-x_{j(i)}^i + C_i) \cap \overline{B}(0, i)$ is less than $1/i$. If we fix some positive $r > 0$, Lemma 2.4 implies

$$\limsup_{i \to \infty} \delta(K_i \cap \overline{B}(0, r), (-x_{j(i)}^i + C_i) \cap \overline{B}(0, r))$$
$$\leqslant \lim_{i \to \infty} \delta(K_i \cap \overline{B}(0, i), (-x_{j(i)}^i + C_i) \cap \overline{B}(0, i)) = 0.$$

Let z_i be the metric projection of $x_{j(i)}^i$ onto C. By assumption we have $|z_i - x_{j(i)}^i| \leqslant \delta(C, C_i) \to 0$. Hence, for any $r > 0$

$$\lim_{i \to \infty} \delta((-x_{j(i)}^i + C_i) \cap \overline{B}(0, r), (-z_i + C) \cap \overline{B}(0, r)) = 0.$$

Finally, Lemma 3.2 implies that the sequence $\{-z_i + C\}_{i \in \mathbb{N}}$ subconverges locally in Hausdorff distance to some $K \in \mathcal{K}(C)$.

An application of the triangle inequality to the sets K_i, $-x_{j(i)}^i + C_i$, $-z_i + C$ and K, intersected with $\overline{B}(0, r)$, yields (ii). $\square$

The following result implies that certain translations of asymptotic cylinders are also asymptotic cylinders. This is not true in general, as shown by horizontal translations by large vectors of asymptotic cylinders of cylindrically bounded convex bodies.

LEMMA 3.5. *Let C be an unbounded convex body. If $K \in \mathcal{K}(C)$ and $z \in K$, then $-z + K \in \mathcal{K}(C)$.*

PROOF. As $K \in \mathcal{K}(C)$, there exists an unbounded sequence $\{x_i\}_{i \in \mathbb{N}}$ of points in C such that $-x_i + C$ converges locally in Hausdorff distance to K. Hence $-(x_i + z) + C$ converges locally in Hausdorff distance to $-z + K$.

Consequently, there exists a sequence $\{y_i\}_{i \in \mathbb{N}}$ such that $y_i \in -x_i + C$ and $y_i \to z$ as $i \to \infty$. Since x_i diverges and y_i converges, the sequence $x_i + y_i \in C$ is divergent, thus we let $C_i = -(x_i + y_i) + C$ and show that C_i converges locally in Hausdorff distance to $-z + K$, as $i \to \infty$. Let us fix $r > 0$ and notice that $0 \in C_i$ for all i, then setting $\delta_r(F, G) = \delta(F \cap \overline{B}(0, r), G \cap \overline{B}(0, r))$ we have

$$\delta_r(C_i, -z + K) \leqslant \delta_r(C_i, -(z - y_i) + C_i) + \delta_r(-z + (-x_i + C), -z + K)$$

so that by Lemmata 2.3 and 2.5 we find that the left-hand side of the above inequality is infinitesimal as $i \to \infty$, which concludes the proof. $\square$

The next Lemma generalizes [**72**, Lemma 6.1]

LEMMA 3.6. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body. Then there exists $K \in \{C\} \cup \mathcal{K}(C)$ and $p \in K$ such that*

$$\alpha(K_p) = \min\{\alpha(L_q) : L \in \{C\} \cup \mathcal{K}(C), q \in L\}.$$

PROOF. For every solid cone $V \subset \mathbb{R}^n$ with vertex p the co-area formula implies that $|V \cap B(p,1)| = n^{-1}\alpha(V)$. Our problem is then equivalent to minimizing $|L_p \cap B(p,1)|$ when $L \in \{C\} \cup \mathcal{K}(C)$ and $p \in L$.

Consider a sequence $K_i \in \{C\} \cup \mathcal{K}(C)$ and a sequence of points $p_i \in K_i$ such that
$$\lim_{i \to \infty} \alpha((K_i)_{p_i}) = \inf\{\alpha(L_q) : L \in \{C\} \cup \mathcal{K}(C), q \in L\}.$$
Let us see that $L_i := -p_i + K_i$ subconverges locally in Hausdorff distance either to a translation of C or to an asymptotic cylinder of C. Assume first that there is a subsequence so that $K_i = C$. If the corresponding subsequence p_i is bounded, it subconverges to some $p \in C$ and then L_i subconverges to $-p + C$. In case p_i is unbounded then L_i subconverges to an asymptotic cylinder L. So we can suppose that $K_i \neq C$ for all i. By Lemma 3.5, $L_i \in \mathcal{K}(C)$ for all i. By Lemma 3.4(ii), L_i subconverges to an asymptotic cylinder L.

Let us denote by L the local limit in Hausdorff distance of a subsequence of L_i. The set L is either $-p + C$, for some $p \in C$, or an asymptotic cylinder of C. Passing again to a subsequence, the tangent cone of L_i at the origin, $(L_i)_0$, locally converges in Hausdorff distance to a convex cone $L' \subset \mathbb{R}^n$ with vertex 0. Because of this convergence and the inclusion $L_i \subset (L_i)_0$, we get $L \subset L'$. Hence $L_0 \subset L'$ since L_0 is the smallest cone including L. By the continuity of the volume with respect to Hausdorff convergence we have
$$|L_0 \cap B(0,1)| \leqslant |L' \cap B(0,1)| \leqslant \lim_{i \to \infty} |(L_i)_0 \cap B(0,1)| = \lim_{i \to \infty} \alpha((K_i)_{p_i}).$$
Thus $\alpha(L_0)$ is a minimum for the solid angle. $\square$

REMARK 3.7. By (2.13) the isoperimetric profiles of tangent cones which are minima of the solid angle function coincide. The common profile will be denoted by $I_{C_{\min}}$. By the above proof we get that
$$I_{C_{\min}} \leqslant I_{K_p},$$
for every $K \in \{C\} \cup \mathcal{K}(C)$ and $p \in K$.

Now we proceed to build an example of unbounded convex body C for which the isoperimetric profile I_C is identically zero. The following result is essential for the construction.

PROPOSITION 3.8 ([**73**, Prop. 6.2]). *Let $C \subset \mathbb{R}^n$ be a convex body and $p \in \partial C$. Then every intrinsic ball in C centered at p has no more perimeter than an intrinsic ball of the same volume in C_p. Consequently*
$$(3.1) \qquad\qquad\qquad I_C(v) \leqslant I_{C_p}(v),$$
for all $0 < v < |C|$.

PROOF. Let $p \in \partial C$, and $0 < v < |C|$. Take $r > 0$ so that $|B_C(p,r)| = v$. Let L_p be the closed cone centered at p subtended by $\overline{B}(p,r) \cap C$. Then $L_p \subset C_p$ and, by convexity, $\overline{B}_{L_p}(p,r) \subset \overline{B}_C(p,r)$. By (2.12) and (2.13) we have
$$\begin{aligned}
I_C(v) \leqslant P_C(B_C(p,r)) &= P_{L_p}(B_{L_p}(p,r)) \\
&= n^{(n-1)/n} \, \alpha(L_p)^{1/n} \, |B_{L_p}(p,r)|^{(n-1)/n} \\
&\leqslant n^{(n-1)/n} \, \alpha(C_p)^{1/n} \, |B_{L_p}(p,r)|^{(n-1)/n} \\
&= I_{C_p}(|B_{L_p}(p,r)|) \leqslant I_{C_p}(v).
\end{aligned}$$

$\square$

REMARK 3.9. An obvious consequence of Proposition 3.8 is that

$$(3.2) \qquad I_C(v) \leqslant I_{C_{\min}}(v), \qquad \text{for all } 0 < v < |C|.$$

REMARK 3.10. A closed half-space $H \subset \mathbb{R}^n$ is a convex cone with the largest possible solid angle. Hence, for any convex body $C \subset \mathbb{R}^n$, we have

$$I_C(v) \leqslant I_H(v),$$

for all $0 < v < |C|$.

REMARK 3.11. Proposition 3.8 implies that $E \cap \partial C \neq \emptyset$ when $E \subset C$ is isoperimetric. Since in case $E \cap \partial C$ is empty, then E is an Euclidean ball. Moreover, as the isoperimetric profile of Euclidean space is strictly larger than that of the half-space, a set whose perimeter is close to the the value of the isoperimetric profile of C must touch the boundary of C.

3.2. Convex bodies of uniform geometry

The isoperimetric profile of an unbounded convex body, i.e. of an unbounded closed convex set with nonempty interior, can be identically zero, as shown by the following example. Note that some asymptotic cylinder has no interior points in this case. Such "bad" asymptotic cylinders are obtained by sequences x_i such that $\min(|x_i|, d(x_i, Q)) \to \infty$, where Q is the half-cylinder.
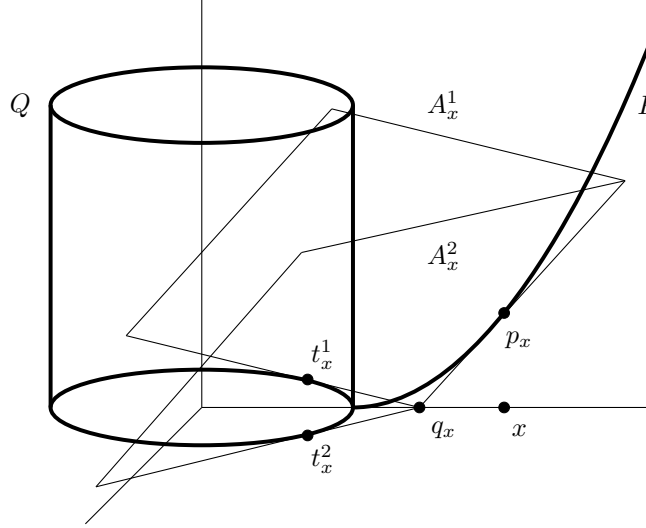

FIGURE 3.1. Example 3.12

EXAMPLE 3.12. We consider in $\mathbb{R}^3$ the half-cylinder

$$Q = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant 1, z \geqslant 0\},$$

and the parabolic curve

$$P = \{(x, y, z) \in \mathbb{R}^3 : z = (x-1)^2, y = 0, x \geqslant 1\}$$

(see Figure 3.1). Let C be the closed convex envelope of $Q \cup P$. For a given coordinate $x > 1$ the corresponding point on the parabola P is denoted by $p_x = (x, 0, (x-1)^2)$. The tangent line to P at p_x contained in the $y = 0$ plane intersects the x-axis at the point $q_x = ((1+x)/2, 0, 0)$, which is of course in the $z = 0$ plane and outside the unit disk $D = \{(x, y, z) : \ x^2 + y^2 \leqslant 1, \ z = 0\}$. Therefore, one can consider the two tangent lines from q_x to the boundary circle ∂D, meeting the circle at the two tangency points $t_x^1 = (2/(1+x), \sqrt{1 - 4/(1+x)^2}, 0)$ and $t_x^2 = (2/(1+x), -\sqrt{1 - 4/(1+x)^2}, 0)$. The (unique) affine planes A_x^1, A_x^2 containing, respectively, the points p_x, q_x, t_x^1 and p_x, q_x, t_x^2 are supporting planes for C. The corresponding half-spaces bounded by A_x^1, A_x^2 and containing C are denoted by H_x^1, H_x^2. It follows in particular that p_x is a boundary point of C. The solid angle of the tangent cone C_{p_x} of C at p_x is smaller than or equal to the solid angle of the wedge $W_x = H_x^1 \cap H_x^2$, which trivially goes to 0 as $x \to \infty$. By Proposition 3.8 we get for any $v > 0$

$$0 \leqslant I_C(v) \leqslant \inf_{x > 1} I_{C_{p_x}}(v) = 0.$$

Note that with some extra work it is possible to prove that $C_{p_x} = W_x$.

In the following proposition we give some conditions equivalent to the nontriviality of the isoperimetric profile.

We shall say that C is a *convex body of uniform geometry* if it is unbounded and for some $r_0 > 0$ there holds

$$(3.3) \qquad\qquad b(r_0) := \inf_{x \in C} |B_C(x, r_0)| > 0 \,.$$

REMARK 3.13. By Lemma 2.7 one immediately deduce that $b(r)^{1/n}$ is a concave function (indeed it is the infimum of a family of concave functions).

PROPOSITION 3.14. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body. The following assertions are equivalent:*

   (i) *for all $r > 0$, $\inf_{x \in C} |B_C(x, r)| = b(r) > 0$;*
   (ii) *C is a convex body of uniform geometry;*
   (iii) *all asymptotic cylinders of C are convex bodies;*
   (iv) *for all $v > 0$, $I_C(v) > 0$;*
   (v) *there exists $v_0 > 0$ such that $I_C(v_0) > 0$.*

Moreover, any of these conditions imply $\inf_{x \in \partial C} \alpha(C_x) = \alpha_0 > 0$.

PROOF. We shall first prove that (i), (ii) and (iii) are equivalent, then prove the implications (iv) $\Rightarrow$ (v), (v) $\Rightarrow$ (iii), and (ii) $\Rightarrow$ (iv).

The fact that (i) implies (ii) is obvious. Let us prove that (ii) implies (iii). Let $K \in \mathcal{K}(C)$ be an asymptotic cylinder obtained as the limit of the sequence $\{-x_i + C\}_{i \in \mathbb{N}}$ under local convergence in Hausdorff distance. Then

$$|K \cap B(0, r)| = \lim_{i \to \infty} |(-x_i + C) \cap B(0, r)| = \lim_{i \to \infty} |C \cap B(x_i, r)| \geqslant b(r) > 0.$$

This implies that K has interior points, i.e., that it is a convex body.

Assume now (iii) holds. Let us prove (i) reasoning by contradiction. Take $r > 0$. In case $b(r) = 0$, we take a sequence $\{x_i\}_{i \in \mathbb{N}}$ so that $\lim_{i \to \infty} |B_C(x_i, r)| = 0$. This sequence is divergent since otherwise we could extract a subsequence converging to some $x \in C$ with $|B_C(x_i, r)|$ subconverging to $|B_C(x, r)| > 0$. Consider the

asymptotic cylinder $K \in \mathcal{K}(C)$ obtained as the limit of a subsequence $\{-x_{i_j} + C\}_{j \in \mathbb{N}}$. Since

$$|K \cap B(0, r)| = \lim_{j \to \infty} |C \cap B(x_{i_j}, r)| = 0,$$

the cylinder K would have not interior points, contradicting assumption (iii). This completes the proof of the equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

The fact that (iv) implies (v) is obvious. Now, we show that (v) implies (iii). To this aim, we argue by contradiction, i.e., we assume the existence of an asymptotic cylinder K of C with empty interior. Therefore, there exists a sequence $x_j \in C$ going off to infinity, such that $C_j = -x_j + C$ converges locally Hausdorff to K, as $j \to \infty$. Now, for $\varepsilon > 0$ small enough, we construct a set $E_\varepsilon \subset C$ such that $|E_\varepsilon| = v_0$ but $P_C(E_\varepsilon) \leqslant \varepsilon$, thus implying $I_C(v_0) = 0$, a contradiction. To this aim we fix $z_0 \in C$ and define $r_\varepsilon = \frac{n v_0}{\varepsilon}$, then we assume ε small enough, so that $|B_C(z_0, r_\varepsilon)| > v_0$. Since $\lim_{j \to \infty} |B_C(x_j, r_\varepsilon)| = 0$, by continuity of the volume of intrinsic balls we can choose $z_\varepsilon \in C$ such that $E_\varepsilon = B_C(z_\varepsilon, r_\varepsilon)$ satisfies $|E_\varepsilon| = v_0$. By comparison with the cone $\mathcal{C}_\varepsilon$ over $\partial E_\varepsilon \cap \mathrm{int}(C)$ with vertex z_ε, taking into account $P_C(E_\varepsilon) = P_{\mathcal{C}_\varepsilon}(B_{\mathcal{C}_\varepsilon}(z_\varepsilon, r_\varepsilon))$ and $|E_\varepsilon| \geqslant |B_{\mathcal{C}_\varepsilon}(z_\varepsilon, r_\varepsilon)|$ we get

$$v_0 = |E_\varepsilon| \geqslant |B_{\mathcal{C}_\varepsilon}(z_\varepsilon, r_\varepsilon)| = \frac{r_\varepsilon}{n} P_{\mathcal{C}_\varepsilon}(B_{\mathcal{C}_\varepsilon}(z_\varepsilon, r_\varepsilon)) = \frac{r_\varepsilon}{n} P_C(E_\varepsilon),$$

whence

$$P_C(E_\varepsilon) \leqslant \frac{n}{r_\varepsilon} v_0 = \varepsilon.$$

This shows that $I_C(v_0) \leqslant \varepsilon$ for all $\varepsilon > 0$, thus $I_C(v_0) = 0$, i.e., a contradiction with (v).

Let us finally prove that (ii) implies (iv). Fix some volume $v > 0$ and consider a set $E \subset C$ of finite relative perimeter and volume $|E| = v$. We shall show that there exists a constant $\Lambda(C, v) > 0$, only depending on the geometry of C and v, such that $P_C(E) \geqslant \Lambda(C, v)$. This would imply $I_C(v) \geqslant \Lambda(C, v) > 0$, as desired. Recall first that (ii) implies the existence of a positive radius $r_0 > 0$ satisfying (3.3). An application of Fubini's Theorem, [**32**, Lemme 6.2] yields

$$(3.4) \qquad \int_C |E \cap B_C(y, r_0)| \, d\mathcal{H}^n(y) = \int_E |B_C(x, r_0)| \, d\mathcal{H}^n(x).$$

Since E has finite volume and $|B_C(x, r_0)| \leqslant \omega_n r_0^n$, the function $f(y) := |E \cap B_C(y, r_0)|$ is in $L^1(C)$. Hence, for any $\varepsilon > 0$ the set $f^{-1}([\varepsilon, +\infty)) = \{y \in C : |E \cap B_C(y, r_0)| \geqslant \varepsilon\}$ has finite volume and we get

$$(3.5) \qquad \inf_{x \in C} |E \cap B_C(x, r_0)| = 0.$$

Let us assume first that there exists $x_0 \in C$ such that

$$\frac{|E \cap B_C(x, r_0)|}{|B_C(x_0, r_0)|} \geqslant \frac{1}{2}.$$

By (3.5), (3.10) and a continuity argument we get a point $z_0 \in C$ so that

$$\frac{|E \cap B_C(z_0, r_0)|}{|B_C(z_0, r_0)|} = \frac{1}{3}.$$

By Lemma 3.16 below, we obtain

$$P_C(E) \geqslant P(E, B_C(z_0, r_0)) \geqslant M\left(\frac{|B_C(z_0, r_0)|}{3}\right)^{\frac{n-1}{n}}$$

(3.6)

$$\geqslant M\left(\frac{\ell_1 r_0^n}{3}\right)^{\frac{n-1}{n}} > 0.$$

Therefore the perimeter of $P_C(E)$ is bounded from below by a constant only depending on the geometry of C. Now assume that

$$\frac{|E \cap B_C(x, r_0)|}{|B_C(x, r_0)|} < \frac{1}{2}$$

holds for all $x \in C$. Let $\{B_C(x_i, r_0/2)\}_{i \in I}$ be a maximal family of disjoint intrinsic open balls centered at points of C. Then the family $\{B_C(x_i, r_0)\}_{i \in I}$ is an open covering of C. The overlapping of sets in this family can be estimated in the following way. For $x \in C$, define

$$A(x) = \{i \in I : x \in B_C(x_i, r_0)\}.$$

When $i \in A(x)$, it is immediate to check that $B_C(x_i, r_0/2) \subset B_C(x, 2r_0)$ (if $y \in B_C(x_i, r_0/2)$, then $d(y, x) \leqslant d(y, x_i) + d(x_i, x) < 2r_0$). Hence $\{B_C(x_i, r_0/2)\}_{i \in A(x)}$ is a disjoint family of balls contained in $B_C(x, 2r_0)$. Coupling the estimate (3.10) in Lemma 3.16 with (2.7), we get

$$\#A(x)\, \ell_1 \left(\frac{r_0}{2}\right)^n \leqslant \sum_{i \in A(x)} |B_C(x_i, r_0/2)| \leqslant |B_C(x, 2r_0)| \leqslant 2^n \omega_n\, r_0^n,$$

which implies the uniform bound $\#A(x) \leqslant K(C, n) := 4^n \omega_n \ell_1^{-1}$. Finally, the overlapping estimate and the relative isoperimetric inequality in $B_C(x_i, r_0)$ (see Theorem 4.11 in [**72**]) imply

$$K(C, n)\, P_C(E) \geqslant \sum_{i \in I} P_C(E, B_C(x_i, r_0))$$

(3.7)

$$\geqslant M \sum_{i \in I} |E \cap B_C(x_i, r_0)|^{\frac{n-1}{n}} \geqslant M\, |E|^{\frac{n-1}{n}}.$$

From (3.6) and (3.7) we obtain

(3.8) $$P_C(E) \geqslant \Lambda(C, v) := \min\left\{M\left(\frac{\ell_1 r_0^n}{3}\right)^{\frac{n-1}{n}}, \frac{M v^{\frac{n-1}{n}}}{K(C, n)}\right\} > 0.$$

This completes the proof of (ii)$\Rightarrow$(iv). We have thus proved the equivalence of the properties (i)–(ii)–(iii)–(iv)–(v).

Finally, assume (i) holds and consider a point $x \in \partial C$. Since

$$0 < b(r) \leqslant |B_C(x, r)| \leqslant |B_{C_x}(x, r)| = \int_0^r \alpha(C_x)\, s^n\, ds = \frac{\alpha(C_x)\, r^n}{n}$$

and owing to the concavity of $b(r)^{1/n}$, one finds that $\alpha(C_x)$ is estimated uniformly from below by the positive constant $\alpha_0 = n\, b(r_0)\, r_0^{-n}$. $\square$

REMARK 3.15. By a slight variant of Example 3.12, one sees that only assuming condition $\inf_{p \in \partial C} \alpha(C_p) > 0$ is not enough to ensure $I_C > 0$: indeed it is sufficient to intersect the unbounded convex body C constructed in Example 3.12 with the one-parameter family of half-spaces A_x having the point $m_x := p_x - (1, 0, 0)$ on their

boundary and inner normal vector $N_x = (2 - 2x, 0, 1)$, for $x > 1$. The resulting set $\hat{C}$ satisfies $\inf_{p \in \partial \hat{C}} \alpha(\hat{C}_p) > 0$ but still has a null isoperimetric profile. This is easy to check using (iii) in Proposition 3.14 since the asymptotic cylinder of $\hat{C}$ obtained as a limit of a convergent subsequence of $-m_x + \hat{C}$ (when x diverges) has empty interior as it is contained in a sequence of wedges with solid angles going to zero. Alternatively, one can take C as the convex hull of $Q \cup P^+ \cup P^-$, where

$$P^\pm = \{(x, y, z) : \ x \geqslant 1, \ z = (x-1)^2, \ y = \pm x^{-1}\} \,.$$

3.3. Density estimates and a concentration lemma

Next we show that whenever C is a convex body of uniform geometry, we can obtain uniform lower (and upper) density estimates for the volume of $B_C(x, r_0)$, as well as uniform relative isoperimetric inequalities on $B_C(x, r_0)$. In what follows, given a bounded set $A \subset \mathbb{R}^n$, we denote by $\mathrm{inr}(A)$ the *inradius* of A, i.e. the largest radius of an open ball contained in A.

LEMMA 3.16. *Let C be a convex body of uniform geometry and let r_0 be fixed. Then*

(i) $\displaystyle\inf_{x \in C} \mathrm{inr}(B_C(x, r_0)) \geqslant \frac{b(r_0)}{n\omega_n r_0^{n-1}}$;

(ii) *there exists $M > 0$ only depending on n, $r_0/b(r_0)$, such that for all $x \in C$, $0 < r \leqslant r_0$, and $0 < v < |B_C(x, r)|$, one finds*

$$(3.9) \qquad I_{B_C(x,r)}(v) \geqslant M \min\{v, |B_C(x, r)| - v\}^{(n-1)/n} ;$$

(iii) *there exist $\ell_1 > 0$ only depending on $n, r_0, b(r_0)$, such that for all $x \in C$ and $0 < r \leqslant r_0$ one has*

$$(3.10) \qquad \ell_1 r^n \leqslant |B_C(x, r)| \leqslant \omega_n r^n \,.$$

PROOF. To prove (i) we let D_t be the set of points in $D := \overline{B}_C(x, r_0)$ whose distance from ∂D is at least t. Then D_t is convex and nonempty for any $t \in [0, \mathrm{inr}(D)]$, while it is empty as soon as $t > \mathrm{inr}(D)$. The coarea formula applied to the distance function from ∂D yields

$$(3.11) \qquad b(r_0) \leqslant |D| = \int_0^{\mathrm{inr}(D)} P(D_t) \, dt \leq P(D) \, \mathrm{inr}(D) \leqslant P(\overline{B}(0, r_0)) \, \mathrm{inr}(D),$$

since $D \subset \overline{B}(0, r_0)$ implies $P(D) \leqslant P(B(0, r_0)) = n\omega_n r_0^{n-1}$. Therefore we find

$$\mathrm{inr}(B_C(x, r_0)) \geqslant \frac{b(r_0)}{n\omega_n r_0^{n-1}} \,,$$

thus proving (i).

In order to prove (ii) we shall use Theorem 4.11 in [72]: if $K \subset \mathbb{R}^n$ is a bounded convex body, $x, y \in K$, and $0 < \rho_1 < \rho_2$ satisfy $\overline{B}(y, \rho_1) \subset K \subset \overline{B}(x, \rho_2)$, then there exists a constant $M > 0$ given as a explicit function of n and ρ_2/ρ_1 such that

$$I_K(v) \geqslant M \min\{v, |K| - v\}^{(n-1)/n},$$

for all $0 \leqslant v \leqslant |K|$. The proof of this result makes use of the bilipschitz map $f : K \to \overline{B}(y, \rho_2)$ defined in [72, (3.9)] (with $r = \rho_1/2$) and the estimates on the lipschitz constants in Corollary 3.9 of [72], that depend on ρ_2/ρ_1. The dependence

of the constant M on the dimension n of the ambient Euclidean space is due to the dependence of M on the optimal constant M_0 in the isoperimetric inequality

$$I_{B(y,\rho_2)}(v) \geqslant M_0 \min\{v, |B(y,\rho_2)| - v\}^{(n-1)/n}, \quad v \in (0, |B(y,\rho_2)|).$$

The constant M_0 is invariant by translations and dilations in $\mathbb{R}^n$ and hence valid for any closed ball.

Using this result, we choose $\rho_1 = b(r_0)$, $K = \overline{B}_C(x,r_0)$, $\rho_2 = r_0$ so that the constant $M > 0$ in the inequality

$$I_{B_C(x,r_0)}(v) \geqslant M \min\{v, |B_C(x,r_0)| - v\}^{(n-1)/n}, \quad v \in (0, |B_C(x,r_0)|,$$

is given explicitly as a function of n and $r_0/b(r_0)$ for any $x \in C$. Fix now some $0 < r \leqslant r_0$ and some $x \in C$, and take $y \in \overline{B}_C(x,r_0)$ such that $\overline{B}(y, b(r_0)) \subset \overline{B}_C(x,r_0)$. Take $\lambda \in (0,1]$ such that $r = \lambda r_0$. Denoting by $h_{x,\lambda}$ the homothety of center x and ratio λ we have

$$\overline{B}(h_{x,\lambda}(y), \lambda b(r_0)) = h_{x,\lambda}(\overline{B}(y, b(r_0))) \subset h_{x,\lambda}(\overline{B}_C(x,r_0)) \subset \overline{B}_C(x,r),$$

the latter inclusion following from the concavity of C. We conclude that a relative isoperimetric inequality holds in $B_C(x,r)$ with a constant given explicitly as a function of n and of $r/\lambda b(r_0) = \lambda r_0/\lambda b(r_0) = r_0/b(r_0)$. This means that $M > 0$ can be taken uniformly for any $x \in C$ and $r \in (0, r_0]$.

We now prove (iii). The upper bound in (3.10) is immediate as $|B_C(x,r)| \leqslant |B(x,r)| = \omega_n r^n$. Then setting $\lambda = r/r_0$ and $\delta = \mathrm{inr}(B_C(x,r_0))$ we have

$$\begin{aligned}
|B(x,r) \cap C| = |B(x, \lambda r_0) \cap C| &= |B(0, \lambda r_0) \cap (-x + C)| \\
&\geqslant |B(0, \lambda r_0) \cap \lambda(-x + C)| = \lambda^n |B_C(x,r_0)| \\
&\geqslant \omega_n(\delta\lambda)^n = \ell_1 r^n,
\end{aligned}$$

where $\ell_1 = \omega_n(\delta/r_0)^n$ only depends on $n, r_0, b(r_0)$. This completes the proofs of (iii) and of the lemma. $\qquad\square$

An immediate consequence of Lemma 3.16 and the argument leading to equation (3.8) is the following corollary:

COROLLARY 3.17. *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry. Then there exist v_0, $c_0 > 0$, depending only on n, on the Ahlfors constant ℓ_1 in (3.10), and on the Poincaré inequality (3.9), such that*

(3.12) $$I_C(v) \geqslant c_0 \, v^{(n-1)/n} \quad \textit{for any} \quad v \leqslant v_0.$$

REMARK 3.18. An alternative proof of (3.11) could be given using Steinhagen's Theorem [80]. The width w of $B_C(x,r_0)$ satisfies

$$w \leqslant A_n \, \mathrm{inr}(B_C(x,r_0)),$$

where $A_n > 0$ is a constant only depending on n. On the other hand,

$$|B_C(x,r_0)| \leqslant \omega_n w r_0^{n-1},$$

where $\omega_n > 0$ is the $\mathcal{H}^n$-measure of the n-dimensional unit disc. Hence we obtain from (3.3)

$$\mathrm{inr}(B_C(x,r_0)) \geqslant (A_n B_n)^{-1} r_0^{1-n} b(r_0) > 0.$$

REMARK 3.19. Assume that $\{C_j\}_{j\in\mathbb{N}}$ converge in (global) Hausdorff distance to an unbounded convex body C as $j \to \infty$. If (3.3) holds for C, then, for $j \in \mathbb{N}$ large enough, one can show that C_j satisfies the thesis of Lemma 3.16 with constants M, ℓ_1 that do not depend on j. Viceversa, if $\{C_j\}_{j\in\mathbb{N}}$ is a sequence of convex bodies satisfying (3.3) uniformly on $j \in \mathbb{N}$, and converging locally in Hausdorff distance to a convex body C, then C necessarily satisfies the thesis of Lemma 3.16 with constants M, ℓ_1 that only depend on n, r_0 and $b(r_0)$. This follows from the 1-Lipschitz continuity of the inradius as a function defined on compact convex bodies endowed with the Hausdorff distance, as shown in Lemma 3.20 below.

Now we proceed to prove that the inradius of a bounded convex body is a 1-Lipschitz function with respect to the Hausdorff distance.

Let $C \subset \mathbb{R}^n$ be a bounded convex body. For any $t > 0$, we define the *inner parallel* at distance t by

$$C_{-t} = \{p \in C : d(p, \partial C) \geqslant t\}.$$

It is well-known that C_{-t} is a convex set.

LEMMA 3.20. *Let C, $K \subset \mathbb{R}^n$ be bounded convex bodies. Then*

$$(3.13) \qquad\qquad |\operatorname{inr}(C) - \operatorname{inr}(K)| \leqslant \delta(C, K).$$

PROOF. We split the proof in two steps.

Step one. We show that $(C_t)_{-t} = C$ for any $t > 0$, whenever $C \subset \mathbb{R}^n$ is a bounded convex body. Let us start proving that $C \subset (C_t)_{-t}$. Arguing by contradiction, we assume that $d(p, \partial C_t) < t$ for some $p \in C$. Then there exists $q \in \partial C_t$ so that $|p-q| = d(p, \partial C_t)$. Choose $r > 0$ small enough so that $|p-q|+r < t$. If $z \in B(q, r)$, then

$$d(z, C) \leqslant |z - p| \leqslant |z - q| + |q - p| < |p - q| + r < t.$$

This implies that $B(q, r) \subset C_t$, a contradiction to the fact that $q \in \partial C_t$. So we have $C \subset (C_t)_{-t}$.

To prove the reverse inequality, we take $p \in (C_t)_{-t}$. If $p \notin C$, then $d(p, C) = d > 0$. Let q be the metric projection of p to C. Being C convex, it turns out that q is also the metric projection of every point in the half-line $\{q + \lambda(p - q) : \lambda \geqslant 0\}$. Let z be the point in this half-line at distance t from C. Then $z \in \partial C_t$, and $d(p, \partial C_t) \leqslant |p - z| = t - |p - q| < t$, a contradiction since p was taken in $(C_t)_{-t}$. So we get $(C_t)_{-t} \subset C$.

Step two. Let $\varepsilon = \delta(C, K)$ and observe that $K \subset C_\varepsilon$. If both $\operatorname{inr}(C), \operatorname{inr}(K)$ are smaller than or equal to ε, then inequality (3.13) is trivial. So let us assume that $\operatorname{inr}(K)$ is the largest inradius and that $r = \operatorname{inr}(K) > \varepsilon$. Take $B(x, r) \subset K \subset C_\varepsilon$. By Step one we find

$$B(x, r - \varepsilon) = B(x, r)_{-\varepsilon} \subset (C_\varepsilon)_{-\varepsilon} = C.$$

So we have $\operatorname{inr}(K) \geqslant \operatorname{inr}(C) \geqslant \operatorname{inr}(K) - \varepsilon$. This implies (3.13). $\qquad\square$

PROPOSITION 3.21. *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry. Then any isoperimetric region in C is bounded.*

PROOF. Let $v > 0$ and $E \subset C$ be such that $|E| = v$ and $P_C(E) = I_C(v)$. Let $x_0 \in C$ and $r_0 > 0$ be chosen in such a way that $P(E, B_C(x_0, r_0)) > 0$. Set $m(r) = |E \setminus B_C(x_0, r)|$. Let $C, \overline{m} > 0$ be the two positive constants given by Theorem 2.12, with $B = B_C(x_0, r_0)$. By the infinitesimality of $m(r)$ as $r \to 0$, and

by possibly taking a larger r_0, we can enforce that $m(r) < \overline{m}$ for all $r \geqslant r_0$. This means that for all $r \geqslant r_0$ there exists a set E_r such that $E_r = E$ outside $B_C(x_0, r)$, $|E_r| = |E| + m(r)$, and $P_C(E_r) \leqslant P_C(E) + C\, m(r)$. If r is large enough, we can entail at the same time that $m(r) < \min(\varepsilon, v_0)$ (where v_0 is as in Corollary 3.17). Therefore we can define $F_r = E_r \setminus B_C(x_0, r)$ and get for almost all $r > 0$

$$P_C(F_r) = P_C(E_r) - 2m'(r) - P_C(E \setminus B_C(x_0, r))$$
$$\leqslant P_C(E) + c\, m(r) - 2m'(r) - c_0 m(r)^{(n-1)/n},$$

where c_0 is as in Corollary 3.17. Since of course $|F_r| = |E|$, by minimality, and up to choosing r large enough, we can find a constant $c_1 > 0$ such that

$$c_1 m(r)^{(n-1)/n} \leqslant c_0 m(r)^{(n-1)/n} - c\, m(r) \leqslant -2m'(r).$$

Let us fix r and take any $R > r$ such that $m(R) > 0$. Then one rewrites the above inequality as

$$c_2 \leqslant -\frac{m'(r)}{m(r)^{(n-1)/n}}$$

for some $c_2 > 0$. By integrating this last inequality between r and R one gets

$$c_3(R - r) \leqslant m(r)^{1/n} - m(R)^{1/n}$$

for some $c_3 > 0$, whence the boundedness of R follows. In conclusion, there exists a largest $R > 0$ such that $E \subset B_C(x_0, R)$, as wanted. $\qquad\square$

We conclude this section with a concentration lemma, that is well-known in $\mathbb{R}^n$ as well as in Carnot groups (see [**41**]). Since the ambient domain here is a convex body C of uniform geometry, it seems worth giving a full proof of the result (notice the use of Tonelli's theorem instead of the covering argument used in the proof of [**41**, Lemma 3.1]).

LEMMA 3.22 (Concentration). *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry. Then, there exists a constant $\Lambda > 0$ only depending on C. with the following property. Choose $0 < r \leqslant 1$ and $m \in (0, \frac{1}{2}]$, and assume*

$$(3.14) \qquad\qquad |E \cap B_C(x, r)| \leqslant m\, |B_C(x, r)|, \qquad \forall x \in C.$$

Then we have

$$(3.15) \qquad\qquad \Lambda\, |E| \leqslant m^{1/n} r\, P_C(E).$$

REMARK 3.23. The constant Λ is defined by

$$(3.16) \qquad\qquad \Lambda := \frac{c_1 b(1)}{\omega_n^{(n+1)/n}},$$

where $\omega_n = |B(0,1)|$, $b(1) = \inf_{x \in C} |B_C(x, 1)|$, and c_1 is the Poincaré constant for the relative isoperimetric inequality in balls of radius $0 < r \leqslant 1$.

PROOF OF LEMMA 3.22. Since C is of uniform geometry, $|B_C(x, 1)| \geqslant b(1) > 0$ for all $x \in C$. By Lemma 2.7, inequality $|B_C(x, r)| \geqslant b(1)\, r^n$ holds for any $r \in (0, 1]$. Using (3.9) with the fact that $m \leqslant \frac{1}{2}$, we deduce that

$$(3.17) \qquad\qquad P_C(E, B_C(x, r)) \geqslant c_1 |E \cap B_C(x, r)|^{\frac{n-1}{n}},$$

where c_1 is a uniform constant. Then Tonelli's Theorem implies

$$\int_C P(E, B_C(x,r)) \, d\mathcal{H}^n(x) = \int_C \left\{ \int_{\text{int}(C)} \chi_{B(x,r)}(y) \, d|D\chi_E|(y) \right\} d\mathcal{H}^n(x)$$

$$= \int_C \left\{ \int_{\text{int}(C)} \chi_{B(y,r)}(x) \, d|D\chi_E|(y) \right\} d\mathcal{H}^n(x)$$

$$= \int_{\text{int}(C)} \left\{ \int_C \chi_{B(y,r)}(x) \, d\mathcal{H}^n(x) \right\} d|D\chi_E|(y)$$

$$= \int_{\text{int}(C)} |B_C(y,r)| \, d|D\chi_E|(y)$$

$$\leqslant \omega_n r^n P_C(E).$$

Then we get

$$\omega_n r^n P_C(E) \geqslant \int_C P(E, B_C(x,r)) \, d\mathcal{H}^n(x)$$

$$\geqslant c_1 \int_C \frac{|E \cap B_C(x,r)|}{|E \cap B_C(x,r)|^{\frac{1}{n}}} \, d\mathcal{H}^n(x)$$

$$\geqslant c_1 \int_C \left(\frac{1}{m\,|B_C(x,r)|} \right)^{\frac{1}{n}} |E \cap B_C(x,r)| \, d\mathcal{H}^n(x)$$

$$\geqslant \frac{m^{-1/n} c_1}{\omega_n^{1/n} r} \int_C |E \cap B_C(x,r)| \, d\mathcal{H}^n(x)$$

$$= \frac{m^{-1/n} c_1}{\omega_n^{1/n} r} \int_E |B_C(y,r)| \, d\mathcal{H}^n(y)$$

$$\geqslant \frac{m^{-1/n} c_1 b(1)}{\omega_n^{1/n} r} r^n |E|,$$

where we have used (3.17) to get the second inequality,(3.14) to obtain the inequality relating the second and third lines, equation (3.4) to get the equality in the fifth line, and the fact that $b(r) \geqslant b(1)r^n$ for all $0 < r < 1$ to get the last inequality. The above chain of inequalities, together with the definition (3.16) of Λ, implies (3.15). $\qquad\square$

The following two corollaries will be used in the sequel. The first one will play an important role in the proof of Theorem 5.1. The second one will be used in Chapter 4.

COROLLARY 3.24. *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry, and $E \subset C$ a set with positive relative perimeter. Then there exists $\Lambda > 0$, only depending on C, such that, for every $r > 0$ satisfying*

$$r < \min \left\{ 2^{1/n} \Lambda \, \frac{|E|}{P_C(E)}, 1 \right\},$$

there exists a point $x \in C$, only depending on r and E, with

$$|E \cap B_C(x,r)| > \frac{|B_C(x,r)|}{2}.$$

PROOF. We simply argue by contradiction using Lemma 3.22 and (3.15) for $m = 1/2$. $\qquad\square$

COROLLARY 3.25. *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry, $v_0 > 0$, and $\{E_i\}_{i \in \mathbb{N}} \subset C$ a sequence such that, denoting by H a generic half-space,*

$$|E_i| \leqslant v_0 \text{ for all } i \in \mathbb{N}, \quad \lim_{i \to \infty} |E_i| = v \in (0, v_0], \quad \liminf_{i \to \infty} P_C(E_i) \leqslant I_H(v).$$

Take $m_0 \in (0, \frac{1}{2}]$ such that

$$m_0 < \min\left\{\frac{1}{2v_0}, \frac{\Lambda^n}{I_H(1)^n}\right\}.$$

Then there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subset C$ such that

$$|E_i \cap B_C(x_i, 1)| \geqslant m_0 v \qquad \text{for i large enough.}$$

PROOF. By contradiction, and up to subsequences, we may assume $|E_i \cap B_C(x, 1)| < m_0 |E_i|$ for all $x \in C$. We apply Lemma 3.22, and in particular (3.15) with $m = m_0 v \leqslant m_0 v_0 \leqslant \frac{1}{2}$, in order to get

$$\Lambda^n |E_i|^n \leqslant m_0 P_C(E_i)^n,$$

From this inequality and our hypotheses, by taking limits we obtain

$$\Lambda^n v^n \leqslant m_0 I_H(v)^n$$

and thus

$$m_0 \geqslant \Lambda^n \frac{v^n}{I_H(v)^n} = \frac{\Lambda^n}{I_H(1)^n},$$

that is, a contradiction to the choice of m_0. $\square$

REMARK 3.26. Corollary 3.25 holds in particular when $\{E_i\}_{i \in \mathbb{N}}$ is a minimizing sequence for volume v, since $\liminf_{i \to \infty} P_C(E_i) \leqslant I_C(v) \leqslant I_H(v)$.

3.4. Examples

Let us give now some examples of unbounded convex bodies of uniform geometry.

EXAMPLE 3.27 (Convex bodies with non-degenerate asymptotic cone are of uniform geometry). Let $C \subset \mathbb{R}^n$ be a convex body with non-degenerate asymptotic cone C_∞. For every $x \in C$ we know that $x + C_\infty$ is contained in C, so that taking any $r > 0$ we have $B_{x+C_\infty}(x, r) \subset B_C(x, r)$. Hence $|B_C(x, r)| \geqslant |B_{C_\infty}(0, r)| = c\, r^n$, for $c = |B_{C_\infty}(0, 1)|$. This implies that C is of uniform geometry.

EXAMPLE 3.28 (Cylindrically bounded convex bodies are of uniform geometry and their asymptotic cylinders are unique up to horizontal translations). Assume that $C \subset \mathbb{R}^n$ is a cylindrically bounded convex body as defined in [73]: the set C is the epigraph of a convex function defined on the interior of a bounded convex body $K \subset \mathbb{R}^n \equiv \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n$, and the intersections of C with the horizontal hyperplanes $\Pi_c := \{x_n = c\}$, for $c \in \mathbb{R}$, projected to Π_0 form an increasing (w.r.t. c) family converging in Hausdorff distance to K.

Let $\{x_i\}_{i \in \mathbb{N}}$ be a diverging sequence in C such that $\{-x_i + C\}_{i \in \mathbb{N}}$ converges locally in Hausdorff distance to an asymptotic cylinder C_∞. Write $x_i = (z_i, t_i) \in \mathbb{R}^n \times \mathbb{R}$. The coordinates t_i are unbounded and the vectors z_i are uniformly bounded. This implies that the sequence $\{x_i/|x_i|\}_{i \in \mathbb{N}}$ converges to the unit vector $v = (0, 1) \in \mathbb{R}^n \times \mathbb{R}$. By construction, the half-lines $\{x + \lambda v : \lambda \geqslant 0\}$ are contained in C for all $x \in C$. It is easy to check that

$$(-x_i + C) \cap \Pi_0 = -z_i + \pi(C \cap \Pi_{t_i}),$$

where π is the orthogonal projection onto the hyperplane Π_0. Since $\pi(C \cap \Pi_{t_i})$ converges to K in Hausdorff distance and $\{z_i\}_{i\in\mathbb{N}}$ is a bounded sequence, we immediately conclude that $(-x_i + C) \cap \Pi_0$ subconverges to a horizontal translation K' of K. Obviously $K' \subset C_\infty$. Since vertical lines passing through a point in C_∞ are contained in C_∞ we immediately obtain that $K' \times \mathbb{R} \subset C_\infty$.

Let us check that $C_\infty = K' \times \mathbb{R}$. If $x \in C_\infty$ then $\pi(x) \in C_\infty$. Then there exists a sequence $\{c_i\}_{i\in\mathbb{N}} \subset C$ such that $\{-x_i + c_i\}_{i\in\mathbb{N}}$ converges to $\pi(x)$. If we write c_i as $(c_i', s_i) \in \mathbb{R}^n \times \mathbb{R}$, then $-z_i + c_i'$ converges to $\pi(x)$ and $-t_i + s_i$ converges to 0. For each i, choose $j(i) > j(i-1)$ such that $t_{j(i)} > s_i$. Then $d_i := (c_i', t_{j(i)}) \in C$ and $-x_{j(i)} + d_i = (-z_{j(i)} + c_i', 0)$ converges to $\pi(x)$. This implies that $\pi(x) \in K' \times \mathbb{R}$ and so $x \in K' \times \mathbb{R}$. Hence $C_\infty \subset K' \times \mathbb{R}$.

The above arguments imply that any asymptotic cylinder of C is a horizontal translation of $K \times \mathbb{R}$. Since $K \times \mathbb{R}$ has non-empty interior, Proposition 3.14 implies that C is of uniform geometry.

EXAMPLE 3.29 (Unbounded convex bodies of revolution are of uniform geometry and the asymptotic cylinders of non-cylindrically bounded convex bodies of revolution are either half-spaces or $\mathbb{R}^n$). Let $\psi : [0, \infty) \to [0, \infty)$ be a continuous concave function satisfying $\psi(0) = 0$ and $\psi(x) > 0$ for all $x > 0$. Consider the convex body of revolution

$$C = C_\psi := \{(z, t) \in \mathbb{R}^n \times [0, \infty) \subset \mathbb{R}^n : |z| \leqslant \psi(t)\}.$$

We shall assume that ψ is unbounded since otherwise C_ψ would be a cylindrically bounded convex body. We remark that we are not assuming any smoothness condition on ψ.

Take a diverging sequence $\{x_i\}_{i\in\mathbb{N}} \subset C$ and assume that $\{-x_i + C\}_{i\in\mathbb{N}}$ converges locally in Hausdorff distance to some asymptotic cylinder C_∞.

Write $x_i = (z_i, t_i) \in \mathbb{R}^n \times [0, \infty)$. The sequence $\{t_i\}_{i\in\mathbb{N}}$ cannot be bounded since otherwise inequality $|z_i| \leqslant \psi(t_i)$ would imply that z_i is also uniformly bounded (and $\{x_i\}_{i\in\mathbb{N}}$ would be bounded). On the other hand, since the function $\psi(t)/t$ is non-increasing by the concavity of ψ, the sequence $|z_i|/t_i \leqslant \psi(t_i)/t_i$ is uniformly bounded. Hence z_i/t_i subconverges to some vector $c \in \mathbb{R}^n$ and so

$$\frac{x_i}{|x_i|} = \frac{(z_i/t_i, 1)}{\sqrt{(|z_i|/t_i)^2 + 1}}$$

subconverges to the vector $v := (c, 1)/\sqrt{c^2 + 1}$, whose last coordinate is different from 0. We conclude that any straight line parallel to v containing a point in C_∞ is entirely contained in C_∞.

The sets $(-x_i + C) \cap \Pi_0$ are closed disks $\overline{D}(w_i, \psi(t_i)) \subset \mathbb{R}^n$ of center $w_i = -z_i \in \mathbb{R}^n$ and radius $\psi(t_i)$. We define $r_i := \psi(t_i) - |w_i| = \psi(t_i) - |z_i| \geqslant 0$.

In case $\{r_i\}_{i\in\mathbb{N}}$ is an unbounded sequence the inclusion $\overline{D}(0, r_i) \subset \overline{D}(w_i, \psi(t_i))$ holds and shows that any point in $\mathbb{R}^n \times \{0\}$ belongs to C_∞. Hence $C_\infty = \mathbb{R}^n$.

If the sequence $\{r_i\}_{i\in\mathbb{N}}$ is bounded, passing to a subsequence we may assume $\lim_{i\to\infty} r_i = c \geqslant 0$ and $\lim_{i\to\infty} w_i/|w_i| = e$. Let us prove first that the set

$$K := \{(z, 0) : \langle z, e \rangle \geqslant -c\}$$

is contained in C_∞. Pick some $(z, 0) \in K$ and choose $r > 0$ large enough so that $z \in \mathrm{int}(\overline{B}(0, r))$. In case $\langle z, e \rangle > -c$ we have

$$\lim_{i \to \infty} \frac{|z - w_i|^2 - \psi(t_i)^2}{|w_i|} = \lim_{i \to \infty} \left(\frac{|z|}{|w_i|} - 2\langle z, \frac{w_i}{|w_i|} \rangle + \frac{|w_i|^2 - \psi(t_i)^2}{|w_i|} \right)$$
$$= -2\langle z, e \rangle - 2c < 0,$$

since $|w_i|$ is unbounded, $\lim_{i \to \infty} w_i / |w_i| = e$, and

$$\lim_{i \to \infty} \frac{\psi(t_i)^2 - |w_i|^2}{|w_i|} = \lim_{i \to \infty} \frac{\psi(t_i) + |w_i|}{|w_i|} r_i = 2c.$$

This implies that, if $\langle z, e \rangle > -c$, there exists $i_0 \in \mathbb{N}$ so that $z \in \overline{D}(w_i, \psi(t_i))$ for all $i \geqslant i_0$. Hence $(z, 0) \in C_\infty$.

In case $\langle z, e \rangle = -c$, we consider a sequence $\{\varepsilon_i\}_{i \in \mathbb{N}}$ of positive real numbers decreasing to 0. For each j, the point $z + \varepsilon_j e$ satisfies $\langle z + \varepsilon_j e, e \rangle = -c + \varepsilon_j > -c$. Hence we can choose $i(j)$ (increasing in j) such that $z + \varepsilon_j e \in \overline{D}(w_i, \psi(t_j))$ for $i \geqslant i(j)$. We construct a sequence $\{m_k\}_{k \in \mathbb{N}}$ of points in $\mathbb{R}^n \times \{0\}$ so that m_k is chosen arbitrarily in $\overline{D}(w_k, \psi(t_k)) \cap \mathrm{int}(\overline{B}(0, r))$ for $1 \leqslant k < i(1)$, and $m_k := z + \varepsilon_j e$ when $k \in [i(j), i(j+1))$. The point m_k lies in $\overline{D}(w_k, \psi(t_k)) \cap \mathrm{int}(\overline{B}(0, r))$ for all $k \in \mathbb{N}$, and the sequence $\{m_k\}_{k \in \mathbb{N}}$ converges to z. In conclusion, $(z, 0) \in C_\infty$ and thus $K \subset C_\infty$ as claimed.

Since $K \subset C_\infty$, and lines parallel to $L(v)$ intersecting C_∞ are contained in C_∞, we conclude that the half-space $K + L(v)$ is contained in C_∞. But then C_∞ itself must be either $\mathbb{R}^n$ or a half-space. This is easy to prove since, in case $C \neq \mathbb{R}^n$, we have $C = \bigcap_{i \in I} H_i$, where $\{H_i\}_{i \in I}$ is the family of all supporting hyperplanes to C. For any i, we would have $H \subset C \subset H_i$, and this would imply that H_i is a half-space parallel to H containing H and so it would be $\bigcap_{i \in I} H_i$.
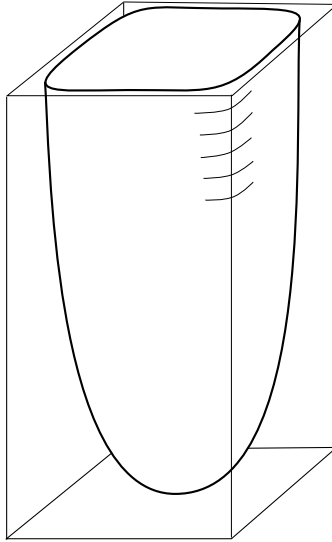

FIGURE 3.2. The convex body in Example 3.30

EXAMPLE 3.30 (An unbounded convex body of uniform geometry with C^∞ boundary, all of whose asymptotic cylinders have non-smooth boundary). Let $C \subset \mathbb{R}^3$ be the upper-graph of the function $u : [-1,1]^2 \to \mathbb{R}$ defined by

$$u(x,y) = \frac{1}{1-x^2} + \frac{1}{1-y^2}\,.$$

Since $u(x,y)$ is smooth, convex, and divergent to $+\infty$ when $|x| \to 1$ or $|y| \to 1$, one easily concludes that any asymptotic cylinder of C coincides with a translation of the set $[-1,1]^2 \times \mathbb{R}$.

EXAMPLE 3.31 (Closed n-dimensional convex cones are of uniform geometry). Let $L \subset \mathbb{R}^n$ be a closed convex cone with non-empty interior and vertex 0. Since $p + L \subset L$ for any $p \in L$, we have $p + \overline{B}_L(0,r) \subset \overline{B}_L(p,r)$ for any $r > 0$. This implies

$$|B_L(p,r)| \geqslant |B_L(0,r)| = |B_L(0,1)|\, r^n,$$

for any $p \in L$ and $r > 0$. Hence L is of uniform geometry.

For convex cones we can prove that tangent cones out of a vertex are always asymptotic cylinders. From Lemma 3.4(ii) we may conclude that the limits of these tangent cones are also asymptotic cylinders. In general, we are able to prove that any asymptotic cylinder contains a tangent cone.

PROPOSITION 3.32. Let $C \subset \mathbb{R}^n$ be a closed n-dimensional convex cone such that $0 \in \partial C$ is a vertex of C. Then

(i) For any $x \in C \setminus \{0\}$ and $\mu > 0$, we have $-x + C_x = -\mu x + C_{\mu x}$. Moreover, the set $-x + C_x$ is a closed convex cylinder.
(ii) For any $x \in C \setminus \{0\}$, $-x + C_x$ is an asymptotic cylinder of C.
(iii) Let $\{x_i\}_{i \in \mathbb{N}}$ be a divergent sequence in ∂C so that $-x_i + C$ converges locally in Hausdorff distance to $K \in \mathcal{K}(C)$. Assume that $z_i := |x_i|^{-1} x_i$ converges to z and that $-x_i + C_{x_i}$ converges to K'. Then $-z + C_z \subset K \subset K'$.

PROOF. To prove (i) we fix $\lambda > 0$ and $c \in C$, so that

$$-x + h_{x,\lambda}(c) = -\mu x + h_{\mu x, \mu^{-1}\lambda}(\mu c) \in -\mu x + C_{\mu x},$$

since $\mu c \in C$. This implies that $-x + h_{x,\lambda}(C) \subset -\mu x + C_{\mu x}$. As $\lambda > 0$ is arbitrary and $-\mu x + C_{\mu x}$ is closed we get $-x + C_x \subset -\mu x + C_{\mu x}$. The reverse inclusion is obtained the same way. The set $-x + C_x$ is trivially closed and convex. Since it contains the line $L(x) = \{tx : t \in \mathbb{R}\}$, it is a convex cylinder by virtue of [**79**, § 1.4].

For the proof of (ii) take $x \in C \setminus \{0\}$ and an increasing diverging sequence of positive real numbers λ_i. Let $x_i := \lambda_i x$. Taking a subsequence if needed we may assume that $-x_i + C$ converges locally in Hausdorff distance to an asymptotic cylinder $K \in \mathcal{K}(C)$. We shall check that $K = -x + C_x$. Take first $z \in K$ so that there exists a sequence of points $c_i \in C$ such that $z = \lim_{i \to \infty} -x_i + c_i$. Since $C \subset C_x$, we get $z \in -x + C_x$. This implies $K \subset -x + C_x$. For the reverse inclusion fix some $\lambda > 0$ and take $z \in -x + h_{x,\lambda}(C)$. Then there exists $c \in C$ such that $z = \lambda(c - x)$ and so

$$z = -\lambda_i x + \lambda_i \big(x + \frac{\lambda}{\lambda_i}(c - x)\big).$$

This implies that $z \in -x_i + C$ for i large since $x + \frac{\lambda}{\lambda_i}(c - x)$ belongs to C for i large enough so that $\lambda/\lambda_i \leqslant 1$. In particular, $z \in K$ and so we obtain $-x + h_{x,\lambda}(C) \subset K$.

As $\lambda > 0$ is arbitrary and K is closed, from the definition of the tangent cone C_x we obtain $-x + C_x \subset K$.

Let us check that (iii) holds. Take $\mu > 0$ and a point $x \in -z + h_{z,\mu}(C)$. Then there exists $c \in C$ such that $x = \mu(c - z)$ and, setting $\lambda_i = |x_i|$, we have

$$x = \mu(c - z) = -x_i + \left(x_i + \frac{\mu}{\lambda_i}(\lambda_i(c - z + z_i) - x_i)\right).$$

We define

$$d_i := x_i + \frac{\mu}{\lambda_i}(\lambda_i c - x_i)$$

and observe that $d_i \in C$ for large i since $\lambda_i c \in C$ and $\mu/\lambda_i \leqslant 1$. We have

$$x - (-x_i + d_i) = \mu(-z + z_i) \to 0$$

when $i \to \infty$. This implies that $x \in K$ and so we get $-z + h_{z,\mu}(C) \subset K$ for all $\mu > 0$. As K is closed we obtain $-z + C_z \subset K$.

Finally, take $x \in K$ and choose a sequence of points $c_i \in C$ such that $x = \lim_{i \to \infty} -x_i + c_i$. Since $C \subset C_{x_i}$, it follows that x can be written as the limit of a sequence of points in $-x_i + C_{x_i}$. This implies that $x \in K'$. $\qquad\square$

Let $C \subset \mathbb{R}^n$ be a closed n-dimensional convex cone so that $0 \in \partial C$ is a vertex of C, and assume that $\partial C \setminus \{0\}$ is of class C^1. Then any asymptotic cylinder of C is either $\mathbb{R}^n$ or a half-space. This is easy to check since, by Remark 3.3 we may obtain $\mathbb{R}^n$ as an asymptotic cone by taking $x \in \text{int}(C)$ and a diverging sequence $\lambda_i x$, with $\lim_{i \to \infty} \lambda_i = +\infty$. On the other hand, if we take a diverging sequence in the boundary of C, Proposition 3.32(iii) implies that the limit asymptotic cylinder K contains a set of the form $-z + C_z$, with $z \in \partial C \setminus \{0\}$. Since $\partial C \setminus \{0\}$ is of class C^1, the set $-z + C_z$ is a half-space. Hence K is a half-space.

In Proposition 3.33 below we show that the smoothness of the asymptotic cone C_∞ implies that all asymptotic cylinders of C are either $\mathbb{R}^n$ or half-spaces. This fact will be of use in Section 6.3: indeed we shall prove in Corollary 6.22 that unbounded convex bodies with a non-degenerate asymptotic cone, that is smooth except that in a vertex, admit isoperimetric solutions for any prescribed volume. The idea of the proof is that all asymptotic cylinders of C_∞ are either $\mathbb{R}^n$ or half-spaces, as we have proved above, thus one has to show that this property can be transferred to C.

PROPOSITION 3.33. *Let C be an unbounded convex body with non degenerate asymptotic cone C_∞. Assume that $\partial C \setminus \{0\}$ is of class C^1. Then any asymptotic cylinder $K \in \mathcal{K}(C)$ is either $\mathbb{R}^n$ or a half-space.*

PROOF. We assume $C \neq \mathbb{R}^n$. By Remark 3.3 it is enough to consider asymptotic cylinders obtained from diverging sequences contained in the boundary of C.

We first prove the following fact: for every $\delta > 0$, for any divergent sequence of points $x_i \in \partial C$, and for any sequence of unit outer normal vectors $u_i \in N(C, x_i)$, whenever $y_i \in \partial C$ satisfies $|y_i - x_i| \leqslant \delta$ then for every sequence of unit vectors $v_i \in N(C, y_i)$ one has

(3.18) $$\lim_{i \to \infty} |u_i - v_i| = 0.$$

To see this we argue by contradiction assuming the existence of positive constants $\delta, \varepsilon > 0$ and of sequences $y_i \in \partial C$ and $v_i \in N(C, y_i)$, $|v_i| = 1$, such that

$$|y_i - x_i| \leqslant \delta \qquad \text{and} \qquad |u_i - v_i| \geqslant \varepsilon, \qquad \text{for all } i\,.$$

Let $t_i = |x_i|$ and define $z_i = t_i^{-1} x_i$, $w_i = t_i^{-1} y_i$. Clearly, as $i \to \infty$ we have $t_i \to +\infty$, $|z_i| = 1$, $|w_i - z_i| \leqslant t_i^{-1} \delta \to 0$, and $t_i^{-1} C \to C_\infty$ locally in Hausdorff distance, owing to the properties of the asymptotic cone C_∞. At the same time, $u_i \in N(t_i^{-1} C, z_i)$ and $v_i \in N(t_i^{-1} C, w_i)$. By compactness, up to extracting a subsequence, we have that $z_i \to z \in \partial C_\infty \cap \partial B(0,1)$, $w_i \to z$, $u_i \to u_\infty$ and $v_i \to v_\infty$ with u_∞, v_∞ unit vectors in $N(C_\infty, z)$. However we have $|u_\infty - v_\infty| = \lim_{i \to \infty} |u_i - v_i| \geqslant \varepsilon$, which contradicts the regularity of ∂C_∞ at z.

Now we prove that any asymptotic cylinder of C is a half-space. Let $K \in \mathcal{K}(C)$ and let $x_i \in \partial C$ be a diverging sequence such that, setting $C_i = -x_i + C$, we have $C_i \to K$ locally in Hausdorff distance. Fix $\delta > 0$ and choose $z \in \partial K$ such that $|z| \leqslant \delta/2$. Choose $u_\infty \in N(K, 0)$ with $|u| = 1$ such that u_∞ is the limit of unit outer normal vectors $u_i \in N(K, x_i)$; then choose $v_\infty \in N(K, z)$ such that $|v| = 1$.

Up to an isometry, and for i large enough, the convex bodies K and C_i locally coincide with the epigraphs of convex functions defined on a relative neighborhood of z in some supporting hyperplane for K at z. Hence by Lemma 2.8 we infer that there exist sequences $y_i \in \partial C_i$, $v_i \in N(K, y_i)$ with $|v_i| = 1$, such that $y_i - x_i \to z$, $|y_i - x_i| \leqslant \delta$ and $\lim_{i \to \infty} v_i = v_\infty$. Now we recall (3.18) and obtain

$$|u_\infty - v_\infty| = \lim_{i \to \infty} |u_i - v_i| = 0,$$

so that $u_\infty = v_\infty$. This means that K is necessarily a half-space and the proof is completed. $\qquad\square$

REMARK 3.34. We remark that Proposition will be used later in the proof of Theorem 6.21

A generalized existence result

Let $C \subset \mathbb{R}^n$ be an unbounded convex body of uniform geometry. The main result of this chapter is Theorem 4.6, which shows existence of minimizers of the relative isoperimetric problem in C in a generalized sense, for any prescribed volume $v > 0$. More precisely we will show that there exists a finite family of sets $(E^0, E^1, \ldots, E^\ell)$, which satisfy $E^0 \subset C$, $E^j \subset K^j$, where K^j is an asymptotic cylinder of C for all $j \geqslant 1$, and

$$\sum_{j=0}^{\ell} |E^j| = v, \qquad P_C(E^0) + \sum_{j=1}^{\ell} P_{K^j}(E^j) = I_C(v).$$

Generalized existence of isoperimetric regions is a previous essential step before proving the concavity of the function $I_C^{n/(n-1)}$ in the next chapter.

4.1. Preliminary results

Before stating and proving Theorem 4.6, we need some preparatory results, collected in this section

LEMMA 4.1. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of convex bodies in $\mathbb{R}^n$ that converges locally in Hausdorff distance to a convex body C. Let $E_i \subset C_i$ be finite perimeter sets such that $\{E_i\}_{i\in\mathbb{N}}$ and $\{P_{C_i}(E_i)\}_{i\in\mathbb{N}}$ are bounded. Then, possibly passing to a subsequence, $E_i \to E$ in $L^1(\mathbb{R}^n)$, where E is a finite perimeter set in C, and*

$$(4.1) \qquad P_C(E) \leqslant \liminf_{i\to\infty} P_{C_i}(E_i).$$

PROOF. Since C_i converges to C locally in Hausdorff distance and $\{E_i\}_{i\in\mathbb{N}}$ is bounded, there exist a Euclidean ball $B \subset \mathbb{R}^n$ so that $E_i \subset\subset B$ for all $i \in \mathbb{N}$. Therefore, up to replacing C_i and C with, respectively, $C_i \cap B$ and $C \cap B$, we can directly assume from now on that C_i and C are contained in B for all $i \in \mathbb{N}$. By [**79**, Cor. 1.3.6], $\mathcal{H}^{n-1}(\partial C_i) \leqslant \mathcal{H}^{n-1}(\partial B)$ for all $i \in \mathbb{N}$. Hence

$$P_{\mathbb{R}^n}(E_i) \leqslant P_{C_i}(E_i) + \mathcal{H}^{n-1}(\partial C_i) \leqslant P_{C_i}(E_i) + \mathcal{H}^{n-1}(\partial B).$$

Since the sequence $\{P_{C_i}(E_i)\}_{i\in\mathbb{N}}$ is bounded by hypotheses, the previous inequality shows that the sequence $\{P_{\mathbb{R}^n}(E_i)\}_{i\in\mathbb{N}}$ is bounded. As $\{E_i\}_{i\in\mathbb{N}}$ is uniformly bounded, a standard compactness result for finite perimeter sets, [**45**, Thm. 12.26], implies the existence of a set $E \subset \mathbb{R}^n$ and a non-relabeled subsequence E_i such that $E_i \to E$ in $L^1(\mathbb{R}^n)$ and χ_{E_i} converges almost everywhere to χ_E. By Kuratowski criterion, [**79**, Thm. 1.8.7], we get $E \subset C$.

Fix an open set $A \subset\subset \text{int}(C)$. Then $A \subset\subset \text{int}(C_i)$ for i large enough and thus by the lower-semicontinuity of the perimeter we infer

$$P(E; A) \leqslant \liminf_{i\to\infty} P(E_i; A) \leqslant \liminf_{i\to\infty} P_{C_i}(E_i).$$

Then, by recalling that $P(E, \cdot)$ is a Radon measure and by taking the supremum over $A \subset\subset \mathrm{int}(C)$ we obtain

$$P_C(E) = \sup_{A \subset\subset \mathrm{int}(C)} P(E, A) \leqslant \liminf_{i \to \infty} P_{C_i}(E_i) \,,$$

thus proving (4.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

LEMMA 4.2. *Let $\{C_i\}_{i \in \mathbb{N}}$ be a sequence of convex bodies in $\mathbb{R}^n$ locally converging in Hausdorff distance to a convex body C. Let $E \subset C$ be a bounded set of finite and positive perimeter and volume $v < |C|$. Assume $v_i \to v$. Then there exists a sequence of bounded sets $E_i \subset C_i$ of finite perimeter such that $E_i \to E$ in $L^1(\mathbb{R}^n)$ and*

    (i) $E_i \to E$ in $L^1(\mathbb{R}^n)$,
    (ii) $|E_i| = v_i$ for all $i \in \mathbb{N}$, and
    (iii) $P_C(E) = \lim_{i \to \infty} P_{C_i}(E_i)$.

PROOF. Assume first that all sets C_i are contained in C. Then the sets $E'_i := E \cap C_i$ converge to E in $L^1(\mathbb{R}^n)$ and have volumes converging to v by the local Hausdorff convergence and the boundedness of E, and $\lim_{i \to \infty} P_{C_i}(E'_i) = P_C(E)$ by the assumption and Lemma 4.1.

Choose now a ball $B' \subset \mathrm{int}(C)$ so that $P(E; B') > 0$. This ball is contained in $\mathrm{int}(C_i)$ for i large enough. We can use Theorem 2.12 to deform $E \cap B' = E_i \cap B'$ inside B' to a set $E_i \subset C_i$ of volume v_i preserving the L^1-convergence. The convergence of perimeters $\lim_{i \to \infty} P_{C_i}(E_i) = P_C(E)$ also follows from Theorem 2.12.

To reduce ourselves to the previous case, we take a closed Euclidean ball $B \subset \mathbb{R}^n$ so that $\overline{E} \subset \mathrm{int}(B)$. We conveniently assume that $C \cap B$ contains the origin, which is always true up to a translation. By local Hausdorff convergence, there exists a sequence of positive real numbers λ_i converging to 1 such that $\lambda_i(C_i \cap B) \subset C \cap B$ for i large enough, see [**79**, Thm. 1.8.16]. As in the previous case, we produce sets $F_i \subset \lambda_i C_i$ of volumes $\lambda_i^n v_i$ converging in $L^1(\mathbb{R}^n)$ to E, and whose perimeters in $\lambda_i C_i$ converge to $P_C(E)$. Then the sets $E_i := \lambda_i^{-1} F_i$ satisfy the requirements.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following two propositions are easily proved by means of Lemma 4.2.

PROPOSITION 4.3. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body, let $K \in \mathcal{K}(C)$ be an asymptotic cylinder of C, and let $E \subset K$ be a bounded set of finite perimeter and volume $v \geqslant 0$. Let $\{v_i\}_{i \in \mathbb{N}}$ be any sequence of positive numbers converging to v. Then there exists a sequence $E_i \subset C$ of bounded sets of finite perimeter with $|E_i| = v_i$ and $\lim_{i \to \infty} P_C(E_i) = P_K(E)$. In particular, this implies that $I_C(v) \leqslant I_K(v)$.*

PROOF. By hypothesis, there exists a diverging sequence $\{x_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n$ so that $C_i = -x_i + C$ converges locally in Hausdorff distance to K. If $v = 0$ we simply take a sequence of balls in C of volumes v_i with perimeters converging to 0. If $v \neq 0$ then K has non-empty interior, and we obtain from Lemma 4.2 the existence of a sequence $F_i \subset C_i$ of sets of finite perimeter and $|F_i| = v_i$ satisfying $\lim_{i \to \infty} P_{C_i}(F_i) = P_K(E)$. Setting $E_i := x_i + F_i \subset C$ we conclude the proof of the first claim.

Finally, to show that $I_C(v) \leqslant I_K(v)$ we fix $\varepsilon > 0$. Proposition 2.13 allows us to take a bounded set $E \subset K$ of volume v such that $P_K(E) \leqslant I_C(v) + \varepsilon$. From the first part we produce a sequence of set $E_i \subset C$ of volume v such that

$\lim_{i\to\infty} P_C(E_i) = P_K(E)$. So we have

$$I_C(v) \leqslant \lim_{i\to\infty} P_C(E_i) = P_K(E) \leqslant I_K(v) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we obtain the desired inequality. $\qquad\square$

PROPOSITION 4.4. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of unbounded convex bodies locally converging in Hausdorff distance to a convex body C. Let $v, v_i > 0$ be such that $v_i \to v$ as $i \to \infty$. Then*

$$(4.2) \qquad \limsup_{i\to\infty} I_{C_i}(v_i) \leqslant I_C(v).$$

PROOF. Owing to Proposition 2.13, for a given $\varepsilon > 0$ we can find a bounded set $E \subset C$, of finite perimeter and volume v, such that

$$(4.3) \qquad P_C(E) \leqslant I_C(v) + \varepsilon.$$

Lemma 4.2 implies the existence of a sequence of sets of finite perimeter $E_i \subset C_i$ with $|E_i| = v_i$ and

$$(4.4) \qquad \lim_{i\to\infty} P_{C_i}(E_i) = P_C(E)$$

Combining (4.3) and (4.4) we get $\limsup_{i\to\infty} I_{C_i}(v_i) \leqslant I_C(v) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain (4.2). $\qquad\square$

The next lemma is a key tool for the proof of Theorem 4.6.

LEMMA 4.5. *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of unbounded convex bodies converging locally in Hausdorff distance to an unbounded convex body C containing the origin. Let $E_i \subset C_i$ be a sequence of measurable sets with volumes v_i converging to $v > 0$ and uniformly bounded perimeter. Assume $E \subset C$ is the $L^1_{loc}(\mathbb{R}^n)$ limit of $\{E_i\}_{i\in\mathbb{N}}$. Then, passing to a (non relabeled) subsequence, there exist diverging radii $r_i > 0$ such that*

$$E_i^d := E_i \setminus B(0, r_i)$$

satisfies

    (i) $|E| + \lim_{i\to\infty} |E_i^d| = v$.
    (ii) $P_C(E) + \liminf_{i\to\infty} P_{C_i}(E_i^d) \leqslant \liminf_{i\to\infty} P_{C_i}(E_i)$.

PROOF. The set E clearly has finite volume less than or equal to v. Take a sequence of increasing radii $\{s_i\}_i$ such that $s_{i+1} - s_i \geqslant i$ for all $i \in \mathbb{N}$. Since E_i converges to E in $L^1_{loc}(\mathbb{R}^n)$, taking a non relabeled subsequence of E_i we may assume

$$\int_{B(0, s_{i+1})} |\chi_{E_i} - \chi_E| \leqslant \frac{1}{i}$$

for all i. By the coarea formula

$$\int_{s_i}^{s_{i+1}} \mathcal{H}^{n-1}(E_i \cap \partial B(0, t)) \, dt \leqslant \int_{\mathbb{R}} \mathcal{H}^{n-1}(E_i \cap \partial B(0, t)) \, dt = |E_i| = v_i.$$

Hence the set of $t \in [s_i, s_{i+1}]$ such that $\mathcal{H}^{n-1}(E_i \cap \partial B(0, t)) \leqslant \frac{2v_i}{i}$ has positive measure. A well-known property of finite perimeter sets implies that we can choose $r_i \in [s_i, s_{i+1}]$ in this set so that

$$P_{C_i}(E_i \cap B(0, r_i)) = P(E_i, \text{int}\,(C_i) \cap B(0, r_i)) + \mathcal{H}^{n-1}(E_i \cap \partial B(0, r_i)),$$
$$P_{C_i}(E_i \setminus B(0, r_i)) = P(E_i, \text{int}\,(C_i \setminus B(0, r_i))) + \mathcal{H}^{n-1}(E_i \cap \partial B(0, r_i)),$$

(e.g see [**45**, Lemma 15.12]). We define

$$E_i^d := E_i \setminus B(0, r_i).$$

Then $|E_i| = |E_i \cap B(0, r_i)| + |E_i^d|$. Since E_i converges to E in $L_{loc}^1(\mathbb{R}^n)$ and E has finite volume, we have $\lim_{i \to \infty} |E_i \cap B(0, r_i)| = |E|$. This proves (i).

On the other hand,

$$P_{C_i}(E_i) \geqslant P(E_i, \text{int}(C_i) \cap B(0, r_i)) + P(E_i, \text{int}(C_i \setminus B(0, r_i)))$$

$$\geqslant P_{C_i}(E_i \cap B(0, r_i)) + P_{C_i}(E_i^d) - \frac{4v_i}{i}.$$

Taking inferior limits we have

$$\liminf_{i \to \infty} P_{C_i}(E_i) \geqslant \liminf_{i \to \infty} P_{C_i}(E_i \cap B(0, r_i)) + \liminf_{i \to \infty} P_{C_i}(E_i^d).$$

By Lemma 4.1 we finally get (ii).   □

4.2. The main result

Before stating (and proving) the main result of this section, i.e. Theorem 4.6 below, we introduce some extra notation.

We say that a finite family $E^0, E^1, \ldots, E^k$ of sets of finite perimeter is a *generalized isoperimetric region in C* if $E^0 \subset C = K^0$, $E^i \subset K^i \in \mathcal{K}(C)$ for $i \geqslant 1$ and, for any family of sets $F^0, F^1, \ldots, F^m$ such that $F^0 \subset C = \tilde{K}^0$, $F^i \subset \tilde{K}^i \in \mathcal{K}(C)$, for $i \geqslant 2$, we have

$$\sum_{i=0}^k P_{K^i}(E^i) \leqslant \sum_{i=0}^m P_{\tilde{K}^i}(F^i).$$

Obviously, each E^i is an isoperimetric region in K^i with volume $|E^i|$.

THEOREM 4.6. *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry and fix $v > 0$. Then there exists $\beta > 0$ and $\ell \in \mathbb{N}$ only depending on C and v, with the following property: for any minimizing sequence $\{F_i\}_i$ for volume v, one can find a (not relabeled) subsequence $\{F_i\}_i$ such that, for every $j \in \{0, \ldots, \ell\}$, there exist*

- *a sequence $\{x_i^j\}_i$, divergent for every $j \geqslant 1$,*
- *a sequence of sets $\{F_i^j\}_i$,*
- *an asymptotic cylinder $K^j \in \mathcal{K}(C)$,*
- *an isoperimetric region $E^j \subset K^j$ with volume larger than β,*

with in particular $x_i^0 = 0$ for all $i \in \mathbb{N}$ and $K^0 = C$, such that

(i) *$F_i^{j+1} \subset F_i^j \subset F_i$ for all $i \in \mathbb{N}$ and $j \in \{0, \ldots, \ell - 1\}$;*
(ii) *$-x_i^j + C$ converges to K^j locally in Hausdorff distance for all $j \in \{1, \ldots, \ell\}$;*
(iii) *$-x_i^j + F_i^j$ converges to $E^j \subset K^j$ in $L_{loc}^1(\mathbb{R}^n)$ for all $j \in \{0, \ldots, \ell\}$;*
(iv) *for any $0 \leqslant q \leqslant \ell$, $E^0, E^1, \ldots, E^q$ is a generalized isoperimetric region of volume $\sum_{j=0}^q |E^j|$.*
(v) *$\sum_{j=0}^\ell |E^j| = v$;*
(vi) *$I_C(v) = \sum_{j=0}^\ell I_{K^j}(|E^j|)$.*

PROOF. We shall split the proof into several steps.

Step one. Here we define the set E^0 as the L_{loc}^1 limit of $\{F_i\}_i$ up to subsequences, then show that it is isoperimetric for its volume. We henceforth assume that C contains the origin. Let $E^0 \subset C$ be the (possibly empty) limit in $L_{loc}^1(\mathbb{R}^n)$ of a

(not relabeled) subsequence of $\{F_i\}_{i\in\mathbb{N}}$. By Lemma 4.5 there exists a sequence of diverging radii $r_i^0 > 0$ so that the set $F_i^1 := F_i \setminus B(0, r_i^0)$ satisfies

$$(4.5) \qquad\qquad |E^0| + \lim_{i\to\infty} |F_i^1| = v,$$

and

$$P_C(E^0) + \liminf_{i\to\infty} P_C(F_i^1) \leqslant \liminf_{i\to\infty} P_C(F_i).$$

In case $|E^0| > 0$ the set E^0 is isoperimetric for its volume: otherwise there would exist a bounded measurable set $G^0 \subset C$ satisfying $|G^0| = |E^0|$ and $P_C(G^0) < P_C(E^0)$. This set can be approximated by a sequence $\{G_i\}_{i\in\mathbb{N}}$ of uniformly bounded sets of finite perimeter satisfying $|G_i| + |F_i^1| = v$ and $\lim_{i\to\infty} P_C(G_i) = P_C(G^0)$. For large i, G_i and F_i^1 are disjoint, $|G_i \cup F_i^1| = v$ and there holds

$$\begin{aligned}
I_C(v_0) &\leqslant \liminf_{i\to\infty} P_C(G_i \cup F_i^1) \\
&= \liminf_{i\to\infty}(P_C(G_i) + P_C(F_i^1)) = P_C(G^0) + \liminf_{i\to\infty} P_C(F_i^1) \\
&< P_C(E^0) + \liminf_{i\to\infty} P_C(F_i^1) \leqslant \liminf_{i\to\infty} P_C(F_i) = I_C(v),
\end{aligned}$$

yielding a contradiction. A similar argument proves the equality

$$(4.6) \qquad\qquad P_C(E^0) + \liminf_{i\to\infty} P_C(F_i^1) = I_C(v).$$

In particular, (iv) is trivially satisfied for $q = 0$.

Step two. We have the following alternative: either $|E^0| = v$ (which means that E^0 is an isoperimetric region of volume v and thus the theorem is verified for $\ell = 0$) or $|E^0| < v$. The latter case corresponds to a "volume loss at infinity". We observe that in this case the sequence $\{F_i^1\}_{i\in\mathbb{N}}$ defined in step one satisfies the hypotheses of Corollary 3.25 since $\lim_{i\to\infty} |F_i^1| = v - |E^0| < v$ and $\liminf_{i\to\infty} P_C(F_i^1) \leqslant I_H(v - |E^0|)$, where I_H is the isoperimetric profile of a half-space in $\mathbb{R}^n$. The last inequality follows by contradiction: in case $\liminf_{i\to\infty} P_C(F_i^1) > I_H(v - |E^0|)$, we could consider the union of the bounded isoperimetric set E^0 with a disjoint ball $B_C(x, r)$ centered at a boundary point $x \in \partial C$ with volume $|B_C(x, r)| = v - |E^0|$. Since $P_C(B_C(x, r)) \leqslant I_H(|B_C(x, r)|)$, we would obtain

$$\begin{aligned}
P_C(E^0 \cup B_C(x, r)) &\leqslant P_C(E^0) + I_H(|B_C(x, r)|) \\
&< P_C(E^0) + \liminf_{i\to\infty} P_C(F_i^1) \leqslant \liminf_{i\to\infty} P_C(F_i) = I_C(v),
\end{aligned}$$

yielding again a contradiction. Since $\{F_i^1\}_{i\in\mathbb{N}}$ satisfies the hypotheses of Corollary 3.25 we can find a sequence of points $x_i^1 \in C$ so that

$$|F_i^1 \cap B_C(x_i^1, 1)| \geqslant m_0 |F_i^1|$$

for all i. The sequence $\{x_i^1\}_{i\in\mathbb{N}}$ is divergent since the sequence $\{F_i^1\}_{i\in\mathbb{N}}$ is divergent. Then, possibly passing again to a subsequence, the convex sets $-x_i^1 + C$ converge to an asymptotic cylinder K^1, and the sets $-x_i^1 + F_i^1$ converge in $L^1_{loc}(\mathbb{R}^n)$ to a set $E^1 \subset K^1$ of volume $v - |E^0| \geqslant |E^1| \geqslant m_0(v - |E^0|)$. We can apply Lemma 4.5 to find a sequence of diverging radii r_i^1 so that the set $F_i^2 \subset F_i^1 \subset C$ defined by

$$-x_i^1 + F_i^2 = (-x_i^1 + F_i^1) \setminus B(0, r_i^1)$$

satisfies

$$|E^1| + \lim_{i\to\infty} |F_i^2| = \lim_{i\to\infty} |F_i^1|,$$

$$P_{K^1}(E^1) + \liminf_{i\to\infty} P_C(F_i^2) \leqslant \liminf_{i\to\infty} P_C(F_i^1),$$

where in the last inequality we have used $P_{-x_i^1+C}(-x_i^1 + F_i^2) = P_C(F_i^2)$. Equation (4.5) then implies

$$|E^0| + |E^1| + \lim_{i\to\infty} |F_i^2| = v,$$

and (4.6) yields

$$P_C(E^0) + P_{K^1}(E^1) + \liminf_{i\to\infty} P_C(F_i^2) \leqslant I_C(v).$$

Arguing in a similar way as in step one, we show that E^1 is isoperimetric for its volume in K^1. We argue by contradiction assuming the existence of a bounded measurable set $G^1 \subset K^1$ with $P_{K^1}(G^1) < P_{K^1}(E^1)$. By Proposition 4.3 we would find a sequence of uniformly bounded subsets $G_i^1 \subset -x_i^1 + C$ with volumes $v - |E^0| - |F_i^2|$ such that $\lim_{i\to\infty} P_{-x_i^1+C}(G_i^1) = P_{K^1}(G^1)$. Since the sets G_i^1 are uniformly bounded and the sequence F_i^2 is divergent, we have $G_i^1 \cap F_i^1 = \emptyset$ for large i. Being the sequence x_i^1 divergent, the sets $x_i^1 + G_i^1$, $x_i^1 + F_i^2$ are disjoint from E^0. So the sets $E^0 \cup (x_i^1 + G_i^1) \cup (x_i^1 + F_i^2)$ have volume v and their perimeters have inferior limit less than or equal to

$$P_C(E_0) + P_{K^1}(G^1) + \liminf_{i\to\infty} P_C(x_i^1 + F_i^2),$$

that, by the choice of G^1, is strictly less than

$$P_C(E_0) + P_{K^1}(E^1) + \liminf_{i\to\infty} P_C(x_i^1 + F_i^2) \leqslant I_C(v),$$

again yielding a contradiction. The same argument shows that

$$P_C(E_0) + P_{K^1}(E^1) + \liminf_{i\to\infty} P_C(x_i^1 + F_i^2) = I_C(v).$$

Arguing similarly, we can then show that (iv) is satisfied for $q = 1$.

   Step three (induction). Assume that after repeating step two q times we have found q asymptotic cylinders K^j, q isoperimetric regions $E^j \subset K^j$ of positive volume, $j = 1, \ldots, q$, and a chain of (non relabeled) subsequences

$$F_i^{q+1} \subset F_i^q \subset \cdots \subset F_i^1 \subset F^1,$$

so that

(4.7) $$|E^0| + |E^1| + \cdots + |E^q| + \lim_{i\to\infty} |F_i^{q+1}| = v$$

(4.8) $$P_C(E^0) + P_{K^1}(E^1) + \cdots + P_{K^q}(E^q) + \liminf_{i\to\infty} P_C(F_i^{q+1}) = I_C(v).$$

If $\lim_{i\to\infty} |F_i^{q+1}| = 0$ we are done. Otherwise, if $\lim_{i\to\infty} |F_i^{q+1}| > 0$ then we claim that the inequality

(4.9) $$\liminf_{i\to\infty} P_C(F_i^{q+1}) \leqslant I_H(v - \sum_{j=0}^{q} |E^j|)$$

must be satisfied. In order to prove (4.9) we reason by contradiction assuming that

$$\liminf_{i\to\infty} P_C(F_i^{q+1}) > I_H(v - \sum_{j=1}^{q} |E^j|) + \varepsilon$$

holds for some positive $\varepsilon > 0$. Recall that each isoperimetric set E^j is bounded and that each asymptotic cylinder K^j is the local limit in Hausdorff distance of a sequence $-x_i^j + C$, where $\{x_i^j\}_i$ is a diverging sequence in C. For each $j \in 1, \ldots, q$, consider a sequence of uniformly bounded sets $G_i^j \subset -x_i^j + C$ of finite perimeter such that

$$\lim_{i \to \infty} P_{-x_i^j + C}(G_i^j) = P_{K^j}(E^j), \text{ and } |G_i^j| = |E^j| \text{ for all } i.$$

Since the sequences $\{x_i^j + G_i^j\}_i$ of subsets of C are divergent, for every $j \in 1, \ldots, q$, we can find $i(j)$ so that the sets $G_j := x_{i(j)}^j + G_{i(j)}^j$ are pairwise disjoint, do not intersect E^0, and

$$P_C(G_j) < P_{K^j}(E^j) + \frac{\varepsilon}{q}.$$

Now choose some $x \in \partial C$ so that the intrinsic ball B centered at x of volume $v - \sum_{j=0}^q |E^j|$ is disjoint from $E^0 \cup \bigcup_{j=1}^q G_j$. We know that $P_C(B) \leqslant I_H(|B|) = I_H(v - \sum_{j=0}^q |E^j|)$. So we finally obtain that $E^0 \cup B \cup \bigcup_{j=1}^q G_j$ has volume v, and

$$P_C\left(E^0 \cup B \cup \bigcup_{j=1}^q G_j\right) \leqslant P_C(E^0) + \sum_{j=1}^q P_{K^j}(E^j) + I_H(|B|) + \varepsilon.$$

$$< P_C(E^0) + \sum_{j=1}^q P_{K^j}(E^j) + \liminf_{i \to \infty} P(F_i^{q+1}) = I_C(v),$$

providing a contradiction and thus proving (4.9).

We can then apply Corollary 3.25 to obtain a divergent sequence of points x_i^{q+1} so that

$$|F_i^{q+1} \cap B_C(x_i^{q+1}, 1)| \geqslant m_0 |F_i^{q+1}|$$

for all i. Possibly passing to a subsequence, the sets $-x_i^{q+1} + C$ converge to an asymptotic cylinder K^{q+1} and the sets $x_i^{q+1} + F_i^{q+1}$ to a set $E^{q+1} \subset K^{q+1}$ satisfying $v - \sum_{j=0}^q |E^j| \geqslant |E^{q+1}| \geqslant m_0 \left(v - \sum_{j=0}^q |E^j|\right)$. We can use again Lemma 4.5 to obtain a sequence of diverging radii r_i^{q+1} such that

$$-x_i^{q+1} + F_i^{q+2} := (-x_i^{q+1} + F_i^{q+1}) \setminus B(0, r_i^{q+1})$$

satisfies

$$|E^{q+1}| + \liminf_{i \to \infty} |F_i^{q+2}| = |F_i^{q+1}|,$$

$$P_{K^{q+1}}(E^{q+1}) + \liminf_{i \to \infty} P_C(F_i^{q+2}) \leqslant \liminf_{i \to \infty} |F_i^{q+1}|.$$

From (4.7) we get

$$\sum_{j=0}^{q+1} |E^j| + \liminf_{i \to \infty} |F_i^{q+2}| = v,$$

and, from (4.8) we obtain

$$P_C(E^0) + \sum_{j=1}^{q+1} P_{K^j}(E^j) + \liminf_{i \to \infty} P_C(F_i^{q+2}) \leqslant I_C(v).$$

Reasoning as above we conclude that E^{q+1} is isoperimetric in K^{q+1} and that equality holds in the above inequality, thus yielding

$$P_C(E^0) + \sum_{j=1}^{q+1} P_{K^j}(E^j) + \liminf_{i \to \infty} P_C(F_i^{q+2}) = I_C(v).$$

Arguing as in step two, we obtain (iv) with $q+1$ in place of q. Moreover it is clear from the procedure illustrated above that (i)–(iii) will be granted at the end of the inductive process.

Step four (finiteness). Let us finally prove that the induction step needs to be repeated only a finite number $\ell - 2$ of times, where ℓ only depends on C and v. The key observation leading to this conclusion is the existence of a constant $\beta > 0$, ultimately depending only on C and v, such that any E^j with $j \geqslant 2$ obtained as in step three (with $|E^j| > 0$) necessarily satisfies

$$|E^j| \geqslant \beta. \tag{4.10}$$

As an immediate consequence of (4.10), one obtains

$$\ell \leqslant 2 + \lfloor v/\beta \rfloor, \tag{4.11}$$

where $\lfloor x \rfloor$ denotes the largest integer $\leqslant x$. The property expressed by (4.11) is actually stronger than a generic finiteness of ℓ, as the right-hand side of (4.11) does not depend upon the specific choices made during each application of step three.

In order to prove (4.10) we first show the existence of a constant $\beta > 0$ depending either on E^0 or on E^1 (more precisely, on the one of the two sets with larger volume), such that (4.10) holds true. Then, in final Step five, we describe how to deduce via a compactness argument (originally due to Almgren [**2**]) that β only depends on C and v, and not on the given minimizing sequence $\{F_i\}_i$. Let us start noticing that

$$\max\{|E^0|, |E^1|\} \geqslant m_0 v/2. \tag{4.12}$$

Indeed, (4.12) follows by arguing as in step two with the additional observation that, owing to Corollary 3.25, the following more precise alternative holds: either $|E^0| \geqslant v/2$, or $\lim_{i \to \infty} |F_i^1| = v - |E^0| \geqslant v/2$ and therefore $|E^1| \geqslant m_0 v/2$. Thus, (4.12) is proved by recalling that $m_0 \leqslant 1$. Let now assume without loss of generality that $|E^1| \geqslant |E^0|$ (indeed, the argument is even simpler in the opposite case).

Using Theorem 2.12 with $E_0 = E^1$ we can find a deformation E_t^1 of E^1, parameterized by $t \in [-\overline{m}, \overline{m}]$ and localized in a fixed ball in the interior of K^1, such that $E_0^1 = E^1$, $|E_t^1| = |E^1| + t$ and

$$P_{K^1}(E_t^1) \leqslant P_{K^1}(E^1) + M|t|, \quad t \in [-\overline{m}, \overline{m}] \tag{4.13}$$

for some positive constants $M, \overline{m}$ depending on E^1.

Let now $\{E^j \subset K^j\}$ for $2 \leqslant j \leqslant q$ be the sets obtained after applying step three $q - 1$ times. Assuming that $v > \sum_{j=0}^{q-1} |E^j|$ (which is clearly the case in order to justify the application of step three $q-1$ times) is equivalent to require that $|E^j| > 0$ for all $j = 2, \ldots, q$. Let now $j \in \{2, \ldots, q\}$ be fixed and assume by contradiction that

$$t_j := |E^j| < \min\{v_0, \delta, \frac{c_0^n}{M^n}\}, \tag{4.14}$$

where c_0 and v_0 are as in Corollary 3.17. Owing to Remark 3.19 one easily deduces that the isoperimetric inequality for small volumes stated in Corollary 3.17 is also

valid for any $K \in \mathcal{K}(C)$ with the same constants c_0 and v_0, thus by (4.14) we get

$$(4.15) \qquad\qquad P_{K^j}(E^j) \geqslant c_0 |E^j|^{(n-1)/n} \,.$$

Next we set $F^1 = E^1_{t_j}$, $F^j = \emptyset$ and $F^i = E^i$ for all $i \in \{0, \dots, q\} \setminus \{1, j\}$. Then we observe that $\sum_{i=0}^q |F^i| = \sum_{i=0}^q |E^i|$. On the other hand, by step three we know that $E^0, \dots, E^q$ is a generalized isoperimetric region and, at the same time, by (4.13), (4.14) and (4.15), we have that

$$\sum_i P_{K^i}(F^i) = \sum_{i \neq j} P_{K^i}(E^i) + M t_i < \sum_{i \neq j} P_{K^i}(E^i) + c_0 t_j^{(n-1)/n} \leqslant \sum_i P_{K^i}(E^i) \,,$$

a contradiction. Setting $\beta = \min\{\delta, \frac{c_0^n}{M^n}\}$, we have thus proved (4.10).

$\quad$ *Step five (uniformity of β).* The fact that β only depends on C and v, but not on $\{F_i\}_i$, can be proved as follows. Assume by contradiction that there exists a countable family $\{E(i) = (E^0(i), \dots, E^{\ell_i}(i))\}_i$ of generalized isoperimetric regions for the prescribed volume v, such that the volume of the smallest component of each region, which can be set as $E^{\ell_i}(i)$ up to a relabeling, is infinitesimal as $i \to \infty$. Similarly, we can assume without loss of generality that $E^0(i)$ is the component with largest volume, so that by (4.12) we have $m_0 v/2 \leqslant |E^0(i)| \leqslant v$ for all i. By applying Steps one and two to the sequence $E^0(i)$ in place of F_i, and arguing as in Step three, we obtain that $-x_i + E^0(i)$ subconverges locally in L^1 to a finite perimeter set $\tilde{E}^1$ contained in an asymptotic cylinder K of C, owing to Lemma 3.4 (ii). Moreover one has without loss of generality that

$$|\tilde{E}^1| \geqslant m_0^2 v/4 \,,$$

hence there exists an open Euclidean ball B contained in K for which $P(\tilde{E}^1, B) > 0$. By Theorem 2.12 applied with $E_0 = \tilde{E}^1$, $E = -x_i + E^0(i)$, and $t = |E^{\ell_i}(i)|$ for i sufficiently large, we construct a competitor for the generalized isoperimetric region $E(i)$, with the same volume v but strictly smaller perimeter, as done in Step four, hence we have reached a contradiction.

$\quad$ Consequently, (v) and (vi) are now satisfied together with (i)–(iv), which concludes the proof of the theorem. $\qquad\square$

$\quad$ We finally state the following result for future reference.

$\quad$ COROLLARY 4.7. *Let $C = K \times \mathbb{R}^k$, where K is an $(n-k)$-dimensional bounded convex body. Then isoperimetric regions exist in C for all volumes.*

$\quad$ PROOF. The proof follows from Theorem 4.6 and the trivial fact that any asymptotic cylinder of C coincides with C up to translation. $\qquad\square$

Concavity of the isoperimetric profile

In this chapter we first prove in Theorem 5.1 the continuity of the isoperimetric profile I_C of an unbounded convex body $C \subset \mathbb{R}^n$ of uniform geometry. Section 5.2 is a technical one where we show that we can approximate in Hausdorff distance any unbounded convex body by unbounded convex bodies such as they, as well as their asymptotic cylinders, have C^∞ boundary. Finally in section 5.3 we show the concavity of the $\frac{n}{n-1}$ power ($\frac{n}{n-1}$ concavity for short) of the isoperimetric profile of an unbounded convex body of uniform geometry. We do it first in Lemma 5.5 for the approximating sets constructed in section 5.2. In Theorem 5.8, the $\frac{n}{n-1}$ concavity of the profile of an arbitrary unbounded convex body with uniform geometry, as well as the connectedness of isoperimetric regions, are proved.

Several subtleties of this chapter should be indicated here. First of all, the continuity of the isoperimetric profile will be needed in Lemma 5.5 to show the $\frac{n}{n-1}$ concavity. On the other hand, our proof of concavity requires the existence of isoperimetric regions, or at least of generalized isoperimetric regions, since it is based on a comparison argument where a suitable global deformation of the minimizer is used.

5.1. Continuity of the isoperimetric profile

THEOREM 5.1 (Continuity of the isoperimetric profile). *Let $C \subset \mathbb{R}^n$ be an unbounded convex body. Then its isoperimetric profile I_C is a continuous function.*

PROOF. We only need to consider the case of uniform geometry since otherwise $I_C \equiv 0$ by Proposition 3.14. We closely follow Gallot's proof [**32**, Lemme 6.2].

We first choose two arbitrary positive volumes satisfying $0 < v < w$. For any $\varepsilon > 0$ we consider a bounded set $E \subset C$ of volume v such that $P_C(E) \leqslant I_C(v) + \varepsilon$. Let $B \subset C$ be a closed intrinsic ball with center at a point in C, at positive distance from E, and such that $|B| = w - v$. Then $P_C(B) \leqslant I_{\mathbb{R}^n}(1)|B|^{(n-1)/n}$, where $I_{\mathbb{R}^n}(1)$ is the isoperimetric constant in $\mathbb{R}^n$, and we get

$$I_C(w) \leqslant P_C(E \cup B) = P_C(E) + P_C(B) \leqslant I_C(v) + I_{\mathbb{R}^n}(1)(w-v)^{(n-1)/n} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we obtain

$$(5.1) \qquad I_C(w) \leqslant I_C(v) + I_{\mathbb{R}^n}(1)(w-v)^{n/(n+1)}.$$

To obtain inequality (5.1) with the roles of $I_C(v)$ and $I_C(w)$ interchanged, it is necessary to restrict the possible values taken by v and w. So we fix $0 < c < d$ and take some $r > 0$ satisfying

$$r < \min\left\{ \frac{2^{1/n}\Lambda c}{I_H(d)+1}, 1 \right\},$$

where Λ is defined in (3.16) and I_H is the isoperimetric profile of a halfspace. If necessary we shrink the interval $[c, d]$ so that the chosen r additionally satisfies the inequality

$$(d - c) \leqslant \frac{b(1)}{2} r^n.$$

This can be done safely since inequality $r < 2^{1/n} \Lambda c / (I_H(d) + 1)$ is respected when reducing the interval. The contraction can be done to include any arbitrary value in the original interval.

With the above choice of $c < d$, take v, w so that $c < v < w < d$. Consider now a set $E \subset C$ with $|E| = w$ and $P_C(E) \leqslant I_C(w) + \varepsilon$, with $0 < \varepsilon < 1$. Since $P_C(E) \leqslant I_C(w) + \varepsilon \leqslant I_H(w) + 1 \leqslant I_H(d) + 1$, we get

$$\frac{2^{1/n} \Lambda c}{I_H(d) + 1} \leqslant \frac{2^{1/n} \Lambda |E|}{P_C(E)}.$$

Then Corollary 3.24, a consequence of the Concentration Lemma 3.22, gives the existence of a point $x \in C$ so that

$$|E \cap B_C(x, r)| \geqslant \frac{|B_C(x, r)|}{2} \geqslant \frac{b(1)}{2} r^n \geqslant (d - c) \geqslant (w - v),$$

By the continuity of the function $\rho \mapsto |E \cap B_C(x, \rho)|$, we find some $s \in (0, r]$ such that $|E \cap B_C(x, s)| = w - v$. Then we have

$$I_C(v) \leqslant P_C(E \setminus (E \cap B_C(x, s))) \leqslant P_C(E) + P_C(B_C(x, s))$$
$$\leqslant I_C(w) + I_{\mathbb{R}^n}(1) (w - v)^{(n-1)/n} + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary we get

(5.2) $$I_C(v) \leqslant I_C(w) + I_{\mathbb{R}^n}(1) (w - v)^{(n-1)/n}.$$

Finally, for any v, w in (c, d), we get from (5.1) and (5.2) the inequality

(5.3) $$|I_C(v) - I_C(w)| \leqslant I_{\mathbb{R}^n}(1) |v - w|^{1-1/n},$$

that implies the continuity of I_C in the interval (c, d). $\qquad\qquad\square$

REMARK 5.2. Indeed inequality (5.3) implies that the isoperimetric profile I_C is a locally $(1 - \frac{1}{n})$-Hölder continuous function in $(0, +\infty)$.

5.2. Approximation by smooth sets

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be the standard symmetric mollifier: the function ρ is radial, has compact support in $\overline{B}(0, 1)$, its integral over $\mathbb{R}^n$ equals 1, and the functions

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right)$$

converge to Dirac's delta when $\varepsilon \to 0$ in the sense of distributions.

Let $C \subset \mathbb{R}^n$ be an unbounded convex body and let $d_C(x)$ denote the distance from $x \in \mathbb{R}^n$ to C. It is well-known (cf. [**79**, § 1.2]) that d_C is a 1-Lipschitz convex function, differentiable at any $x \in \mathbb{R}^n \setminus C$, and such that $\nabla d_C(x)$ is a unit vector in $\mathbb{R}^n \setminus C$. For every $\varepsilon > 0$ we define the smooth, convex, non-negative function

$$g_{C,\varepsilon}(x) := \int_{\mathbb{R}^n} \rho_\varepsilon(x - y) d_C(y) dy,$$

then we set

$$C^\varepsilon := g_{C,\varepsilon}^{-1}([0, \varepsilon]).$$

The following lemma allows us to approximate in local Hausdorff distance an unbounded convex body C by unbounded convex bodies with C^∞ boundaries. The approximation is strong in the sense that any asymptotic cylinder of C is also approximated by asymptotic cylinders of the approaching convex bodies.

LEMMA 5.3. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body, then*

(i) $C \subset C^\varepsilon \subset C + 2\varepsilon B$, *where* $B = \overline{B}(0,1)$.

(ii) C^ε *is an unbounded convex body with C^∞ boundary.*

(iii) *For each $w \in \mathbb{R}^n$, we get $w + C^\varepsilon = (w + C)^\varepsilon$.*

(iv) *Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of unbounded convex bodies that converges to an unbounded convex body C locally in Hausdorff distance. Then also $(C_i)^\varepsilon$ converges to C^ε locally in Hausdorff distance, as $i \to \infty$.*

(v) $\mathcal{K}(C^\varepsilon) = (\mathcal{K}(C))^\varepsilon := \{K^\varepsilon : K \in \mathcal{K}(C)\}$. *In particular, any cylinder in $\mathcal{K}(C^\varepsilon)$ has C^∞ boundary.*

PROOF. Let $x \in C$, then $d_C(x) = 0$ and since d_C is 1-Lipschitz we get

$$g_{C,\varepsilon}(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x - y) \, d_C(y) \, dy \leqslant \int_{\mathbb{R}^n} \rho_\varepsilon(x - y) \, |d_C(y) - d_C(x)| \, dy$$

$$\leqslant \int_{B(x,\varepsilon)} \rho_\varepsilon(x - y) \, |y - x| \, dy \leqslant \varepsilon \int_{B(x,\varepsilon)} \rho_\varepsilon(x - y) \, dy = \varepsilon.$$

Consequently, $x \in C^\varepsilon$ and so $C \subset C^\varepsilon$. Assume now that $x \in C^\varepsilon$. Then $g_{C,\varepsilon}(x) \leqslant \varepsilon$ and we get

$$d_C(x) - g_{C,\varepsilon}(x) \leqslant \int_{B(x,\varepsilon)} \rho_\varepsilon(x - y) \, |d_C(x) - d_C(y)| \, dy \leqslant \varepsilon.$$

Thus, $d_C(x) \leqslant 2\varepsilon$. Consequently, $x \in C + 2\varepsilon B$ and so $C^\varepsilon \subset C + 2\varepsilon B$. This proves (i).

We now prove (ii). By (i) we get that C^ε is convex since it is the sublevel set of a convex function. As it contains C, it is necessarily unbounded. If $x \in C$, then

$$g_{C,\varepsilon}(x) = \int_{B(x,\varepsilon)} \rho_\varepsilon(x - y) \, d_C(y) \, dy \leqslant \int_{B(x,\varepsilon)} \rho_\varepsilon(x - y) \, |x - y| \, dy < \varepsilon,$$

thus ε is not the minimum value of $g_{C,\varepsilon}$. Consequently, $\nabla g_{C,\varepsilon}(z) \neq 0$ for every $z \in \partial C^\varepsilon$, hence ∂C^ε is smooth.

Item (iii) follows easily from the equalities $d_{w+C}(x) = d_C(x - w)$ and $g_{C,\varepsilon}(x - w) = g_{w+C,\varepsilon}(x)$.

We now prove (iv). We fix $R > 0$ and check that $(C_i)^\varepsilon \cap \overline{B}(0,R)$ converges in Hausdorff distance to $C^\varepsilon \cap \overline{B}(0,R)$. To this aim, we exploit Kuratowski criterion [**79**, Thm. 1.8.7]. First, let $x \in C^\varepsilon \cap \overline{B}(0,R)$. We need to check that x is the limit of a sequence of points in $(C_i)^\varepsilon \cap \overline{B}(0,R)$. If $x \in \mathrm{int}(C^\varepsilon) \cap \overline{B}(0,R)$ then $g_{C,\varepsilon}(x) < \varepsilon$ and thus $g_{C_i,\varepsilon}(x) < \varepsilon$ (henceforth $x \in (C_i)^\varepsilon$) for i large enough. Otherwise we approximate x by a sequence $\{y_j\}_{j\in\mathbb{N}} \subset \mathrm{int}(C^\varepsilon) \cap \overline{B}(0,R)$, then, arguing as above, for any $j \in \mathbb{N}$ we select $i_j \in \mathbb{N}$ with the property that $i_{j+1} > i_j$ for all j and $y_j \in \mathrm{int}(C_i^\varepsilon)$ for all $i \geqslant i_j$. In order to build a sequence of points $x_i \in C_i$ that converge to x, we proceed as follows. First we arbitrarily choose $x_i \in C_i$ for $i = 1, \ldots, i_1 - 1$. Then we set $x_i = y_j$ for all $i_j \leqslant i < i_{j+1}$ (notice that the definition is well-posed, thanks to the fact that $\{i_j\}_{j\in\mathbb{N}}$ is strictly increasing). It is then easy to check that the sequence $\{x_i\}_{i\in\mathbb{N}}$ has the required properties.

Second, let $x_{i_k} \in C_{i_k}^\varepsilon \cap \overline{B}(0,R)$ converge to some point x. Since $g_{C_{i_k},\varepsilon}(x_{i_k}) \leqslant \varepsilon$ and $g_{C_{i_k},\varepsilon}$ uniformly converges in compact sets to $g_{C,\varepsilon}$, we have $g_{C,\varepsilon}(x) \leqslant \varepsilon$ and so $x \in C^\varepsilon$.

Now we prove (v). Let $K \in \mathcal{K}(C^\varepsilon)$. Then there exists a divergent sequence $\{x_i\}_{i \in \mathbb{N}} \subset C^\varepsilon$ so that, by (iii), $-x_i + C^\varepsilon = (-x_i + C)^\varepsilon \to K$ in local Hausdorff distance. Let y_i be the metric projection of x_i onto C. We have $|x_i - y_i| \leqslant 2\varepsilon$ and so the sequence $\{y_i\}_{i \in \mathbb{N}} \subset C$ is divergent. Hence $(-y_i + C)^\varepsilon \to K$ in local Hausdorff distance. Since $-y_i + C$ subconverges to some $K' \in \mathcal{K}(C)$, (iv) implies $(K')^\varepsilon = K$ and so $K \in (\mathcal{K}(C))^\varepsilon$. The proof of the reverse inclusion is similar. $\qquad\square$

5.3. Concavity of the isoperimetric profile

Now we proceed to show in Lemma 5.5 that the function $I_C^{n/(n-1)}$ is concave when C is a convex body of uniform geometry with C^∞ boundary, such that all its asymptotic cylinders have also C^∞ boundary. The general case of C convex but not necessarily smooth will then follow by approximation. Lemma 5.4 is a technical result that will be needed in the proof of Lemma 5.5.

LEMMA 5.4. *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry. Let $K^j \in \mathcal{K}(C) \cup \{C\}$ for $j = 0, \ldots, m$. Then for all $v_0, \ldots, v_m \geqslant 0$ we have*

$$I_C(v_0 + \ldots + v_m) \leqslant \sum_{j=0}^{m} I_{K^j}(v_j).$$

PROOF. For every j, consider a bounded set $E^j \subset K^j$ of volume v_j such that $P_{K^j}(E^j) < I_{K^j}(v_j) + \varepsilon$. Using Proposition 4.3, we get a set $F^j \subset C$ of volume v_j such that $P_C(F^j) < P_{K^j}(E^j) + \varepsilon$. The sets F^j can be taken disjoint. Then we have

$$\begin{aligned} I_C(v_0 + \ldots + v_m) &\leqslant P_C(F^0 \cup \ldots \cup F^m) = P_C(F^0) + \ldots + P_C(F^m) \\ &\leqslant P_{K^0}(E^0) + \ldots + P_{K^m}(E^m) + (m+1)\,\varepsilon \\ &\leqslant I_{K^0}(v_0) + \ldots + I_{K^m}(v_m) + 2\,(m+1)\,\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the result. $\qquad\square$

LEMMA 5.5. *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry with C^∞ boundary. Assume that all its asymptotic cylinders have also C^∞ boundary. Then $I_C^{n/(n-1)}$ is a concave function, hence in particular I_C is strictly concave.*

PROOF. Fix some positive volume $v_0 > 0$. By Theorem 4.6, there exist $m \in \mathbb{N}$, $K^j \in \mathcal{K}(C)$, $1 \leqslant j \leqslant m$, and isoperimetric regions $E^j \subset K^j$, such that

$$(5.4) \qquad \sum_{j=0}^{m} |E^j| = v_0 \quad \text{and} \quad I_C(v_0) = \sum_{j=0}^{m} I_{K^j}(|E^j|),$$

where $K^0 = C$. Denote by S^j the regular part of $\partial_{K^j} E^j$ and assume that S^j is nonempty for all $j = 0, \ldots, m$ (otherwise we may restrict the summation to those indices j such that this property holds true). Since $E^j \subset K^j$ are isoperimetric sets in K^j, a standard first variation argument implies that S^j has constant mean curvature and that S^j intersects orthogonally ∂K^j.

Let us check that all S^j have the same constant mean curvature. Otherwise, there exist S^{j_1}, S^{j_2}, $j_1, j_2 \in \{0, \ldots, m\}$, with different mean curvatures. A standard

first variation argument allows us to deform E^{j_1} and E^{j_2} to get $F^{j_1} \subset K^{j_1}$ and $F^{j_2} \subset K^{j_2}$ satisfying

(5.5)
$$|F^{j_1}| + |F^{j_2}| = |E^{j_1}| + |E^{j_2}|,$$
$$P_{K^{j_1}}(F^{j_1}) + P_{K^{j_2}}(F^{j_2}) < P_{K^{j_1}}(E^{j_1}) + P_{K^{j_2}}(E^{j_2}).$$

Moreover the sets F^{j_1}, F^{j_2} are bounded since they are nice deformations of isoperimetric regions, which are bounded by Proposition 3.21. Letting $F_i = E_i$ when $i \neq j_1, j_2$, we get from (5.4) and (5.5)

$$\sum_{j=0}^{m} |F^j| = v_0 \quad \text{and} \quad I_C(v_0) > \sum_{j=0}^{m} I_{K^j}(|F^j|),$$

Using Proposition 4.3, we can approximate the sets $F^j \subset K^j$ by sets in C of volumes $|F^j|$ and relative perimeters in C as close as we wish to $P_{K^j}(F^j)$. This way we get a finite perimeter set $\Omega \subset C$ so that

$$|\Omega| = v_0 \quad \text{and} \quad I_C(v_0) > P_C(\Omega),$$

yielding a contradiction.

Let us now prove that $I_C^{n/(n-1)}$ is a concave function. We shall do it by deforming an isoperimetric region through a continuous family of competitors with monotone volume, and comparing the perimeters with the corresponding values of the isoperimetric profile. If we knew that the isoperimetric boundaries are regular we could take outer parallels, corresponding to a locally constant test function in the second variation. Because of the presence of singularities, we shall approximate the function 1 on each isoperimetric boundary by a family of cut-off functions vanishing near the singular set.

Since the singular set of E^j has very small Hausdorff dimension, given $\varepsilon > 0$, we cover this set by a finite family of Euclidean balls with radii satisfying $\sum_i r_i^{n-3} < \varepsilon$, and we construct a function φ_ε^j on S^j as a product of piecewise linear functions vanishing on the singular set of $\partial_{K^j} E^j$. These functions satisfy $0 \leqslant \varphi_\varepsilon^j \leqslant 1$, and φ_ε^j converges to the constant function 1 on S^j both pointwise and in the Sobolev norm when $\varepsilon \to 0$. We refer the reader to Lemma 3.1 in [8] for the proof of this standard result.

Fix $\varepsilon > 0$ and consider a C^∞ vector field X_ε^j in K^j obtained by extending the vector field $\varphi_\varepsilon^j N^j$ in S^j, where N^j is the outer unit normal to E^j, to $\mathbb{R}^n$ in such a way that the extension is tangent to ∂K^j. The associated flow $\{\psi_{\varepsilon,t}^j\}_{t \in \mathbb{R}}$ preserves the boundary of K^j. The derivative of the volume for the variation associated with the vector field X_ε^j is equal to

$$\int_{S^j} \varphi_\varepsilon^j \, d\mathcal{H}^{n-1} > 0,$$

and so there exists a function $P_\varepsilon^j(v)$ assigning to v close to $|E^j|$ the perimeter of the set $\psi_{\varepsilon,t(v)}^j(E^j)$ of volume v. Trivially we have

(5.6)
$$I_{K^j}(v) \leqslant P_\varepsilon^j(v).$$

For v close to v_0, we define the function

$$P_\varepsilon(v) = \sum_{j=0}^{m} P_\varepsilon^j(|E^j| + \lambda_j(v - v_0)),$$

where
$$\lambda_j = \frac{\mathcal{H}^{n-1}(S^j)}{\mathcal{H}^{n-1}(S^0) + \ldots + \mathcal{H}^{n-1}(S^m)} = \frac{\mathcal{H}^{n-1}(S^j)}{I_C(v_0)}.$$

Observe that $\sum_{j=0}^m \lambda_j = 1$. Using Lemma 5.4 and (5.6) we have

$$I_C(v)^{n/(n-1)} \leqslant \left(\sum_{j=0}^m I_{K^j}(|E^j| + \lambda_j(v - v_0)) \right)^{n/(n-1)}$$

$$\leqslant \left(\sum_{j=0}^m P_\varepsilon^j(|E^j| + \lambda_j(v - v_0)) \right)^{n/(n-1)} = P_\varepsilon(v)^{n/(n-1)}.$$

By Lemma 5.6 below, it is enough to show

$$\limsup_{\varepsilon \to 0} \left(\frac{d^2}{dv^2}\bigg|_{v=v_0} P_\varepsilon(v)^{n/(n-1)} \right) \leqslant 0$$

to prove the concavity of $I_C^{n/(n-1)}$. Observe that

$$\frac{d^2}{dv^2}\bigg|_{v=v_0} P_\varepsilon(v)^{n/(n-1)} = \left(\frac{n}{n-1} \right) P_\varepsilon^{1/(n-1)}(v_0) \left\{ \frac{1}{n-1} \frac{P_\varepsilon'(v_0)^2}{P_\varepsilon(v_0)} + P_\varepsilon''(v_0) \right\}.$$

Note that

$$P_\varepsilon'(v_0) = \sum_{j=0}^m \lambda_j (P_\varepsilon^j)'(|E^j|) = \left(\sum_{j=0}^m \lambda_j \right) H = H,$$

where H is the common constant mean curvature of S^j for all $j = 0, \ldots, m$, and that

$$P_\varepsilon''(v_0) = \sum_{j=0}^m \lambda_j^2 (P_\varepsilon^j)''(|E^j|)$$

$$= \sum_{j=0}^m \lambda_j^2 \left(\int_{S^j} \varphi_\varepsilon^j \right)^{-2} \left\{ \int_{S^j} |\nabla_{S^j} \varphi_\varepsilon^j|^2 - |\sigma^j|^2 (\varphi_\varepsilon^j)^2 - \int_{\partial S^j} \mathrm{II}^j(N^j, N^j)(\varphi_\varepsilon^j)^2 \right\},$$

where ∇_{S^j} is the gradient in S^j, $|\sigma^j|^2$ is the squared norm of the second fundamental form σ^j of S^j, and II^j is the second fundamental form of ∂K^j. Taking limits when $\varepsilon \to 0$ we get as in [8, (3.7)]

$$\limsup_{\varepsilon \to 0} P_\varepsilon''(v_0) \leqslant -\sum_{j=0}^m \frac{\lambda_j^2}{\mathcal{H}^{n-1}(S^j)^2} \left\{ \int_{S^j} |\sigma^j|^2 + \int_{\partial S^j} \mathrm{II}^j(N^j, N^j) \right\}$$

$$\leqslant -\sum_{j=0}^m \frac{\lambda_j^2}{\mathcal{H}^{n-1}(S^j)^2} \int_{S^j} |\sigma^j|^2.$$

Hence we have

$$\limsup_{\varepsilon \to 0} \left\{ \frac{1}{n-1} \frac{P_\varepsilon'(v_0)^2}{P_\varepsilon(v_0)} + P_\varepsilon''(v_0) \right\} \leqslant \frac{1}{n-1} \frac{H^2}{I_C(v_0)} - \sum_{j=0}^m \frac{\lambda_j^2}{\mathcal{H}^{n-1}(S^j)^2} \int_{S^j} |\sigma^j|^2$$

$$= \sum_{j=0}^m \frac{1}{I_C(v_0)^2} \int_{S^j} \frac{H^2}{n-1} - \sum_{j=0}^m \frac{\lambda_j^2}{\mathcal{H}^{n-1}(S^j)^2} \int_{S^j} |\sigma^j|^2$$

$$= \frac{1}{I_C(v_0)^2} \sum_{j=0}^m \int_{S^j} \left(\frac{H^2}{n-1} - |\sigma^j|^2 \right) \leqslant 0,$$

from the definition of λ_j. This shows that $I_C^{n/(n-1)}$ is a concave function.

Finally we deduce that I_C is strictly concave, being the composition of $I_C^{n/(n-1)}$ with the strictly concave non-increasing function $x \mapsto x^{n/(n+1)}$. $\qquad\square$

LEMMA 5.6. *Let $f : I \to \mathbb{R}$ be a continuous function defined on an open interval $I \subset \mathbb{R}$. Assume that for all $x \in I$ there is a family of smooth functions $(f_{x,\varepsilon})_{\varepsilon>0}$, each one defined in a neighborhood of x, such that $f \leqslant f_{x,\varepsilon}$, $f(x) = f_{x,\varepsilon}(x)$, and $\limsup_{\varepsilon\to 0} f''_{x,\varepsilon}(x) \leqslant 0$. Then f is a concave function.*

PROOF. If f is not concave, then there exists $\delta > 0$ such that the function $f_\delta(x) := f(x) - \delta x^2$ is not concave. To see this, represent the subgraph of f as the closure of the union of the increasing family (when $\delta \to 0$) of the subgraphs of f_δ. If the subgraph of f is not convex, then some of the subgraphs of f_δ are not convex.

As f_δ is not concave, there exist two points $x_1 < x_2$ on I such that the function $L(x) - f_\delta(x)$ has a positive maximum $x_0 \in (x_1, x_2)$. Here $L(x)$ is the linear function passing through $(x_1, f_\delta(x_1))$ and $(x_2, f_\delta(x_2))$. Then each one of the smooth functions $L(x) - f_{x_0,\varepsilon}(x) + \delta x^2$ has a maximum at x_0. Hence $f''_{x_0,\varepsilon}(x_0) \geqslant 2\delta > 0$, contradicting the assumption. $\qquad\square$

LEMMA 5.7. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body of bounded geometry such that $I_K^{n/(n-1)}$ is concave for any $K \in \{C\} \cup \mathcal{K}(C)$. Then, for any prescribed volume $v > 0$, any generalized isoperimetric region in C of volume v consists of a single, connected set E contained in $K \in \{C\} \cup \mathcal{K}(C)$. Moreover, the diameter of E is bounded above by a constant only depending on v and on the constants $n, r_0, b(r_0)$ appearing in (3.3).*

PROOF. *Step one.* We assume by contradiction that for some positive prescribed volume we can find a generalized isoperimetric region $E^0, E^1, \ldots, E^q$ such that $|E^j| > 0$ at least for two distinct indices $j_1 \neq j_2$ in $\{0, \ldots, q\}$. Let $v^j = |E^j|$ and set $v = v^{j_1} + v^{j_2}$. Then combining Proposition 4.3 with Theorem 4.6 (iv) we get

$$(5.7) \qquad I_C(v) = I_{K^{j_1}}(v^{j_1}) + I_{K^{j_2}}(v^{j_2}) \geqslant I_C(v^{j_1}) + I_C(v^{j_2}).$$

On the other hand, the strict concavity of $I_C(v)$ implies strict subadditivity, hence we find

$$I_C(v) < I_C(v^{j_1}) + I_C(v^{j_2}),$$

which is in contradiction with (5.7).

Step two. In order to prove the last part of the statement, we observe that any minimizer $E \subset C$ (or, respectively, $E \subset K$ for some $K \in \mathcal{K}(C)$) for a prescribed volume $v > 0$ must satisfy a uniform density estimate depending only on the dimension n and on the ratio $I_C(v)/v$. Owing to Remark 3.19, we can consider without loss of generality the case $E \subset C$, fix $x \in E$ and, for any $r > 0$, consider the set $E_r = E \setminus B_C(x, r)$. Set $m(r) = |E \cap B_C(x, r)|$ and notice that, by concavity of I_C, one has

$$(5.8) \qquad P_C(E) = I_C(v) \leqslant I_C(v - m(r)) + \frac{I_C(v)}{v} m(r) \leqslant P_C(E_r) + \frac{I_C(v)}{v} m(r).$$

On the other hand, for almost all $0 < r < r_0$ such that $m(r) \leqslant |B_C(x, r)|/2$, owing to the relative isoperimetric inequality (3.9) one has

$$(5.9) \qquad \begin{aligned} P_C(E_r) &= P_C(E) - P_C(E \cap B_C(x, r)) + 2m'(r) \\ &\leqslant P_C(E) - M \, m(r)^{(n-1)/n} + 2m'(r). \end{aligned}$$

Hence combining (5.8) and (5.9) we get

$$
(5.10) \qquad M\, m(r)^{(n-1)/n} - \frac{I_C(v)}{v} m(r) \leqslant 2m'(r)\,.
$$

Now, assume that for some $0 < r_1 < r_0$ we have $m(r_1/2) > 0$ and $m(r_1) \leqslant |B_C(x, r_1/2)|/2$, then of course we have $m(r) \leqslant m(r_1) \leqslant |B_C(x, r)|/2$ for all $r_1/2 < r < r_1$, hence (5.10) holds true for almost all $r \in (r_1/2, r_1)$. Moreover, up to choosing r_1 small enough depending on the ratio $I_C(v)/v$ and on the isoperimetric constant $M > 0$ appearing in (5.10) (we recall that, by Lemma 3.16, M depends only on $n, r_0, b(r_0)$), we can entail that

$$
(5.11) \qquad \frac{M}{4} \leqslant \frac{m'(r)}{m(r)^{(n-1)/n}} \qquad \text{for almost all } r \in (r_1/2, r_1)\,.
$$

Arguing as in the proof of Proposition 3.21, i.e. integrating (5.11) between $r_1/2$ and r_1, we conclude that

$$
(5.12) \qquad m(r_1) \geqslant \left(\frac{M}{8n}\right)^n r_1^n\,,
$$

i.e., that a uniform lower bound for the volume of E holds in balls of radius smaller than r_0 centered at Lebesgue points of the characteristic function of E. Note that this lower bound is uniform, while the one given by Proposition 3.21 is a-priori dependent on the set E.

   Finally, by combining (5.12) with the connectedness of E proved in step one, we eventually obtain a uniform lower bound on the diameter of E. To prove this we fix a maximal family $\mathcal{B}$ of disjoint balls of radius $r = r_0/2$ centered at Lebesgue points of the characteristic function of E, so that the union of the concentric balls with radius $2r$ covers E. By (5.12) we obtain $|E \cap B| > c_0$ for any $B \in \mathcal{B}$ and for a constant c_0 only depending on $n, r_0, b(r_0)$ and v. Consequently the cardinality of $\{\mathcal{B}\}$ cannot exceed v/c_0. On the other hand the union of the concentric balls with radius $2r$ must be connected (otherwise E would be disconnected) and thus the diameter of E is necessarily bounded by the sum of the diameters of these balls, i.e., by $4r_0 v/c_0$. □

   THEOREM 5.8. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body of uniform geometry. Then $I_C^{n/(n-1)}$ is concave. Moreover, any generalized isoperimetric region for $I_C(v)$ is associated with a connected set $E \subset K$ with $|E| = v$, for $K \in \{C\} \cup \mathcal{K}(C)$ suitably chosen, so that it holds $P_K(E) = I_K(v) = I_C(v)$.*

   PROOF. We notice that, by the assumption on C, any $K \in \mathcal{K}(C)$ is an unbounded convex cylinder of uniform geometry. According to Lemma 5.3 we approximate C by a sequence of smooth unbounded convex bodies C^i converging to C in global Hausdorff distance as $i \to \infty$. It is then immediate to prove that C^i is of uniform geometry. Moreover, thanks to Lemma 5.3, any $K \in \mathcal{K}(C^i)$ has smooth boundary and is of uniform geometry. Therefore we deduce by Lemma 5.5 that $I_K^{n/(n-1)}$ is concave for all $K \in \{C^i\} \cup \mathcal{K}(C^i)$ and for all i.

   To deduce the concavity of $I_C^{n/(n-1)}$, it is enough to show that $\lim_{i \to \infty} I_{C^i}(v) = I_C(v)$ for all $v > 0$. Owing to Proposition 4.4, it remains to prove the lower semicontinuity of the isoperimetric profile, i.e. that

$$
(5.13) \qquad \liminf_{i \to \infty} I_{C^i}(v) \geqslant I_C(v)\,.
$$

To this aim, by Lemma 5.7 we find $K^i \in C^i \cup \mathcal{K}(C^i)$ and $E^i \subset K^i$ with $|E^i| = v$, such that $P_{K^i}(E^i) = I_{C^i}(v)$ for all i. By Remark 3.19 and Lemma 5.7, the diameter of E^i is uniformly bounded by some uniform constant $d > 0$, hence we can assume that, up to translations, $E^i \subset B(0, d)$ for all i. Up to subsequences, the corresponding translates of K^i converge in local Hausdorff sense to a limit convex set K that necessarily belongs to $\{C\} \cup \mathcal{K}(C)$ up to a translation, see Lemmata 3.4 and 3.5. By the uniform boundedness of the perimeters of the E^i's (see Remark 3.10), up to a further extraction of a subsequence we can assume that E^i converge to a limit set $E \subset K \cap B(0, d)$ in L^1, whence $|E| = v$. By Lemma 4.1, we deduce that $P_K(E) \leqslant \liminf_{i \to \infty} P_{K^i}(E^i)$, hence by the inequality $I_C(v) \leqslant I_K(v)$ (see Proposition 4.3) we finally deduce (5.13). This concludes the proof of the theorem. $\qquad\square$

COROLLARY 5.9. *Let $C \subset \mathbb{R}^n$ be a convex body and $\lambda < 1$. Then $I_C(v) \geqslant I_{\lambda C}(v)$*

PROOF. According to (2.11), the renormalized isoperimetric profile $Y_C = I_C^{n/(n-1)}$ satisfies

$$Y_{\lambda C}(v) = \lambda^{n+1} Y_C\left(\frac{v}{\lambda^{n+1}}\right).$$

Then, as Y_C is concave with $Y_C(0) = 0$ and $\lambda < 1$, the proof follows. $\qquad\square$

CHAPTER 6

Sharp isoperimetric inequalities and isoperimetric rigidity

In section 3 some upper estimates of the isoperimetric profile like $I_C \leqslant I_{C_{\min}}$, where $C_{\min}$ is the tangent cone with the smallest aperture in $\{C\} \cup \mathcal{K}(C)$, or the weaker $I_C \leqslant I_H$, where H is a closed half-space, have been proved (see Proposition 3.8 and Remark 3.10). In this section, we shall obtain some new isoperimetric inequalities, like $I_C \geqslant I_{C_\infty}$ (see Theorem 6.3) for a convex body C with non-degenerate asymptotic cone C_∞, recall known ones, and prove rigidity results for the equality cases.

6.1. Convex bodies with non-degenerate asymptotic cone

We begin by studying the relative isoperimetric problem in convex bodies with non-degenerate asymptotic cone (see Chapter 2 for the corresponding definition). We shall first need some notation.

We denote by $\mathcal{C}_0^n$ the family of unbounded convex bodies in $\mathbb{R}^n$ that contain the origin and have non-degenerate asymptotic cone. We regard this set equipped with the topology of local convergence in Hausdorff distance. Let $\Gamma_k^n, 0 \leqslant k \leqslant n$, be the set of convex bodies $C \in \mathcal{C}_0^n$ so that $C = \tilde{C} \times \mathbb{R}^{n-k}$ up to an isometry, where $\tilde{C} \subset \mathbb{R}^k$ is a line-free convex body, or just the origin in the case $k = 0$. Observe that $\Gamma_0^n = \{\mathbb{R}^n\}$ and that $\Gamma_1^n = \{H : H \text{ is a half-space}\}$. The latter identity is easy to prove since $\tilde{C}$ in the above decomposition is 1-dimensional and line-free, hence a half-line or a segment. However the case of the segment is excluded, as slabs in $\mathbb{R}^n$ have degenerate asymptotic cones. We also notice that Γ_n^n is the set of line-free convex bodies of $\mathbb{R}^n$ with non-degenerate asymptotic cone.

Let $C \subset \mathbb{R}^n$ be an unbounded convex body. Note that if C contains a line L through the origin, then $C = \tilde{C} \oplus L$. Therefore a sequence of translations of C by a divergent sequence of points belonging to L converges to C, hence C belongs to the space $\mathcal{K}(C)$ of its asymptotic cylinders. This implies that for all $k = 0, \ldots, n$ and $C \in \Gamma_k^n$ one has $C \in \mathcal{K}(C)$.

Given $C \in \mathcal{C}_0^n$ there exist two orthogonal projections $\pi_C, \pi_C^\perp$ such that $V_C = \pi_C(C)$ is an $(n-k)$-dimensional linear space, $\pi_C^\perp(C)$ is either $\{0\}$ when $k = 0$, or a line-free k-dimensional convex body contained in the orthogonal complement $V^\perp$ of V_C, and $C = V_C \oplus \pi_C^\perp(C)$. If C is line-free then there holds $V_C = \{0\}$, thus $\pi_C(x) = 0$ and $\pi_C^\perp(x) = x$ for all $x \in C$.

Now we set for $0 \leqslant m \leqslant n$

(6.1)
$$\mathcal{C}_{0,m}^n = \bigcup_{k=0}^m \Gamma_k^n$$

and note that $\mathcal{C}_{0,n}^n = \mathcal{C}_0^n$. The family $\mathcal{C}_{0,m}^n$ consists of those sets in $\mathcal{C}_0^n$ possessing a Euclidean factor of dimension larger than or equal to $(n-m)$. We shall prove in Theorem 6.3 that the isoperimetric profile I_C of a convex body C with non-degenerate asymptotic cone C_∞ is bounded below by I_{C_∞}. Moreover, in Theorem 6.14 we show a rigidity result for the equality case in $I_C \geqslant I_{C_\infty}$, which states that if $I_C(v_0) = I_{C_\infty}(v_0)$ for some $v_0 > 0$, then C and C_∞ are isometric. We also prove that $I_C(v)$ and $I_{C_\infty}(v)$ are asymptotic for $v \to +\infty$, and that isoperimetric regions exist for large volumes. First we need the following two lemmas.

LEMMA 6.1.
  (i) *The set $\mathcal{C}_{0,m}^n$ is closed under local convergence in Hausdorff distance for every $0 \leqslant m \leqslant n$.*
  (ii) *For every $k = 0, \ldots, n$, if $C \in \Gamma_k^n$ then $C_\infty \in \Gamma_k^n$, where C_∞ is the asymptotic cone of C.*
  (iii) *Let $C \in \Gamma_m^n$ for some $m \in \{1, \ldots, n\}$, and let $x_i \in C$, $i \in \mathbb{N}$. Suppose that $-x_i + C \to K$ locally in Hausdorff distance. If $\pi_C^\perp(x_i) \to x \in C$ as $i \to \infty$ then $K = -x + C$. If $\pi_C^\perp(x_i)$ diverges, then $K \in \mathcal{C}_{0,m-1}^n$.*

PROOF. To show (i), let $K_i \in \mathcal{C}_{0,m}^n$ and assume that $K_i \to K$ locally in Hausdorff distance. Since the family of linear spaces $V_i = V_{K_i}$ is relatively compact in the local Hausdorff topology, and since $\dim(V_i) \geqslant n - m$ for all i, we conclude that K must contain a local Hausdorff limit of (a subsequence of) $\{V_i\}_{i \in \mathbb{N}}$, which is necessarily a linear space of dimension at least $n - m$. This shows that $K \in \mathcal{C}_{0,m}^n$, as wanted.

We now prove (ii). Up to a rigid motion of $\mathbb{R}^n$, we may assume that $C = \tilde{C} \times \mathbb{R}^{n-k}$, where $\tilde{C}$ contains no lines. It is easy to check that $C_\infty = \tilde{C}_\infty \times \mathbb{R}^{n-k}$, where $\tilde{C}_\infty$ is the asymptotic cone of $\tilde{C}$. Since $\tilde{C}$ is line-free and $\tilde{C}_\infty \subset \tilde{C}$, then $\tilde{C}_\infty$ is line-free. Thus $C_\infty \in \Gamma_k^n$.

Finally we prove (iii). Since $-\pi_C(x_i) + C = C$ for all $i \in \mathbb{N}$, we get that $-\pi_C^\perp(x_i) + C \to K$. Thus, if $\{\pi_C^\perp(x_i)\}_{i \in \mathbb{N}}$ converges to $x \in C$, then $K = -x + C$. Assume now that $\{\pi_C^\perp(x_i)\}_{i \in \mathbb{N}}$ diverges and that $\pi_C^\perp(x_i)/|\pi_C^\perp(x_i)|$ converges to some vector $v \in \mathbb{S}^n$. Since $\{\pi_C^\perp(x_i)\}_{i \in \mathbb{N}} \subset C \cap V_C^\perp$, the argument in the proof of Lemma 3.2 yields that $\{tv : t \in \mathbb{R}\} \subset K \cap V_C^\perp$. Since $C \in \Gamma_m^n$ we get $-\pi_C^\perp(x_i) + C \in \Gamma_m^n$. Thus by (ii) we get $K \in \mathcal{C}_{0,m}^n$. Since $K \cap V_C^\perp$ contains a line we finally get $K \in \mathcal{C}_{0,m-1}^n$. $\square$

LEMMA 6.2. *Let $C \in \Gamma_m^n$, $m \in \{1, \ldots, n\}$, with asymptotic cone C_∞. Let $\{\lambda_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers such that $\lambda_i \downarrow 0$ and take $y_i \in \lambda_i C$ for all $i \in \mathbb{N}$.*
  (i) *If $-y_i + \lambda_i C \to M$ locally in Hausdorff distance, then $C_\infty \subset M_\infty$ and, furthermore, $-\pi_C^\perp(y_i) + \lambda_i C \to M$ locally in Hausdorff distance. In case $\{\pi_C^\perp(y_i)\}_{i \in \mathbb{N}}$ subconverges to $y \in C_\infty$, then $M = -y + C_\infty$. If $\{\pi_C^\perp(y_i)\}_{i \in \mathbb{N}}$ diverges, then $M \in \mathcal{C}_{0,m-1}^n$ and one has the strict inequality*

$$(6.2) \qquad\qquad I_{M_\infty}(v) > I_{C_\infty}(v), \quad \text{for all } v > 0.$$

  (ii) *Assume that $K_i \in \mathcal{K}(\lambda_i C)$, that $K_i \in \mathcal{C}_{0,m-1}^n$, and that $K_i \to K$ locally in Hausdorff distance. Then $K \in \mathcal{C}_{0,m-1}^n$, $C_\infty \subset K_\infty$, and*

$$(6.3) \qquad\qquad I_{K_\infty}(v) > I_{C_\infty}(v), \quad \text{for all } v > 0.$$

PROOF. Let $C = \tilde{C} \times V$, where $V \subset \mathbb{R}^n$ is a linear subspace of dimension $(n - m)$ and $\tilde{C} \subset V^\perp$ is a line-free convex set.

We first prove (i). Since C_∞ is the asymptotic cone of $\lambda_i C$ for all i, and $y_i + C_\infty \subset \lambda_i C$ for all $i \in \mathbb{N}$, we have $C_\infty \subset -y_i + \lambda_i C$, and so $C_\infty \subset M$. As M_∞ is the largest convex cone with vertex 0 included in M, we have $C_\infty \subset M_\infty$.

Setting $y_i = \pi_C(y_i) + \pi_C^\perp(y_i)$, and noticing that $-\pi_C(y_i) + \lambda_i C = \lambda_i C$ for all $i \in \mathbb{N}$, we have that $-\pi_C^\perp(y_i) + \lambda_i C$ converges to M locally in Hausdorff distance, as $i \to \infty$. If $\pi_C^\perp(y_i)$ subconverges to y then $M = -y + C$ and, as $\pi_C^\perp(y_i) \in \lambda_i C \cap V^\perp$, we have $y \in C_\infty \cap V^\perp \subset C_\infty$. Assume now that $\pi_C^\perp(y_i)$ diverges. Eventually passing to a subsequence, we can assume that $\frac{\pi_C^\perp(y_i)}{|\pi_C^\perp(y_i)|} \to v$. By Lemma 3.1 the line $L(v)$ is contained in $M \cap V^\perp$. Since $C \in \Gamma_m^n$ then $-\pi_C^\perp(y_i) + \lambda_i C \in \Gamma_m^n$. Thus by Lemma 6.1(i) we get $M \in \mathcal{C}_{0,m}^n$ and, since $M \cap V^\perp$ contains a line, we obtain that $M \in \mathcal{C}_{0,m-1}^n$. Lemma 6.1(ii) then implies that $M_\infty \in \mathcal{C}_{0,m-1}^n$. Since $C \in \Gamma_m^n$ by hypothesis, Lemma 6.1(ii) implies that $C_\infty \in \Gamma_m^n$. Hence the inclusion $C_\infty \subset M_\infty$ is strict, consequently $\alpha(M_\infty) > \alpha(C_\infty)$. Thus by (2.13) we get $I_{M_\infty}(v) > I_{C_\infty}(v)$ for every $v > 0$.

We now prove (ii). Note that each K_i is a local limit in Hausdorff distance of translations of $\lambda_i C$ and $C_\infty \subset \lambda_i C$ for all $i \in \mathbb{N}$. Then $C_\infty \subset K_i$ for all $i \in \mathbb{N}$. Since $K_i \to K$ locally in Hausdorff distance, we get that $C_\infty \subset K$. As K_∞ is the largest convex cone included in K we have $C_\infty \subset K_\infty$. Since $K_i \in \mathcal{C}_{0,m-1}^n$ and $K_i \to K$ locally in Hausdorff distance, Lemma 6.1(i) implies that $K \in \mathcal{C}_{0,m-1}^n$ and, by Lemma 6.1(ii), we have $K \in \mathcal{C}_{0,m-1}^n$. Arguing as in the proof of (i), we get $C_\infty \in \Gamma_m^n$ and thus obtain the strict inclusion $C_\infty \subset K_\infty$, which gives (6.3) at once. $\qquad \square$

THEOREM 6.3. *Let C be a convex body with non-degenerate asymptotic cone C_∞. Then*

$$\text{(6.4)} \qquad I_C(v) \geqslant I_{C_\infty}(v) \qquad \text{for all } v > 0$$

and

$$\text{(6.5)} \qquad \lim_{v \to \infty} \frac{I_C(v)}{I_{C_\infty}(v)} = 1.$$

Moreover, isoperimetric regions exist in C for sufficiently large volumes, and any sequence of isoperimetric regions with volumes tending to infinity converges up to a rescaling to a geodesic ball centered at a vertex in the asymptotic cone C_∞.

PROOF. *Step one.* We first show some inequalities involving the isoperimetric profiles of C, C_∞, and of rescalings of C, as well as a uniform bound on the diameter of generalized isoperimetric regions. We fix $v > 0$ and consider a sequence $\lambda_i \downarrow 0$. Since $\lambda_i C \to C_\infty$ locally in Hausdorff distance, Proposition 4.4 implies

$$\text{(6.6)} \qquad \limsup_{i \to \infty} I_{\lambda_i C}(v) \leqslant I_{C_\infty}(v)$$

for any $v > 0$. Owing to Corollary 5.9, inequalities $I_{\lambda_i C} \leqslant I_C$ hold for large i, and taking limits we obtain

$$\text{(6.7)} \qquad \limsup_{i \to \infty} I_{\lambda_i C}(v) \leqslant I_C(v)$$

for all $v > 0$.

Let E_i be generalized isoperimetric regions in $\lambda_i C$ of volume v. Recall that the sets E_i are connected by Theorem 5.8. We shall prove that E_i have uniformly

bounded diameter. Fix $r_0 > 0$ and set

$$\mathcal{L} = \bigcup_{0 < \lambda \leqslant 1} \{\lambda C\} \cup \mathcal{K}(\lambda C).$$

By Lemma 5.7 and Theorem 5.8, it suffices to show that

(6.8) $$\inf_{L \in \mathcal{L}, x \in L} |B_L(x, r_0)| > 0.$$

Fix $\lambda > 0$ and let $L \in \{\lambda C\} \cup \mathcal{K}(\lambda C)$. Recall that C_∞ is the asymptotic cone of λC, and so $C_\infty \subset \lambda C$. The set L is a local Hausdorff limit of translations of λC. Hence there is a (possibly diverging) sequence $\{z_i\}_{i \in \mathbb{N}} \subset \lambda C$ so that $-z_i + \lambda C \to L$ locally in Hausdorff distance. Since $C_\infty \subset -z_i + \lambda_i C$ we get $C_\infty \subset L$. Now let $x \in L$. As $x + C_\infty \subset L$, we get $\overline{B}_{x+C_\infty}(x, r) \subset \overline{B}_L(x, r)$. Since C_∞ is non-degenerate, we can pick $\delta > 0$ and $y \in C_\infty$ so that $B(y, \delta) \subset \overline{B}_{C_\infty}(0, r_0)$. Hence

$$B(x + y, \delta) \subset \overline{B}_{x+C_\infty}(x, r_0) \subset \overline{B}_L(x, r),$$

which gives (6.8) at once, and shows that E_i has a uniformly bounded diameter.

Step two. We prove (6.4) by an induction argument. Note that the elements of $\mathcal{C}_{0,1}^n$ are only $\mathbb{R}^n$ and half-spaces, hence (6.4) holds trivially in this case. Fix $2 \leqslant m \leqslant n$. Assuming the validity of (6.4) for every set in $\mathcal{C}_{0,m-1}^n$ we shall prove that it holds for any set in Γ_m^n. Then the proof would follow from (6.1). So we assume that

(6.9) $$I_K \geqslant I_{K_\infty} \quad \text{for every } K \in \mathcal{C}_{0,m-1}^n.$$

Take $C \in \Gamma_m^n$ and consider a sequence of generalized isoperimetric regions E_i in $\lambda_i C$ of volume $v > 0$, as in Step one. The sets E_i are connected with uniformly bounded diameter. We shall distinguish three cases.

Case 1: Assume that $E_i \subset \lambda_i C$ and that $\{E_i\}_{i \in \mathbb{N}}$ is uniformly bounded. By Lemma 4.1, there exists a finite perimeter set $E \subset C_\infty$ of volume v so that

(6.10) $$I_{C_\infty}(v) \leqslant P_{C_\infty}(E) \leqslant \liminf_{i \to \infty} P_{\lambda_i C}(E_i) = \liminf_{i \to \infty} I_{\lambda_i C}(v).$$

Owing to (6.7) and (6.10) we get $I_{C_\infty}(v) \leqslant I_C(v)$ and this concludes the proof of (6.4) in this case.

Case 2: Assume that $E_i \subset \lambda_i C$ and that $\{E_i\}_{i \in \mathbb{N}}$ diverges. Let $x_i \in E_i$ and observe that, by step one, $-x_i + E_i$ is uniformly bounded. If $\pi_C^\perp(x_i)$ converges, then we get that $-x_i + \lambda_i C \to -x + C_\infty$ for some $x \in C_\infty$ and, by noticing that $I_{-x+C_\infty} = I_{C_\infty}$, this sub-case is reduced to the previous one. So we assume that $\pi_C^\perp(x_i)$ diverges. Passing to a subsequence we may assume that $-x_i + \lambda_i C \to K$ locally in Hausdorff distance. By Lemma 6.2(ii) we have $K \in \mathcal{C}_{0,m-1}^n$. Now by Lemma 4.1 there exists a finite perimeter set $E \subset K$ so that

(6.11) $$I_K(v) \leqslant P_K(E) \leqslant \liminf_{i \to \infty} P_{(-x_i + \lambda_i C)}(-x_i + E_i)$$
$$= \liminf_{i \to \infty} P_{\lambda_i C}(E_i) = \liminf_{i \to \infty} I_{\lambda_i C}(v).$$

Then (6.9), (6.11) and (6.2) imply

$$I_{C_\infty}(v) < I_{K_\infty}(v) \leqslant I_K(v) < \liminf_{i \to \infty} I_{\lambda_i C}(v)$$

which gives a contradiction with (6.6). So Case 2 cannot hold.

Case 3. We assume that $E_i \subset K_i$, where K_i is an asymptotic cylinder of $\lambda_i C$. From Lemma 6.1(iii) we have the following alternative for each i: either K_i is a translation of $\lambda_i C$, or $K_i \in \mathcal{C}_{0,m-1}^n$. If the first possibility holds for infinitely many i,

we can treat this case as in the previous ones. If the second holds for infinitely many i, passing to a subsequence we may assume that K is the local limit in Hausdorff distance of the sequence K_i. Then Lemma 6.1(ii) implies that $K \in \mathcal{C}_{0,m-1}^n$ and (6.9) implies

$$(6.12) \qquad\qquad I_{K_\infty} \leqslant I_K.$$

Since the sets $\{E_i\}_{i \in \mathbb{N}}$ have uniformly bounded diameter, Lemma 4.1 implies the existence of a finite perimeter set $E \subset K$ with volume v so that

$$(6.13) \qquad I_K(v) \leqslant P_K(E) \leqslant \liminf_{i \to \infty} P_{K_i}(E_i) = \liminf_{i \to \infty} I_{K_i}(v) = \liminf_{i \to \infty} I_{\lambda_i C}(v).$$

Then, (6.12), (6.13) and (6.3) imply

$$I_{C_\infty}(v) < \liminf_{i \to \infty} I_{\lambda_i C}(v),$$

which gives a contradiction with (6.6) showing that Case 3 cannot hold.

We now prove (6.5). Since C_∞ is the asymptotic cone of each $\lambda_i C$ then (6.4) holds for every $\lambda_i C, i \in \mathbb{N}$. Taking limits we conclude

$$I_{C_\infty}(v) \leqslant \liminf_{i \to \infty} I_{\lambda_i C}(v).$$

Thus, by (6.6), we get

$$(6.14) \qquad\qquad I_{C_\infty}(v) = \lim_{i \to \infty} I_{\lambda_i C}(v).$$

From (6.14), Lemma 2.10 and the fact that C_∞ is a cone we deduce

$$1 = \lim_{\lambda \to 0} \frac{I_{\lambda C}(1)}{I_{C_\infty}(1)} = \lim_{\lambda \to 0} \frac{\lambda^n I_C(1/\lambda^n)}{\lambda^n I_{C_\infty}(1/\lambda^n)} = \lim_{v \to \infty} \frac{I_C(v)}{I_{C_\infty}(v)},$$

which shows (6.5).

Step three. We prove the existence of isoperimetric regions for large volumes. We argue by contradiction assuming that there exists a sequence $v_i \uparrow \infty$ such that no generalized isoperimetric region of volume v_i is realized in C. If $v > 0$ is fixed and $\lambda_i = (v/v_i)^{1/n}$. Then no generalized isoperimetric regions for volume v are realized in $\lambda_i C$, that is, the generalized isoperimetric regions of $\lambda_i C$ are contained in asymptotic cylinders of $\lambda_i C$. Arguing exactly as in Case 3 of Step two, we get a contradiction.

Step four. We show the last part of the statement, i.e., that suitable rescalings of sequences of isoperimetric regions with diverging volumes must converge to isoperimetric regions in C_∞, i.e. to geodesic balls centered at the vertex of C_∞, owing to the results in [**29**]. To this end we argue again by contradiction. By virtue of Lemma 4.1, we can assume there exists $v > 0$ so that a sequence $E_i \subset \lambda_i C$ of isoperimetric regions of volume v diverges. Then arguing exactly as in Case 2 of Step two we get a contradiction. As a consequence, only Case 1 in Step two holds. Then the sets E_i are uniformly bounded and thus every subsequence of E_i converges to some E, where $E \subset C_\infty$ is an isoperimetric region by (6.6) and (6.7), hence it must be a geodesic ball centered at some vertex of C_∞. $\qquad\square$

From (6.5) and (2.13) we easily get

COROLLARY 6.4. *Let $C, C' \subset \mathbb{R}^n$ be unbounded convex bodies with non-degenerate asymptotic cone satisfying $\alpha(C_\infty) > \alpha(C'_\infty)$. Then for $v > 0$ sufficiently large we have $I_C(v) > I_{C'}(v)$.*

REMARK 6.5. Let $C \subset \mathbb{R}^n$ be an unbounded convex body. By (6.4), (2.12), and the fact that C_∞ is the largest cone included in C, we get that if C contains a solid convex cone K, then $I_C \geqslant I_K$.

REMARK 6.6. Let $C \subset \mathbb{R}^n$ be an unbounded convex body with non-degenerate asymptotic cone C_∞. Theorem 6.3 and (2.13) imply that the isoperimetric dimension of C is n. Furthermore

$$I_{C_\infty}(1) = \sup\{a > 0 : \ I_C(v) \geqslant av^{(n-1)/n} \text{ for every } v > 0\}$$

or, equivalently,

$$I_{C_\infty}(1) = \inf_{v > 0} \frac{I_C(v)}{v^{(n-1/n)}}.$$

6.2. The isoperimetric profile for small volumes

Theorem 6.9 below states that the isoperimetric profile of an unbounded convex body of uniform geometry is asymptotic to $I_{C_{\min}}$ for small volumes. In the case of a bounded convex body this result was stated and proved in [**72**, Thm. 6.6]. To accomplish this we shall need the following Lemma.

LEMMA 6.7. *Let $\{L_i\}_{i \in \mathbb{N}}$ be a sequence of convex bodies converging locally in Hausdorff distance to a convex body L. Assume that $0 \in L_i$ for all $i \in \mathbb{N}$. Let λ_i be a sequence of positive real numbers diverging to $+\infty$ and assume that $\lambda_i L_i$ converges locally in Hausdorff distance to an unbounded convex body M. Then $L_0 \subset M$.*

PROOF. As $L_0 = \mathrm{cl}(\cup_{\lambda > 0} \lambda L)$ and M is a closed set, it is enough to prove that $\cup_{\lambda > 0} \lambda L \subset M$. Take a point $z \in \cup_{\lambda > 0} \lambda L$, and also $\lambda > 0$, $z' \in L$ so that $z = \lambda z'$. Since $L_i \to L$ locally in Hausdorff distance, there exists a sequence $z_i \in L_i$ so that $z_i \to z'$. Consequently

$$(6.15) \qquad z = \lambda z' = \lim_{i \to \infty} \lambda z_i = \lim_{i \to \infty} \lambda_i \left(\frac{\lambda}{\lambda_i} z_i\right).$$

Since λ_i is a diverging sequence, inequality $\frac{\lambda}{\lambda_i} < 1$ holds for i large enough. Since $z_i \in L_i$ and the sets L_i are convex and contain the origin, we have $\frac{\lambda}{\lambda_i} z_i \in L_i$. By (6.15) and the local convergence in Hausdorff distance of $\lambda_i L_i$ to M we conclude that $z \in M$. $\qquad \square$

EXAMPLE 6.8. In general the set M is not a cone and can be different from L_0. Take $L := [0,1]^3 \subset \mathbb{R}^3$, $x_i := (i^{-1}, 0, 0)$ and $L_i := -x_i + L$. Then $0 \in L_i$ for all $i \in \mathbb{N}$ and $L_i \to L$ in Hausdorff distance. However, if we take $\lambda_i := i$, then $\lambda_i L_i$ converges locally in Hausdorff distance to the set $[-1, +\infty) \times [0, +\infty) \times [0, +\infty)$, different from $L_0 = [0, +\infty)^3$.

THEOREM 6.9. *Let C be a convex body (if unbounded, we further assume that it is of uniform geometry). Then $I_{C_{\min}}(v) > 0$ for all $v \in (0, |C|)$ and*

$$(6.16) \qquad \lim_{v \to 0} \frac{I_C(v)}{I_{C_{\min}}(v)} = 1.$$

Moreover, any sequence of generalized isoperimetric regions with volumes tending to zero subconverges to a point either in C or in some $K \in \mathcal{K}(C)$, where the minimum of the solid angle function is attained.

PROOF. First observe that $I_{C_{\min}}(v) > 0$ for all $v \in (0, |C|)$. This is trivial in the bounded case and, in the unbounded uniform geometry case, it follows from Proposition 3.14 and (2.13).

Let $v_i \downarrow 0$ and E_i be generalized isoperimetric regions of volumes v_i in C, for $i \in \mathbb{N}$. Let $\lambda_i \uparrow \infty$ so that, for all $i \in \mathbb{N}$, $|\lambda_i E_i| = 1$. Then $\lambda_i E_i$ are generalized isoperimetric regions of volume 1 in $\lambda_i C$. Recall that the sets E_i are connected by Theorem 5.8.

First we prove that the sets $\lambda_i E_i$ have uniformly bounded diameter. Fix $r_0 >$ and set

$$\mathcal{M} = \bigcup_{\lambda \geqslant 1} \{\lambda C\} \cup \mathcal{K}(\lambda C).$$

By Lemma 5.7 and Theorem 5.8, it suffices to prove that

$$(6.17) \qquad \inf_{M \in \mathcal{M}, x \in M} |B_M(x, r_0)| > 0.$$

Fix $\lambda > 1$ and let $M \in \{\lambda C\} \cup \mathcal{K}(\lambda C)$. Since C is bounded or of uniform geometry, Proposition 3.14 yields

$$b(r_0) = \inf_{x \in C} |B_C(x, r_0)| > 0.$$

If $K \in \mathcal{K}(C)$ then it is a local limit in Hausdorff distance of translations of C. Hence

$$(6.18) \qquad \inf_{x \in K} |B_K(x, r_0)| \geqslant b(r_0).$$

If $M \in \mathcal{K}(\lambda C)$, with $\lambda > 1$, then $M = \lambda K$ for some $K \in \mathcal{K}(C)$. Since M is convex then $h_{x, \lambda^{-1}}(M) \subset M$, for every $x \in M$. Consequently

$$(6.19) \qquad |B_M(x, r_0)| \geqslant |B(x, r_0) \cap h_{x, \lambda^{-1}}(M)|$$

As $h_{x, \lambda^{-1}}(M)$ is isometric to K then (6.18) and (6.19) imply

$$\inf_{x \in M} |B_M(x, r_0)| \geqslant b(r_0).$$

This concludes the proof of (6.17)

Since $\{\lambda_i E_i\}_{i \in \mathbb{N}}$ has uniformly bounded diameter we shall distinguish two cases.

Case 1. Assume that E_i is contained in $K_i \in \mathcal{K}(C)$ for infinitely many indices i. Possibly passing to a subsequence we may assume that $K_i \to K$ and $\lambda_i K_i \to K'$ in local Hausdorff distance. Applying Lemma 3.4(ii) for the particular case $C_i = C$ we get that $K \in \mathcal{K}(C)$. By Lemma 3.5 we may assume that $0 \in E_i$ for all i. As $\{\operatorname{diam} \lambda_i E_i\}_{i \in \mathbb{N}}$ is uniformly bounded, Lemma 4.1 yields a finite perimeter set $E \subset K'$, with $|E| = 1$, such that

$$(6.20) \qquad I_{K'}(1) \leqslant P_{K'}(E) \leqslant \liminf_{i \to \infty} P_{\lambda_i K_i}(\lambda_i E_i).$$

Now by Lemma 6.7 and Remark 6.5 we get

$$(6.21) \qquad I_{K_0} \leqslant I_{K'},$$

where K_0 is the tangent cone of K at 0. Since $\lambda_i E_i \subset \lambda_i K_i$ are isoperimetric regions of volume 1, we get by (6.20) and (6.21),

$$(6.22) \qquad I_{K_0}(1) \leqslant \liminf_{i \to \infty} I_{\lambda_i K_i}(1).$$

Owing to (6.22), (2.13), Lemma 2.10, the fact that $I_{K_i}(v_i) = I_C(v_i)$, and that $\lambda K_0 = K_0$ (recall that $v_i = 1/\lambda_i^n$) we obtain

$$\liminf_{i\to\infty} \frac{I_C(v_i)}{I_{K_0}(v_i)} = \liminf_{i\to\infty} \frac{\lambda_i^n I_C(1/\lambda_i^n)}{\lambda_i^n I_{K_0}(1/\lambda_i^n)}$$
$$= \liminf_{i\to\infty} \frac{I_{\lambda_i C}(1)}{I_{K_0}(1)} = \liminf_{i\to\infty} \frac{I_{\lambda_i K_i}(1)}{I_{K_0}(1)} \geqslant 1.$$

Owing to Remark 3.7 we have

$$I_{C_{\min}} \leqslant I_{K_0}.$$

Thus, recalling also that $I_C(v_i) \leqslant I_{C_{\min}}(v_i)$ by (3.2), we obtain

$$\limsup_{i\to\infty} \frac{I_C(v_i)}{I_{K_0}(v_i)} \leqslant \limsup_{i\to\infty} \frac{I_C(v_i)}{I_{C_{\min}}(v_i)} \leqslant 1 \leqslant \liminf_{i\to\infty} \frac{I_C(v_i)}{I_{K_0}(v_i)} ,$$

and consequently

$$\lim_{i\to\infty} \frac{I_C(v_i)}{I_{C_{\min}}(v_i)} = 1.$$

Case 2. Assume that E_i is contained in C for infinitely many indices i. Let $x_i \in E_i$ be such that $(-x_i + C) \to K$ locally in Hausdorff distance, up to a subsequence. If $\{x_i\}_{i\in\mathbb{N}}$ subconverges to $x \in C$, then $K = -x + C$. Otherwise $\{x_i\}_{i\in\mathbb{N}}$ is unbounded and, by the definition of asymptotic cylinder, we get that $K \in \mathcal{K}(C)$. Possibly passing to a subsequence, $\lambda_i(-x_i + C) \to K'$ locally in Hausdorff distance. Now by Lemma 6.7 and Remark 6.5 we get

$$I_{K_0} \leqslant I_{K'}.$$

Arguing as in the previous case we get a finite perimeter set $E \subset K'$ with $|E| = 1$, such that

$$I_{K_0}(1) \leqslant I_{K'}(1) \leqslant P_{K'}(E)$$
$$\leqslant \lim_{i\to\infty} P_{\lambda_i(-x_i+C)}(\lambda_i(-x_i + E_i)) = \lim_{i\to\infty} I_{\lambda_i C}(1).$$

Now we continue as in the final part of the proof of step one to conclude the proof of (6.16). The proof of the last part of the statement is a simple consequence of the previous arguments and of the fact that $\text{diam}(E_i) \to 0$ since $\{\text{diam}(\lambda_i E_i)\}_{i\in\mathbb{N}}$ is bounded. □

From the Theorem 6.9 and (2.13) we easily get

COROLLARY 6.10. *Let $C, C' \subset \mathbb{R}^n$ be unbounded convex bodies satisfying*

$$\alpha(C_{\min}) < \alpha(C'_{\min}).$$

Then for sufficient small volumes we have

$$I_C < I_{C'}.$$

Note that in a polytope or a prism, i.e the product of a polytope with a Euclidean space, isoperimetric regions exist, for all volumes, by compactness and Corollary 4.7 respectively.

COROLLARY 6.11 ([**72**, Theorem 6.8], [**71**, Theorem 3.8]). *Let $C \subset \mathbb{R}^n$ be a polytope or a prism. Then for sufficient small volumes isoperimetric regions are geodesic balls centered at vertices of the tangent cone with minimum solid angle.*

PROOF. According to Theorem 6.9, a sequence $\{E_i\}_{i \in \mathbb{N}}$ of isoperimetric regions of volumes going to zero collapses to p, where C_p attains the minimum of the solid angle function. Since C is a polytope or a prism then, for sufficient large $i \in \mathbb{N}$, $E_i \subset C_p$. Then the proof follows by the fact that the only isoperimetric regions in C_p are geodesic balls centered at p, see [**29**]. $\square$

COROLLARY 6.12 ([**60**]). *Let $C \subset \mathbb{R}^n$ be a slab. Then for sufficiently small volumes isoperimetric regions are half-balls.*

PROOF. Since C is the product of a segment with a Euclidean space then Corollary 4.7 implies that isoperimetric regions exist for all volumes. Since C is a slab then all points on the boundary of C attain the minimum of the solid angle function. As shown in the proof of Theorem 6.9, the diameter of a sequence $\{E_i\}_{i \in \mathbb{N}}$ of isoperimetric regions of volumes going to zero, also goes to zero. Consequently, for sufficient large $i \in \mathbb{N}$, E_i belong to a half-space and since in a half-space the only isoperimetric regions are half-balls, the proof follows. $\square$

REMARK 6.13. In [**60**] Pedrosa and Ritoré studied the isoperimetric problem in a slab of $\mathbb{R}^n$ by means of Alexandrov reflection and the characterization of stable free boundary hypersurfaces of revolution connecting two parallel hyperplanes. They showed that up to dimension $n = 8$ the only isoperimetric regions are half-balls and tubes. The case $n = 9$ is still undecided, while for $n \geqslant 10$ isoperimetric regions of undouloid type may appear. See the remarks after Proposition 5.3 in [**60**].

6.3. Isoperimetric rigidity

We consider the following, general question: assuming that a relative isoperimetric inequality holds for a convex body C, does equality for some prescribed volume imply some geometric characterization of C? Whenever this happens, we will say that the isoperimetric inequality is *rigid*. In the following we will provide answers to this question in various cases of interest, see Theorems 6.14, 6.16, and 6.18, and Corollary 6.17. A key tool for proving these rigidity results is Theorem 6.9. The first rigidity result we present is Theorem 6.14, which can be seen as a refinement of Theorem 6.3.

THEOREM 6.14. *Let C be a convex body with non-degenerate asymptotic cone C_∞. If equality holds in the isoperimetric inequality (6.4) for some volume $v_0 > 0$, that is $I_{C_\infty}(v_0) = I_C(v_0)$, then C is isometric to C_∞.*

PROOF. Assume that $I_C(v_0) = I_{C_\infty}(v_0)$ for some $v_0 > 0$. Since Y_C is concave by Theorem 5.8, Y_{C_∞} is linear by (2.13), and $Y_C \geqslant Y_{C_\infty}$ by (6.4), the function $Y_C - Y_{C_\infty} : \mathbb{R}^+ \to \mathbb{R}^+$ is concave and non-negative, thus non-decreasing. Hence $Y_C(v_0) = Y_{C_\infty}(v_0)$ implies $Y_C(v) - Y_{C_\infty}(v) = 0$ for all $v > 0$. Thus

(6.23) $$I_C(v) = I_{C_\infty}(v), \qquad \text{for all } v > 0.$$

By Lemma 3.6 there exist $K \in \{C\} \cup \mathcal{K}(C)$ and $p \in K$ so that $I_{K_p} = I_{C_{\min}}$. Assume first that K is an asymptotic cylinder of C not isometric to C. As $K_\infty \subset K \subset K_p$ we obtain $I_{K_\infty} \leqslant I_{K_p} = I_{C_{\min}}$ by (2.13). By Lemma 6.1(iii) we have $K \in \mathcal{C}_{0,m-1}^n$, hence owing to Lemma 6.2(ii) we get $I_{C_\infty} < I_{K_\infty}$. Combining the two last inequalities we obtain $I_{C_\infty} < I_{C_{\min}}$, and since $I_{C_\infty}^{n/(n-1)}$ and $I_{C_{\min}}^{n/(n-1)}$ are linear functions we get

(6.24) $$\lim_{v \to 0} \frac{I_{C_\infty}(v)}{I_{C_{\min}}(v)} < 1.$$

Now by (6.23), (6.24) and Theorem 6.9

$$1 = \lim_{v \to 0} \frac{I_C(v)}{I_{C_{\min}}(v)} = \lim_{v \to 0} \frac{I_{C_\infty}(v)}{I_{C_{\min}}(v)} < 1,$$

yielding a contradiction. Consequently $K = C$. Then by (6.23), Theorem 6.9 and (2.13) we get

$$1 = \lim_{v \to 0} \frac{I_C(v)}{I_{C_{\min}}(v)} = \lim_{v \to 0} \frac{I_{C_\infty}(v)}{I_{C_p}(v)} = \frac{\alpha(C_\infty)}{\alpha(C_p)}.$$

and since

$$p + C_\infty \subset C \subset C_p,$$

we conclude that $C = p + C_\infty$. □

As a consequence of the previous results, we are able to show the following asymptotic property of isoperimetric regions of small volume.

COROLLARY 6.15. *Let $C \subset \mathbb{R}^n$ be a convex body. Take any sequence of isoperimetric regions with volumes converging to zero and rescale the isoperimetric sets to have volume one. Then a subsequence converges to a geodesic ball centered at a vertex in a tangent cone of minimum solid angle.*

PROOF. Take a sequence of isoperimetric regions E_i in C with $|E_i| \to 0$, and let $\lambda_i > 0$ such that $|\lambda_i E_i| = 1$. Consider a sequence $x_i \in E_i$ and assume that, up to passing to subsequences, both sequences $-x_i + C$ and $\lambda_i(-x_i + C)$ converge in local Hausdorff sense to $K \in \{C\} \cup \mathcal{K}(C)$ and K', respectively. Let K_0 be the tangent cone at 0 of K. On the one hand, we have

$$(6.25) \qquad I_{K_0}(1) \leqslant I_{K'}(1) = \liminf_{i \to \infty} I_{\lambda_i C}(1).$$

On the other hand, by (3.1) we have

$$\frac{I_{\lambda_i C}(1)}{I_{K_0}(1)} = \frac{I_C(|E_i|)}{I_{K_0}(|E_i|)} \leqslant \frac{I_C(|E_i|)}{I_{C_{\min}}(|E_i|)} \leqslant 1,$$

so that

$$\limsup_{i \to \infty} I_{\lambda_i C}(1) \leqslant I_{C_{\min}}(1) \leqslant I_{K_0}(1).$$

Comparing this equation with (6.25) we get

$$I_{K_0}(1) = I_{C_{\min}}(1) = I_{K'}(1) = \lim_{i \to \infty} I_{\lambda_i C}(1).$$

In particular, K_0 is a tangent cone of $-x + C$ with the smallest possible solid angle. Let K'_∞ be the asymptotic cone of K'. Since it is the largest cone included in K' we get $K_0 \subset K'_\infty$ and so $I_{K_0} \leqslant I_{K'_\infty}$. Due to Theorem 6.3 there holds $I_{K'} \geqslant I_{K'_\infty}$, and consequently $I_{K'}(1) = I_{K'_\infty}(1)$. Then Theorem 6.14 implies that $K' = K_0$ and this concludes the proof. □

We now prove three rigidity results describing the equality cases in the isoperimetric inequalities (6.26).

THEOREM 6.16. *Let $C \subset \mathbb{R}^n$ be a convex body, either bounded or of uniform geometry. Then, for every $v \in (0, |C|)$, $\lambda < 1$ and $w \in [v, |C|)$ we have*

$$(6.26) \qquad I_C(v) \leqslant I_{C_{\min}}(v), \quad I_C(v) \geqslant I_{\lambda C}(v), \quad I_C(v) \geqslant \frac{I_C(w)}{w^{(n-1)/n}} v^{(n-1)/n}.$$

If equality holds in any of the inequalities in (6.26) for some $v_0 > 0$, then $I_C(v) = I_{C_{\min}}(v)$ for every $v \leqslant v_0$. Moreover, for every $p \in K$, $K \in \{C\} \cup \mathcal{K}(C)$, where the infimum of the solid angle function is attained, there holds $K_p \cap B(p, r_0) = K \cap B(p, r_0)$, where r_0 is defined by $|B_K(p, r_0)| = v_0$ and geodesic balls $B_K(p, r)$, with $r \leqslant r_0$, realize I_C for $v \leqslant v_0$.

PROOF. The first inequality in (6.26) has been already discussed as an immediate consequence of Proposition 3.8, see Remark 3.9. Assume now there exists $v_0 > 0$ such that $I_C(v_0) = I_{C_{\min}}(v_0)$. Recall that $Y_C = I_C^{n/(n-1)}$. From (3.2) we get $Y_C \leqslant Y_{C_{\min}}$ and, since Y_C is concave by Theorem 5.8 and $Y_{C_{\min}}$ is linear by (2.13), the non-negative function $Y_{C_{\min}} - Y_C$ is convex and satisfies $Y_{C_{\min}}(0) - Y_C(0) = 0$, therefore it is non-decreasing. Hence $Y_C(v_0) = Y_{C_{\min}}(v_0)$ implies $Y_C(v) = Y_{C_{\min}}(v)$ for all $v \leqslant v_0$. Consequently $I_C(v) = I_{C_{\min}}(v)$ for all $v \leqslant v_0$.

Choose $r_0 > 0$ such that $|B(p, r_0) \cap K| = v_0$ and let L_p be the closed cone centered at p subtended by $\partial B(p, r_0) \cap K$. By Proposition 4.3, the fact that $I_C(v_0) = I_{C_{\min}}(v_0) = I_{K_p}(v_0)$, and the inequality (3.1) in the proof of Proposition 3.8 applied to K_p, we get $\alpha(L_p) = \alpha(K_p)$. Since $L_p \subset K_p$, we have

$$B(p, r_0) \cap L_p = B(p, r_0) \cap K_p,$$

and since

$$B(p, r_0) \cap L_p \subset B(p, r_0) \cap K \subset B(p, r_0) \cap K_p,$$

we deduce

$$B(p, r_0) \cap K = B(p, r_0) \cap K_p.$$

Moreover, since $I_C(v) = I_{K_p}(v)$ for all $v \leqslant v_0$ then by (2.13) we get

$$I_C(|B_{K_p}(p, r)|) = P_{K_p}(B_{K_p}(p, r)), \quad \text{for every} \quad r \leqslant r_0.$$

This concludes the proof of the equality case in the first inequality of (6.26).

Note that the second inequality in (6.26) has already been proved in Corollary 5.9. We shall caracterize the equality case. If there exists $v_0 > 0$ such that $I_C(v_0) = I_{\lambda C}(v_0)$, then $Y_C(v_0) = Y_{\lambda C}(v_0)$. Hence Y_C is linear for $v \leqslant v_0$. Since $Y_{C_{\min}}$ is linear, then by Theorem 6.9 we have $Y_{C_{\min}} = Y_C$ for every $v \leqslant v_0$ and we proceed as above to conclude the proof.

We now prove the third inequality in (6.26). Since Y_C is concave we have

$$Y_C(v) \geqslant \frac{Y_C(w)}{w} v$$

for every $0 < v \leqslant w$. Raising to the power $(n-1)/n$ we get the desired inequality. Now if equality holds for some $0 < v_0 < w$ then Y_C is linear for $0 < v < v_0$ and we proceed as before to conclude the proof. $\square$

COROLLARY 6.17. *Let $C \subset \mathbb{R}^n$ be a convex body. Then, for every $v \geqslant 0$,*

$$(6.27) \qquad\qquad I_C(v) \leqslant I_H(v),$$

where $H \subset \mathbb{R}^n$ is a closed half-space. If equality holds in the above inequality for some $v_0 > 0$ then C is a closed half-space or a slab and isoperimetric regions for volumes $v \leqslant v_0$ are half-balls.

PROOF. Note that inequality (6.27) has already been proved in Remark 3.10. Therefore it only remains to prove the rigidity property. Assume

$$I_C(v_0) = I_H(v_0), \quad \text{for some } v_0 > 0.$$

Since $I_C(v_0) = I_H(v_0) > 0$, then, owing to Proposition 3.14, C is of uniform geometry in case it is unbounded. Owing to the first inequality in (6.26) and the previous equality we get

$$I_H(v_0) = I_C(v_0) \leqslant I_{C_{\min}}(v_0) \leqslant I_{C_p}(v_0) \leqslant I_H(v_0),$$

for any $p \in \partial C$ since C_p is a convex cone. Hence $I_C(v_0) = I_{C_p}(v_0) = I_{C_{\min}}(v_0)$ for all $p \in \partial C$. Theorem 6.16 then implies that every point $p \in \partial C$ has a neighborhood in ∂C which is a part of a hyperplane. It turns out that each connected component of ∂C is a hyperplane of $\mathbb{R}^n$ and so C is a closed half-space or a slab. $\qquad\square$

In [**21**] Choe, Ghomi and Ritoré investigated the isoperimetric profile outside a convex body L with smooth boundary showing that

$$I_{\mathbb{R}^n \setminus L}(v) \geqslant I_H(v), \quad \text{for all} \quad v > 0.$$

In the following theorem we first show that the above inequality holds for a convex body C without any regularity assumption, and then characterize the case of equality holding for some $v_0 > 0$.

THEOREM 6.18. *Let $C \subsetneq \mathbb{R}^n$ be a convex body, and let $I_{\mathbb{R}^n \setminus C}$ denote the isoperimetric profile of $\mathbb{R}^n \setminus C$. Then we have*

$$(6.28) \qquad\qquad I_{\mathbb{R}^n \setminus C}(v) \geqslant I_C(v),$$

for every $0 < v < |C|$. If equality holds in (6.28) for some $v_0 > 0$ then C is a closed half-space or a slab.

PROOF. By the proof of Lemma 5.3 we can find a sequence $\{C_i\}_{i \in \mathbb{N}}$ of convex bodies so that $C_{i+1} \subset C_i$, for all $i \in \mathbb{N}$, and $C_i \to C$ in Hausdorff distance.

Thanks to Proposition 2.13, for a given $\varepsilon > 0$ we can find $r = r(\varepsilon) > 0$ and a finite perimeter set $E \subset (\mathbb{R}^n \setminus C) \cap \overline{B}(0, r)$, of volume v, such that

$$(6.29) \qquad\qquad P_{\mathbb{R}^n \setminus C}(E) \leqslant I_{\mathbb{R}^n \setminus C}(v) + \varepsilon.$$

Let $\Omega_i = E \cap (\mathbb{R}^n \setminus C_i)$ and let B_i be Euclidean geodesic balls outside $\overline{B}(0, r)$ having volumes $|E| - |\Omega_i|$. Set $E_i = \Omega_i \cup B_i$. Then

$$(6.30) \qquad\qquad |E_i| = v \quad \text{and} \quad \lim_{i \to \infty} P_{\mathbb{R}^n \setminus C_i}(E_i) = P_{\mathbb{R}^n \setminus C}(E)$$

Combining (6.29) and (6.30) we get

$$(6.31) \qquad \limsup_{i \to \infty} I_{\mathbb{R}^n \setminus C_i}(v) \leqslant \lim_{i \to \infty} P_{\mathbb{R}^n \setminus C_i}(E_i) \leqslant I_{(\mathbb{R}^n \setminus C)}(v) + \varepsilon.$$

As the set C_i have smooth boundary for all i, inequality (6.28) holds for all C_i by the result of [**21**] and, since $\varepsilon > 0$ is arbitrary, we get by (6.31)

$$I_{\mathbb{R}^n \setminus C}(v) \geqslant I_H(v).$$

Combining this with (6.27) we get

$$I_{\mathbb{R}^n \setminus C}(v) \geqslant I_C(v), \quad \text{for all} \quad 0 < v < |C|.$$

Suppose now that equality holds for some $v_0 > 0$ in the above inequality then

$$I_{\mathbb{R}^n \setminus C}(v_0) = I_H(v_0) = I_C(v_0).$$

Consequently Corollary 6.17 implies that C is a closed half-space or a slab. $\qquad\square$

REMARK 6.19. Let $C \subset \mathbb{R}^n$ be a convex body. Then the first inequality in (6.26) combined with Theorem 6.9 and (2.13) imply

$$I_{C_{\min}}(1) = \inf\{a > 0 : \ I_C(v) \leqslant a \, v^{(n-1)/n}, \text{ for every } v > 0\}.$$

Equivalently

$$I_{C_{\min}}(1) = \sup_{v > 0} \frac{I_C(v)}{v^{(n-1)/n}}.$$

We recall that an unbounded convex body C is *cylindrically bounded* if it is contained in a right circular cylinder (the tubular neighborhood of a straight line, the axis, in $\mathbb{R}^n$). Up to rigid motions, we may assume that the axis is the vertical coordinate axis. Let us denote by π the orthogonal projection onto the hyperplane $\{x_n = 0\}$. The closure of the projection $\pi(C)$ is a convex body $K \subset \{x_n = 0\}$. By Example 3.28, the cylinder $C_\infty = K \times \mathbb{R}$ is, up to horizontal translations, the only asymptotic cylinder of C. In case C contains a line, it is a cylinder and coincides with C_∞. Otherwise, we may assume, eventually composing with a reflection with respect to $\{x_n = 0\}$ and a vertical translation, that C is contained in the half-space $x_n \geqslant 0$. Before going on we introduce some further notation. First, we set $C_\infty^+ = K \times [0, +\infty)$. Then for any $v > 0$ we let $\tau_C(v)$ be the unique real number such that the Lebesgue measure of the set $\{x \in C : \ x_n \leqslant \tau_C(v)\}$ is equal to v, and we denote such a set by $\Omega(v)$.

Theorem 6.20 (i) has been proved in [70] under the additional hypotheses that the boundaries of both the convex body and its asymptotic cylinder are $C^{2,\alpha}$. The rigidity in Theorem 6.20 (iii) is a new result.

THEOREM 6.20. *Let $C \subset \mathbb{R}^n$ be a cylindrically bounded convex body, and assume that C is not a cylinder. Then*

(i) *Isoperimetric regions exist in C for sufficiently large volumes.*
(ii) *There exists $v_0 > 0$ so that $I_C(v) \leqslant I_{C_\infty^+}(v)$ for every $v \geqslant v_0$.*
(iii) *If equality $I_C(v_1) = I_{C_\infty^+}(v_1)$ holds for some $v_1 \geqslant v_0$, then $I_C(v) = I_{C_\infty^+}(v)$ for every $v \geqslant v_1$. Moreover $C \setminus \Omega(v_1) = C_\infty^+ \setminus \Omega(v_1)$ and so $\Omega(v)$ are isoperimetric regions in C for $v \geqslant v_1$.*

PROOF. Let us prove (i) first. By [70, Theorem 3.9] there exists $v_0 > 0$ so that the slabs $K \times I$, where $I \subset \mathbb{R}$ is a compact interval, are the only isoperimetric regions of volume larger than or equal to v_0 in C_∞. So, for $v \geqslant v_0$, we get

$$(6.32) \qquad I_C(v) \leqslant P_C(\Omega(v)) \leqslant \mathcal{H}^n(K) < 2\mathcal{H}^n(K) = I_{C_\infty}(v).$$

Thus, by Theorem 5.8, for every $v \geqslant v_0$ there exists an isoperimetric region of volume v in C.

We now prove (ii). By [70, Corollary 3.10] there exists $v_0 > 0$ so that the half-slabs $K \times [0, b]$ are the only isoperimetric regions in C_∞^+ of volume larger than or equal to v_0. Then we obtain

$$(6.33) \qquad I_C(v) \leqslant P_C(\Omega(v)) \leqslant \mathcal{H}^n(K) = I_{C_\infty^+}(v), \quad \text{for every } v \geqslant v_0.$$

We now prove (iii). We know that I_C is non-decreasing by Theorem 5.8, and that $I_{C_\infty^+}(v) = \mathcal{H}^n(K)$ for $v \geqslant v_0$. Then we get $I_C(v) = I_{C_\infty^+}(v)$ for every $v \geqslant v_1$. Furthermore (6.33) provides $P_C(\Omega(v)) = \mathcal{H}^n(K)$ for every $v \geqslant v_1$, yielding

$$C \cap (\mathbb{R}^n \times \{t(v)\}) = C_\infty^+ \cap (\mathbb{R}^n \times \{t(v)\})$$

for every $v \geqslant v_1$. Hence $C \setminus \Omega(v_1) = C_\infty^+ \setminus \Omega(v_1)$. $\qquad \square$

We now conclude the section with two applications of the rigidity results shown before.

THEOREM 6.21. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body, different from a half-space, such that any asymptotic cylinder of C is either a half-space or $\mathbb{R}^n$. Then C is of uniform geometry and any generalized isoperimetric region must lie in C for any given volume.*

PROOF. The unbounded convex body C is of uniform geometry by Proposition 3.14(iii) since any asymptotic cylinder has non-empty interior. Now Theorem 5.8 implies that, for any given $v_0 > 0$, there exists a generalized isoperimetric region either in C or in an asymptotic cylinder K. Assume the latter case holds. Since any asymptotic cylinder of C is either a half-space H or $\mathbb{R}^n$ we have $I_C(v_0) \geqslant I_H(v_0)$. By Theorem 6.18, $I_C(v_0) \leqslant I_H(v_0)$ and so equality $I_C(v_0) = I_H(v_0)$ holds. By the rigidity result of Theorem 6.18, C is a half-space or a slab. The second case cannot hold since any asymptotic cylinder of a slab is again a slab. This contradiction shows that any isoperimetric region of volume v_0 must be contained in C.   □

COROLLARY 6.22. *Let $C \subset \mathbb{R}^n$ be an unbounded convex body satisfying at least one of the following properties:*

- *C is a non-cylindrically bounded convex body of revolution;*
- *C has a non-degenerate asymptotic cone C_∞ such that ∂C_∞ is of class C^1 with the only exception of a vertex.*

Then any generalized isoperimetric region must lie in C for any given volume.

PROOF. We observe that in both cases any asymptotic cylinder of C is either a half-space or $\mathbb{R}^n$, see Example 3.29 and Proposition 3.33. The conclusion is thus achieved by applying Theorem 6.21.   □

Asymptotic behavior of the isoperimetric profile of an unbounded convex body

The ultimate goal of this chapter is to obtain estimates of the isoperimetric dimension of an unbounded convex body C of uniform geometry, defined as the number $\alpha > 0$ such that there exist $0 < \lambda_1 < \lambda_2$ and $v_0 > 0$ satisfying

$$\lambda_1 v^{(\alpha-1)/\alpha} \leqslant I_C(v) \leqslant \lambda_2 v^{(\alpha-1)/\alpha} \qquad \forall\, v > v_0\,.$$

As a first step, we shall give the following estimate of the isoperimetric profile I_C in Corollary 7.7:

$$n\frac{v}{\phi(v)} \geqslant I_C(v) \geqslant 3^{-1}8^{-n}\frac{v}{\phi(v)}\,.$$

Here ϕ is the reciprocal function of $V(r) := \inf_{x \in C} |B_C(x, r)|$. The above displayed inequalities imply that the functions $I_C(v)$ and $v/\phi(v)$ are asymptotically equivalent. Hence in order to derive an (asymptotic) estimate of $I_C(v)$ in terms of some explicit function of v one would need to compute $V(r)$ with sufficient precision, which is not an easy task for a generic convex body C. The first section of this chapter will be devoted to obtain this aymptotic estimate.

In the second section we focus on computing this asymptotic estimate for a special class of convex bodies of revolution. We consider the sets $C_f = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \geqslant f(|x|)\}$, where $f : [0, \infty) \to [0, \infty)$ is a strictly convex function satisfying $f(0) = 0$ and $\lim_{s \to \infty} s^{-1}f(s) = \infty$. The asymptotic cone of any such C is a half-line, so that one expects an isoperimetric dimension strictly smaller than 3. If one further assumes that f is of class $C^3(0, +\infty)$ and satisfies $f''' \leqslant 0$ on $(0, +\infty)$, then Theorem 7.10 proves that $b(r) = |B_C(0, r)|$. In the 3-dimensional case, condition $f''' \leqslant 0$ is a condition on the derivative of the Gauss curvature of ∂C.

Finally, in the third section we compute in Example 7.11 the isoperimetric dimension of

$$C_a = \{(x, y) \in \mathbb{R}^3 : y \geqslant |x|^a\}$$

for $1 < a \leqslant 2$ and show that it is given by $\frac{a+2}{a}$, thus the isoperimetric dimension continuously changes from 3 to 2 as the parameter a goes from 1 to 2.

7.1. An asymptotic isoperimetric inequality

Given a convex body $C \subset \mathbb{R}^n$ of uniform geometry, we shall prove in Theorem 7.4 a relative isoperimetric inequality on C depending on the growth rate of the volume of geodesics balls in C. We shall follow the arguments by Coulhon and Saloff-Coste [24], who established similar inequalities for graphs, groups and manifolds. Their approach makes use of a non-decreasing function $V : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

   (i) $|B_C(x, r)| \geqslant V(r)$ for all $x \in C$ and $r > 0$, and

(ii) $\lim_{r \to \infty} V(r) = \infty$.

The *reciprocal function* of V, $\phi_V : \mathbb{R}^+ \to \mathbb{R}^+$, is defined by

$$\phi_V(v) := \inf\{r \in \mathbb{R}^+ : V(r) \geqslant v\}.$$

It is immediate to check that ϕ is a non-decreasing function. Moreover, if $V_1 \geqslant V_2$ then, for any $v > 0$, $\{r \in \mathbb{R}^+ : V_2(r) \geqslant v\} \subset \{r \in \mathbb{R}^+ : V_1(r) \geqslant v\}$, and so $\phi_{V_1} \leqslant \phi_{V_2}$.

When C is a convex body of uniform geometry, we know that the quantity $b(r) = \inf_{x \in C} |B_C(x, r)|$ is positive for all $r > 0$ by Proposition 3.3. Let us check that the function $b(r)$ is non-decreasing and satisfies (i) and (ii).

When $0 < r < s$, it follows that $b(r) \leqslant |B_C(x, r)| \leqslant |B_C(x, s)|$ for all $x \in C$. This implies $b(r) \leqslant \inf_{x \in C} |B_C(x, s)| = b(s)$ and so the function $b(s)$ is non-decreasing. Property (i) is immediate from the definition of $b(r)$. It remains to show that $\lim_{r \to \infty} b(r) = \infty$. To prove this, we select $x \in C$ and another point $y \in C$ such that $|x - y| = 2r$. Then $B(x, 3r) \supseteq B(x, r) \cup B(y, r)$ and hence $|B_C(x, 3r)| \geqslant 2b(r)$. Varying x we get $b(3r) \geqslant 2b(r)$. This implies that $b(3^i) \geqslant 2^i b(1)$ for all $i \in \mathbb{N}$ and, since b is non-decreasing, we get that $b(r) \to \infty$ when $r \to \infty$.

REMARK 7.1. It is worth noting that, when C is an arbitrary convex body, the asymptotic behaviour of the volume of balls centered at a given point is independent of the point. More precisely we have

(7.1) $$\lim_{r \to \infty} \frac{|B_C(x, r)|}{|B_C(y, r)|} = 1,$$

for any pair of points $x, y \in C$. To prove this, fix two points $x, y \in C$ and let d be the Euclidean distance between x and y. Observe first that Lemma 2.6 implies, for any $z \in C$,

$$1 \leqslant \liminf_{r \to \infty} \frac{|B_C(z, r + d)|}{|B_C(z, r)|} \leqslant \limsup_{r \to \infty} \frac{|B_C(z, r + d)|}{|B_C(z, r)|} \leqslant \lim_{r \to \infty} \frac{(r + d)^n}{r^n} = 1.$$

Since $B_C(x, r) \subset B_C(y, r + d)$, we get

$$\limsup_{r \to \infty} \frac{|B_C(x, r)|}{|B_C(y, r)|} \leqslant \limsup_{r \to \infty} \frac{|B_C(y, r + d)|}{|B_C(y, r)|} = 1.$$

By symmetry we get (7.1).

In addition to the existence of the lower bound $V(r)$ for the volume of metric balls in C, essential ingredients in the proof of the isoperimetric inequality are the existence of a doubling constant, given by Lemma 2.6, and the following uniform Poincaré inequality for convex sets, proven by Acosta and Durán [1] using the "reduction to one-dimensional problem technique" introduced by Payne and Weinberger [59].

THEOREM 7.2 ([1, Thm. 3.2]). *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter d and let $u \in W^{1,1}(\Omega)$ with $\int_\Omega u = 0$. Then*

$$\|u\|_{L^1(\Omega)} \leqslant \frac{d}{2} \|\nabla u\|_{L^1(\Omega)}.$$

Moreover the constant $1/2$ is optimal.

In geometric form, this inequality reads ([26, § 5.6])

LEMMA 7.3. *Let $K \subset \mathbb{R}^n$ be a bounded convex body with diameter d and let $E \subset K$ be a set of finite perimeter. Then*

$$\frac{d}{2} P_K(E) \geqslant \min\{|E|, |K \setminus E|\}.$$

In particular, if $C \subset \mathbb{R}^n$ is an unbounded convex body, $E \subset C$ has locally finite perimeter in C and $r > 0$, then

(7.2) $$r P(E, \text{int}(B_C(x, r))) \geqslant \min\{|E \cap \bar{B}_C(x, r)|, |\bar{B}_C(x, r) \setminus E|\}.$$

Using Lemma 7.3 we can prove the following isoperimetric inequality on a convex body of uniform geometry.

THEOREM 7.4. *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry. Let $V : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function satisfying*

(i) $|B_C(x, r)| \geqslant V(r)$ *for all $x \in C$ and $r > 0$, and*
(ii) $\lim_{r \to \infty} V(r) = +\infty$.

Let ϕ be the reciprocal function of V. Then for any set $E \subset C$ of finite perimeter we have

(7.3) $$P_C(E) \geqslant 8^{-n} \frac{|E|}{\phi(2|E|)},$$

and so

(7.4) $$I_C(v) \geqslant 8^{-n} \frac{v}{\phi(2v)}.$$

PROOF. Fix $r > 0$ such that $2|E| \leqslant V(r)$. With this choice, $|E| \leqslant |B_C(x, r)|/2$ for any $x \in C$. Moreover, from the definition of ϕ, the quantity $\phi(2|E|)$ is equal to the infimum of all $r > 0$ such that inequality $V(r) \geqslant 2|E|$ holds.

Consider a maximal family $\{x_j\}_{j \in J}$ of points in C such that $|x_j - x_k| \geqslant r$ for all $j, k \in J$, $j \neq k$. Then $C = \bigcup_{j \in J} B_C(x_j, r)$ and the balls $B_C(x_j, r/2)$ are disjoint.

By Lemma 2.6, the number of balls $B_C(x_i, r)$ that contain a given point $x \in C$ is uniformly bounded: if $J(x) := \{j \in J : x \in B_C(x_j, r)\}$ then $B_C(x_j, r) \subset B_C(x, 2r) \subset B_C(x_j, 4r)$ when $j \in J(x)$ and

$$|B_C(x, 2r)| \geqslant \sum_{j \in J(x)} |B_C(x_j, r/2)| \geqslant 8^{-n} \sum_{j \in J(x)} |B_C(x_j, 4r)|$$
$$\geqslant 8^{-n} \# J(x) |B_C(x, 2r)|,$$

so that

$$8^n \geqslant \# J(x).$$

Then we have

$$|E| \leqslant \sum_{j \in J} |E \cap B_C(x_j, r)| \leqslant \sum_{j \in J} r P(E, \text{int}(B_C(x_j, r))) \leqslant 8^n r P_C(E).$$

Here we have used (7.2) and the fact that $V(r) \geqslant 2|E|$ implies that $|E \cap B_C(x, r)| \leqslant |B_C(x, r) \setminus E|$. Finally, taking infimum over all $r > 0$ such that $V(r) \geqslant 2|E|$ we get

$$|E| \leqslant 8^n \phi(2|E|) P_C(E),$$

equivalent to (7.3). Equation (7.4) follows from the definition of the isoperimetric profile. $\square$

REMARK 7.5. Let $C \subset \mathbb{R}^n$ be a convex body with non-degenerate asymptotic cone C_∞. We know from Example 3.27 that C is of uniform geometry and that $V(r) \geqslant c\, r^n$ for all $r > 0$. Theorem 7.4 then implies $I_C(v) \geqslant c' v^{(n-1)/n}$ for every $v > 0$ and for some positive constant $c' > 0$. See also Remark 6.6 for the optimal constant.

REMARK 7.6. Equation (7.4) in Theorem 7.4 provides a lower estimate of the isoperimetric profile of C whenever there is a lower estimate $V(r)$ of the volume of metric balls in C. For any such function V we have $V(r) \leqslant b(r) = \inf_{x \in C} |B_C(x,r)|$. Hence $\phi_V \geqslant \phi_b$ and

$$I_C(v) \geqslant 8^{-n} \frac{v}{\phi_b(2v)} \geqslant 8^{-n} \frac{v}{\phi_V(2v)}.$$

Hence the best function ϕ we can choose in (7.4) corresponds to the reciprocal function of b.

COROLLARY 7.7. *Let $C \subset \mathbb{R}^n$ be a convex body of uniform geometry, and let ϕ be the reciprocal function of b. Then the following inequalities*

$$(7.5) \qquad\qquad n\, \frac{v}{\phi(v)} \geqslant I_C(v) \geqslant 3^{-1} 8^{-n} \frac{v}{\phi(v)}$$

hold.

PROOF. To prove the left side inequality, we pick $x \in C$, $r > 0$ so that the ball $B_C(x,r)$ has a given volume v, and we consider the cone with vertex x over $\partial B(x,r) \cap C$ to obtain

$$nv \geqslant r P_C(B_C(x,r)) \geqslant r I_C(v).$$

Since $\phi(v)$ is the reciprocal function of $b(r)$ we have $\phi(v) = \inf\{r > 0 : b(r) \geqslant v\}$. Hence, for any radius $r > 0$ such that $|B_C(x,r)| = v$, we get $r \geqslant \phi(v)$. So we obtain

$$n\, \frac{v}{\phi(v)} \geqslant I_C(v),$$

as claimed.

We now prove the right side inequality of (7.5) using (7.4) and a relation between $\phi(v)$ and $\phi(2v)$ obtained in the following way: consider a vector w with $|w| = 1$ so that the half-line $\{x + \lambda w : \lambda \geqslant 0\}$ is contained in C for all $x \in C$. Take $x \in C$ and $r > 0$. Then $2rw + B_C(x,r)$ is a subset of C disjoint from $B_C(x,r)$. Since $B_C(x,r) \cup \left(2rw + B_C(x,r)\right)$ is contained in the ball $B_C(x,3r)$, we get the estimate

$$2|B_C(x,r)| \leqslant |B_C(x,3r)|,$$

for any $x \in C$ and $r > 0$.

Fix now $v > 0$ and take a sequence of radii $\{r_i\}_{i \in \mathbb{N}}$ so that $b(r_i) \geqslant v$ and $\lim_{i \to \infty} r_i = \phi(v)$. For every $x \in C$ and $i \in \mathbb{N}$ we have

$$|B_C(x,3r_i)| \geqslant 2|B_C(x,r_i)| \geqslant 2b(r_i) \geqslant 2v.$$

Taking infimum on $x \in C$ when r_i is fixed we obtain $b(3r_i) \geqslant 2v$. From the definition of ϕ we have $\phi(2v) \leqslant 3r_i$ and taking limits we get

$$\phi(2v) \leqslant 3\phi(v).$$

Hence, from (7.4) we get

$$I_C(v) \geqslant 3^{-1} 8^{-n} \frac{v}{\phi(v)},$$

as desired. □

REMARK 7.8. Let $C \subset \mathbb{R}^{n+1}$ be an unbounded convex body. For every $x \in C$, Lemma 2.6 implies that

$$\frac{|B_C(x,s)|}{s^{n+1}} \leqslant \frac{|B_C(x,r)|}{r^{n+1}}, \qquad 0 < r < s.$$

In particular the function

$$r \mapsto \frac{|B_C(x,r)|}{r^{n+1}}$$

is non-increasing.

Taking $s > 0$ fixed, the above inequality implies

$$\frac{|B_C(x,r)|}{r^{n+1}} \geqslant \frac{|B_C(x,s)|}{s^{n+1}} \geqslant \frac{b(s)}{s^{n+1}}, \qquad 0 < r < s.$$

Taking the infimum over $x \in C$ we get

$$b(r) \geqslant \frac{b(s)}{s^{n+1}} \, r^{n+1} = C_s r^{n+1}, \qquad 0 < r < s.$$

and so $\phi(v) \leqslant C_s^{1/(n+1)} v^{1/(n+1)}$ for v in the interval $(0, C_s^{1/(n+1)} s^{1/(n+1)})$.

Hence (7.4) implies

$$I_C(v) \geqslant 8^{-(n+1)} C_s^{-1/(n+1)} \frac{v}{v^{1/(n+1)}} = 8^{-(n+1)} C_s^{-1/(n+1)} \, v^{n/(n+1)},$$

for v in the interval $(0, C_s^{1/(n+1)} s^{1/(n+1)})$. This way we recover inequality (3.12) in Corollary 3.17.

7.2. Estimates on the volume growth of balls

Our aim now is to obtain accurate estimates of $b(r)$ for given special convex sets in order to understand the behavior of the isoperimetric profile for large volumes using (7.5). While $b(r)$ is easy to compute in homogeneous spaces [61], it is harder to estimate in unbounded convex bodies. The following argument will be of crucial importance to study the behavior of the volume function $x \in C \mapsto |B_C(x,r)|$ for $r > 0$ fixed.

Recall that, given a set E of locally finite perimeter in $\mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $t, r \geqslant 0$, then for all $x \in \mathbb{R}^n$ one has

$$(7.6) \quad |E \cap B(x + t\xi, r)| = |E \cap B(x, r)| - \int_0^t \int_{\partial^* E \cap B(x + s\xi, r)} \xi \cdot \nu_E \, d\mathcal{H}^{n-1} \, ds,$$

where ν_E denotes the weak exterior normal to $\partial^* E$. The proof of (7.6) can be found in [33, Lemma 4.5]. We notice that the function

$$s \mapsto \int_{\partial^* E \cap B(x + s\xi, r)} \xi \cdot \nu_E \, d\mathcal{H}^{n-1}$$

is in $L^\infty(0, t)$, consequently the function

$$t \mapsto |E \cap B(x + t\xi, r)|$$

is Lipschitz-continuous and thus by (7.6) and for almost all $t > 0$

$$\frac{d}{dt} |E \cap B(x + t\xi, r)| = - \int_{\partial^* E \cap B(x + t\xi, r)} \xi \cdot \nu_E \, d\mathcal{H}^{n-1}.$$

On the other hand, by integrating $0 = \operatorname{div} \xi$ on $E \cap B(x + t\xi, r)$ and applying Gauss-Green's Theorem we get for almost all $t > 0$

$$\frac{d}{dt}|E \cap B(x + t\xi, r)| = \int_{E \cap \partial B(x+t\xi,r)} \xi \cdot \nu_{B(x+t\xi,r)} \, d\mathcal{H}^{n-1}.$$

Finally, if $x = x(z) \in \mathbb{R}^n$ is a smooth parametric curve, $z \in \mathbb{R}$, then the composition

$$z \mapsto |E \cap B(x(z), r)|$$

is Lipschitz and by the chain rule one gets for almost all $z \in \mathbb{R}$

(7.7) $$\frac{d}{dz}|E \cap B(x(z), r)| = \int_{E \cap \partial B(x(z),r)} \xi(z) \cdot \nu_{B(x+t\xi,r)} \, d\mathcal{H}^{n-1},$$

where $\xi(z)$ denotes the velocity of $x(z)$.

To compute the integral in (7.7) the following lemma will be extremely useful

LEMMA 7.9. *Let S be the sphere $\partial B(x, r)$, ν the outer unit normal to S. For $\xi \in \mathbb{S}^n$, let $\sigma_\xi : S \to S$ be the reflection with respect to the hyperplane orthogonal to ξ passing through x. Let $f_\xi : S \to \mathbb{R}$ be the function $f_\xi(z) = \langle \nu(z), \xi \rangle$, and let $H_\xi^+ := \{z \in S : f_\xi(z) \geqslant 0\}$ and $H_\xi^- := \{z \in S : f_\xi(z) \leqslant 0\}$.*

Let $\Omega \subset S$ be a measurable set and $\Omega_\xi^+ := \Omega \cap H_\xi^+$, $\Omega_\xi^- := \Omega \cap H_\xi^-$. If $\sigma_\xi(\Omega_\xi^-) \subset \Omega_\xi^+$ then

$$\int_\Omega f_\xi \geqslant 0.$$

PROOF. Let us drop the subscript ξ. The proof easily follows from the equality $f \circ \sigma = -f$ and the area formula:

$$\int_\Omega f = \int_{\Omega^+} f + \int_{\Omega^-} f = \int_{\Omega^+} f - \int_{\sigma(\Omega^-)} f = \int_{\Omega^+ \setminus \sigma(\Omega^-)} f \geqslant 0. \qquad \square$$

Now we restrict ourselves to a class of rotationally symmetric unbounded convex bodies. Take a strictly convex function $f : [0, +\infty) \to [0, +\infty)$ of class C^1 such that $f(0) = f'(0) = 0$. We shall assume that $f''(x)$ exists and is positive for $x > 0$, and that $f'''(x) \leqslant 0$ for $x > 0$. For instance, the functions $f(x) := x^a$, with $1 < a \leqslant 2$, satisfy these conditions. The function f determines the unbounded convex body

$$C_f := \{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t \geqslant f(|z|)\}.$$

The asymptotic cone of the epigraph of f is the half-line $\{(0, t) : t \geqslant 0\}$ if and only if

$$\lim_{s \to \infty} \frac{f(s)}{s} = +\infty.$$

This limit exists since the quantity $f(s)/s$ is increasing in s (because of the convexity of f and equality $f(0) = 0$). The boundary of C_f is the graph of the function $z \in \mathbb{R}^{n-1} \mapsto f(|z|)$. From now on, we shall assume that C satisfies this hypothesis.

The function

$$\kappa(s) := \frac{f''(s)}{(1 + f'(s)^2)^{3/2}}, \qquad s > 0,$$

is the geodesic curvature of the planar curve determined by the graph of f. It is also the principal curvature of the meridian curves of the graph of f. The function κ is decreasing when $s > 0$ since

$$\kappa' = \frac{-3f'(f'')^2 + f'''(1 + (f')^2)}{(1 + (f')^2)^{5/2}} < 0.$$

The principal curvatures of the parallel curves of the graph of f are given by

$$\frac{f'(s)}{s\,(1 + f'(s)^2)^{1/2}}, \quad s > 0.$$

This function is also decreasing when $s > 0$ as

$$\left(\frac{f'}{s(1 + (f')^2)^{1/2}}\right)' = \frac{-f'(1 + (f')^2) + sf''}{s^2(1 + (f')^2)^{3/2}}$$

and $sf'' \leqslant f'$ because of the concavity of f' and the fact that $f'(0) = 0$.

For the convex set C_f we are going to prove that $b(r) = |B_C(0, r)|$ for all $r > 0$. Thus we can easily estimate the reciprocal function $\phi(v)$ to obtain accurate estimates of the isoperimetric profile of C_f using inequalities (7.5).

THEOREM 7.10. *Let* $f : [0, \infty) \to [0, \infty)$ *be a* C^1 *function such that* $f(0) = f'(0) = 0$. *Assume that* f *is of class* C^3 *in* $(0, \infty)$ *with* $f'' > 0$, $f''' \leqslant 0$, *and* $\lim_{s \to \infty}(f(s)/s) = +\infty$. *Consider the convex body of revolution in* $\mathbb{R}^n$ *given by*

$$C_f := \{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t \geqslant f(|z|)\}.$$

Then $|B_C(0, r)| = b(r)$ *for all* $r > 0$.

PROOF. Let $C = C_f$. For any $x_0 = (z_0, f(|z_0|)) \in \partial C_f \setminus \{0\}$, consider the meridian vector

$$v_{x_0} := \frac{(\frac{z_0}{|z_0|}, f'(|z_0|))}{(1 + f'(|z_0|)^2)^{1/2}}$$

and let σ_{x_0} be the orthogonal symmetry with respect to the hyperplane

$$H_{x_0} := \{x \in \mathbb{R}^n : \langle x - x_0, v_{x_0} \rangle = 0\}.$$

Define $H_{x_0}^- := \{x \in \mathbb{R}^n : \langle x - x_0, v_{x_0} \rangle \leqslant 0\}$, $H_{x_0}^+ := \{x \in \mathbb{R}^n : \langle x - x_0, v_{x_0} \rangle \geqslant 0\}$. By (7.7) and Lemma 7.9, it is enough to prove

$$(7.8) \qquad \qquad \sigma_{x_0}(C \cap H_{x_0}^-) \subset C \cap H_{x_0}^+$$

for any $x_0 \in \partial C \setminus \{0\}$.

To prove (7.8) we shall use a deformation argument similar to Alexandrov Reflection. Let

$$w_\theta := \left(\sin\theta \frac{z_0}{|z_0|}, \cos\theta\right), \qquad \theta \in [0, \theta_0]$$

where

$$\theta_0 := \arccos\left(\frac{f'(|z_0|)}{(1 + f'(|z_0|)^2)^{1/2}}\right) < \frac{\pi}{2}.$$

When θ moves along $[0, \theta_0]$, the vector w_θ varies from $(0, 1)$ to v_{x_0}. Let us consider the hyperplanes

$$H_\theta := \{x \in \mathbb{R}^n : \langle x - x_0, w_\theta \rangle = 0\},$$

and $H_\theta^- := \{x \in \mathbb{R}^n : \langle x - x_0, w_\theta \rangle \leqslant 0\}$, $H_\theta^+ := \{x \in \mathbb{R}^n : \langle x - x_0, w_\theta \rangle \geqslant 0\}$.

We are proving first the two dimensional case.

Here ∂C is a strictly convex curve and so $\sigma_\theta(\partial C \cap \text{int}(H_\theta^-))$ is strictly convex and its tangent vector rotates monotonically. If this tangent vector is never vertical then the curve $\sigma_\theta(\partial C \cap \text{int}(H_\theta^-))$ is the graph of a function over the z-axis lying in $C \cap H_\theta^+$, and so $\sigma_\theta(\partial C \cap \text{int}(H_\theta^-))$ is trivially contained in $\text{int}(C)$.

So assume there is a point x_v in $\sigma_\theta(\partial C \cap \text{int}(H_\theta^-))$ with vertical tangent vector. A straightforward computation shows that x_v is the image $\sigma_\theta(x_1)$ of a point $x_1 \in \partial C \cap \text{int}(H_\theta^-)$ with $x_1 = (z_1, f(z_1))$ and

$$\frac{(1, f'(z_1))}{(1 + f'(z_1)^2)^{1/2}} = (\sin(2\theta), \cos(2\theta)).$$

This implies that $z_1 > 0$. Define the curves $\Gamma_2 := \partial C \cap \{z_1 \leqslant z < z_0\}$, $\Gamma_2 := \partial C \cap \{z > z_0\}$ and let κ_1, κ_2 be their geodesic curvatures. Then $\kappa_1(y_1) > \kappa_2(y_2)$ for every pair of points $y_1 \in \Gamma_1$, $y_2 \in \Gamma_2$. This implies that $\sigma_\theta(\Gamma_1)$ and Γ_2 are graphs over a line orthogonal to L_θ and $\sigma_\theta(\Gamma_1)$ lies above Γ_2. Since both curves are contained in the half-space $\{z \geqslant 0\}$ we conclude that $\sigma_\theta(\Gamma_1)$ is contained in $\text{int}(C)$. The curve $\sigma_\theta((C \cap H_\theta^-) \setminus \Gamma_1)$ has no vertical tangent vector and so it is a graph over the z-axis lying over the line L_θ. So it is also contained in $\text{int}(C)$. This concludes the proof in the two dimensional case.

Now we prove the general case by reducing it to the two dimensional one.

Let us check first that the set $C \cap H_\theta^-$ is bounded. The use of hypothesis $\lim_{s \to \infty}(f(s)/s) = +\infty$ is essential here. Any point $(z, t) \in C \cap H_\theta^-$ satisfies the inequalities

(7.9) $$\left\langle z - z_0, \frac{z_0}{|z_0|} \right\rangle \sin \theta + (t - t_0) \cos \theta \leqslant 0, \qquad t \geqslant f(|z|).$$

We reason by contradiction, assuming there is a sequence of points $x_i = (z_i, t_i)$ $(i \in \mathbb{N})$ in $C \cap H_\theta^-$ with $\lim_{i \to \infty} |x_i| = +\infty$. The sequence $|z_i|$ converges to $+\infty$ since, from (7.9) and Schwarz inequality

$$0 \leqslant t_i \cos \theta \leqslant t_0 \cos \theta + |z_0| \sin \theta + |z_i| \cos \theta.$$

Hence boundedness of a subsequence of $|z_i|$ would imply boundedness of the corresponding subsequence of $|t_i|$, contradicting that $\lim_{i \to \infty} |x_i| = +\infty$. On the other hand, inequalities (7.9), together with Schwarz inequality, imply

$$\frac{f(|z_i|)}{|z_i|} \cos \theta \leqslant \frac{t_0 \cos \theta + |z_0| \sin \theta}{|z_i|} + \sin \theta.$$

Taking limits when $i \to \infty$ we get a contradiction since $|z_i|$ and $f(|z_i|)/|z_i|$ converge to ∞.

Now we start with a deformation procedure. For $\theta = 0$, we have the inclusion $\sigma_\theta(C \cap H_\theta^-) \subset C \cap H_\theta^+$ since H_θ is a horizontal hyperplane and C is the epigraph of a function defined onto this hyperplane. Let $\bar{\theta}$ be the supremum of the closed set

$$\{\theta \in [0, \theta_0] : \sigma_\theta(C \cap H_\theta^-) \subset C \cap H_\theta^+\}.$$

If $\bar{\theta} = \theta_0$ we are done. Otherwise let us assume that $\bar{\theta} < \theta_0$.

Let us check that, for any $\theta \in [0, \theta_0)$ and $x \in \partial C \cap H_\theta$, we have

(7.10) $$\langle \bar{\sigma}_\theta(v_x), n_x \rangle > 0,$$

where $\bar{\sigma}_\theta$ is the orthogonal symmetry with respect to the linear hyperplane of vectors orthogonal to w_θ and n_x is the outer unit normal to ∂C at x given by

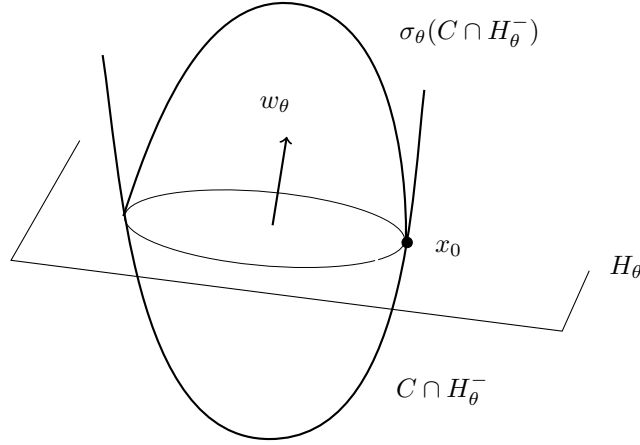$$n_x = \frac{(f'(|z|)\frac{z}{|z|}, -1)}{(1 + f'(|z|)^2)^{1/2}}.$$

FIGURE 7.1. Sketch of the reflection procedure

Since $\bar{\sigma}_x(v_x) = v_x - 2\langle v_x, w_\theta \rangle w_\theta$ we have $\langle \bar{\sigma}_\theta(v_x), n_x \rangle = -2\langle v_x, w_\theta \rangle \langle w_\theta, n_x \rangle$, so that

$$(7.11) \quad \langle \bar{\sigma}_\theta(v_x), n_x \rangle = \frac{-2}{1 + f'(|z|)^2} \left(\frac{\langle z, z_0 \rangle}{|z||z_0|} \sin\theta + f'(|z|) \cos\theta \right)$$
$$\times \left(f'(|z|) \frac{\langle z, z_0 \rangle}{|z||z_0|} \sin\theta - \cos\theta \right)$$

Observe that, if $x \in \partial C \cap H_\theta$, then $|z| > |z_0|$ unless $z = z_0$. This is easy to prove since

$$0 = \langle z - z_0, \frac{z_0}{|z_0|} \rangle \sin\theta + (f(|z| - f(|z_0|)) \cos\theta$$
$$\leqslant (|z| - |z_0|) \sin\theta + (f(|z| - f(|z_0|)) \cos\theta.$$

In case $|z| < |z_0|$ then $f(|z|) < f(|z_0|)$ and we get a contradiction. If $|z| = |z_0|$ then equality holds in the first inequality and so $z = \lambda z_0$ for some positive λ which must be equal to one.

To prove (7.10), let us analyze the sign of the factors between parentheses in (7.11). For the first factor, when $x \in \partial C \cap H_\theta$ we get

$$\frac{\langle z, z_0 \rangle}{|z||z_0|} \sin\theta = \frac{|z_0|}{|z|} \sin\theta - \frac{f(|z|) - f(|z_0|)}{|z|} \cos\theta$$
$$\geqslant \frac{|z_0|}{|z|} \sin\theta - \frac{f(|z|)}{|z|} \cos\theta$$
$$\geqslant \frac{|z_0|}{|z|} \sin\theta - f'(|z|) \cos\theta$$

where for the last inequality we have used $f(x)/x \leqslant f'(x)$, a consequence of the convexity of f. Hence

$$\frac{\langle z, z_0 \rangle}{|z||z_0|} \sin\theta + f'(|z|) \cos\theta \geqslant \frac{|z_0|}{|z|} \sin\theta$$

We thus infer that the quantity in the left-hand side is positive when $\sin\theta > 0$ and, when $\sin\theta = 0$, it is equal to $f'(|z|)$, which is also positive as $|z| \geqslant |z_0| > 0$.

For the second factor in (7.11) we have, for $x \in \partial C \cap H_\theta$, that the quantity

$$f'(|z_0|)\frac{\langle z, z_0\rangle}{|z||z_0|}\sin\theta - \cos\theta,$$

equal to

$$\frac{|z_0|}{|z|}f'(|z|)\sin\theta - \frac{f(|z|) - f(|z_0|)}{|z|}f'(|z|)\cos\theta - \cos\theta$$

is strictly smaller than

$$\frac{1}{(1 + f'(|z_0|)^2)^{1/2}}\left(\frac{|z_0|}{|z|}f'(|z|) - \frac{f(|z|) - f(|z_0|)}{|z|}f'(|z|)f'(|z_0|) - f'(|z_0|)\right).$$

This quantity is negative since $f'(|z|)/|z| \leqslant f'(|z_0|)/|z_0|$ by the concavity of f'. In case $\theta = \theta_0$, it is also negative when $z \neq z_0$ since, in this case, $|z| > |z_0|$ and so $f(|z|) > f(|z_0|)$.

So we have proved that the sign of the first factor between parentheses in (7.11) is positive and the sign of the second factor is negative. This proves (7.10).

Inequality (7.10) guarantees that $\sigma_\theta(C\cap H_\theta^-)$ is strictly contained in $C\cap H_\theta^+$ near H_θ when $\theta < \theta_0$. As in the proof of Alexandrov Reflection principle, it shows that $\sigma_{\bar\theta}(\partial C \cap H_{\bar\theta}^-)$ and $\partial C \cap H_{\bar\theta}^+$ have a tangential contact at some point $x_2 \in \partial C \cap H_{\bar\theta}^+$. The point x_2 is the image by $\sigma_{\bar\theta}$ of a point $x_1 \in \partial C \cap H_{\bar\theta}^-$ and must lie in the interior of $H_{\bar\theta}^+$. Since

$$x_2 = x_1 - 2\langle x_1 - x_0, w_{\bar\theta}\rangle w_{\bar\theta}, \qquad n_{x_2} = n_{x_1} - 2\langle n_{x_1}, w_{\bar\theta}\rangle w_{\bar\theta},$$

we have

$$z_2 = z_1 - 2\langle x_1 - x_0, w_{\bar\theta}\rangle \sin\theta\frac{z_0}{|z_0|},$$

(7.12)
$$\frac{f'(|z_2|)\,z_2}{|z_2|(1 + f'(|z_2|)^2)^{1/2}} = \frac{f'(|z_1|)\,z_1}{|z_1|(1 + f'(|z_1|)^2)^{1/2}} - 2\langle n_{x_1}, w_{\bar\theta}\rangle \sin\theta\frac{z_0}{|z_0|}$$

$$\frac{-1}{(1 + f'(|z_2|)^2)^{1/2}} = \frac{-1}{(1 + f'(|z_1|)^2)^{1/2}} - 2\langle n_{x_1}, w_{\bar\theta}\rangle \cos\theta.$$

Replacing the value of z_1 in the second equation using the first one we get

$$\left(\frac{f'(|z_2|)}{|z_2|(1 + f'(|z_2|)^2)^{1/2}} - \frac{f'(|z_1|)}{|z_1|(1 + f'(|z_1|)^2)^{1/2}}\right) z_2$$

$$= 2\sin\theta\left(\frac{f'(|z_1|)}{|z_1|(1 + f'(|z_1|)^2)^{1/2}}\langle x_1 - x_0, w_{\bar\theta}\rangle - \langle n_{x_1}, w_{\bar\theta}\rangle\right)\frac{z_0}{|z_0|}.$$

As the function $s \mapsto f'(s)/(s(1 + f'(s)^2)^{1/2})$ is strictly decreasing, the constant multiplying z_2 is different from zero if and only if $|z_1| \neq |z_2|$. In this case z_2 is proportional to z_0 and hence x_2 (and so x_1) belongs to the plane generated by x_0 and $(0, 1)$. By the proof of the two-dimensional case, this is not possible. Let us check that the case $|z_1| = |z_2|$ is not possible. From the first equation in (7.12) we get

$$|z_2|^2 = |z_1|^2 + 4\langle x_1 - x_0, w_{\bar\theta}\rangle \sin\bar\theta\left(\langle x_1 - x_0, w_{\bar\theta}\rangle \sin\bar\theta - \frac{\langle z_1, z_0\rangle}{|z_1||z_0|}\right).$$

If $|z_1| = |z_2|$ then

$$\langle x_1 - x_0, w_{\bar\theta} \rangle \sin\bar\theta - \frac{\langle z_1, z_0 \rangle}{|z_1||z_0|} = 0$$

and, in particular, $\langle z_1, z_0 \rangle < 0$. Hence

$$\langle n_{x_1}, w_{\bar\theta} \rangle = \frac{1}{(1 + f'(|z_1|)^2)^{1/2}} \left(f'(|z_1|) \frac{\langle z_1, z_0 \rangle}{|z_1||z_0|} \sin\bar\theta - \cos\bar\theta \right) < 0.$$

From the third equation in (7.12) we get

$$\frac{-1}{(1 + f'(|z_2|)^2)^{1/2}} > \frac{-1}{(1 + f'(|z_1|)^2)^{1/2}},$$

and, as the function $s \mapsto -s/(1 + f'(s)^2)^{1/2}$ is strictly increasing, we conclude that $|z_2| > |z_1|$, a contradiction.

$\square$

7.3. Examples

EXAMPLE 7.11. We consider the convex body or revolution $C = \{(x, y, z) : z \geqslant f(\sqrt{x^2 + y^2})\}$ determined by a convex function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, $f \in C^3(\mathbb{R})$, $f''' \leqslant 0$, and

$$\lim_{s \to \infty} \frac{f(s)}{s} = \infty.$$

This condition implies that the asymptotic cone of C has empty interior. Observe that C cannot be cylindrically bounded since f is defined on the whole real line.

For every $r > 0$, consider the unique point $(x(r), z(r))$, with $x(r) > 0$, in the intersection of the graph of f and the circle of center 0 and radius r. Let $\alpha(r)$ be the angle between the vectors $(x(r), z(r))$ and $(0, 1)$. Since $z(r) = f(x(r))$ we have

$$(7.13) \qquad \cos(\alpha(r)) = \frac{f(x(r))}{r}, \quad \sin(\alpha(r)) = \frac{x(r)}{r}.$$

It is immediate to check that $x(r)$ is an increasing function of r. On the other hand, the function $\alpha(s)$ is a decreasing function of s because of the convexity of f. An easy application of the coarea formula implies that the volume $V(r)$ of the ball $B_C(0, r)$ is given by

$$V(r) = 2\pi \int_0^r s^2 (1 - \cos\alpha(s)) \, ds$$

Writing $1 - \cos\alpha = \sin^2\alpha/(1 + \cos\alpha)$ and taking into account that $\alpha(s) \in [0, \frac{\pi}{2}]$ and that $s \sin\alpha(s) = x(s)$, we get

$$(7.14) \qquad \pi \int_{r_0}^r x(s)^2 \, ds \leqslant V(r) - V(r_0) \leqslant 2\pi \int_{r_0}^r x(s)^2 \, ds,$$

for all $0 < r_0 < r$.

From (7.13) we get $f(x(s))^2 + x(s)^2 = s^2$ and so

$$1 + \frac{x(s)^2}{f(x(s))^2} = \frac{s^2}{f(x(s))^2}.$$

Taking into account that $\lim_{s \to \infty} x(s) = \infty$ we get $\lim_{s \to \infty} \frac{x(s)}{f(x(s))} = 0$ and so

$$(7.15) \qquad \lim_{s \to \infty} \frac{s}{f(x(s))} = 1.$$

Since $\sin \alpha(s) = f(x(s))/s$ is an increasing function of s, we can find $r_0 > 0$ so that

(7.16)
$$\frac{1}{2} < \frac{f(x(s))}{s} \leqslant 1, \quad \text{for } s > r_0.$$

From now on we assume that $f(t) = t^a$, with $a \in (1, 2]$. Then (7.16) implies that

$$\frac{1}{2^{1/a}} s^{1/a} \leqslant x(s) \leqslant s^{1/a}, \quad s > r_0.$$

From (7.14) we have

$$\frac{\pi}{2^{2/a}} \int_{r_0}^{r} s^{2/a} ds \leqslant V(r) - V(r_0) \leqslant 2\pi \int_{r_0}^{r} s^{2/a} ds,$$

and

$$\int_{r_0}^{r} s^{2/a} ds = \left(\frac{a}{a+2}\right) \left(r^{(a+2)/a} - r_0^{(a+2)/a}\right).$$

We now make use of the properties

(i) $V_1 \leqslant V_2$ implies that $\phi_{V_1} \geqslant \phi_{V_2}$,

(ii) $V_1 \leqslant c + V_2$, $c \in \mathbb{R}$, implies that $\phi_{V_2}(v - c) \geqslant \phi_{V_1}(v)$.

to show the existence of constants $0 < \lambda < \Lambda$ such that

$$\Lambda v^{a/(a+2)} > \phi_V(v) > \lambda v^{a/(a+2)}, \quad v > v_0.$$

Hence there exists constants $0 < \lambda_1 < \lambda_2$ so that

$$\lambda_2 v^{2/(a+2)} > I_C(v) \geqslant \lambda_1 v^{2/(a+2)}, \quad v > v_0,$$

and this shows that the unbounded convex body

$$C_a = \{(x, y, z) \in \mathbb{R}^3 : z \geqslant (x^2 + y^2)^{a/2}\}$$

has isoperimetric dimension equal to $\frac{a+2}{a}$ for all $a \in (1, 2]$.

Bibliography

1. Gabriel Acosta and Ricardo G. Durán, *An optimal Poincaré inequality in L^1 for convex domains*, Proc. Amer. Math. Soc. **132** (2004), no. 1, 195–202 (electronic). MR 2021262 (2004j:26031)
2. F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc. **4** (1976), no. 165, viii+199. MR 0420406
3. F. Alter, V. Caselles, and A. Chambolle, *A characterization of convex calibrable sets in $\mathbb{R}^N$*, Math. Ann. **332** (2005), no. 2, 329–366. MR 2178065 (2006g:35091)
4. Maria Athanassenas, *A variational problem for constant mean curvature surfaces with free boundary*, J. Reine Angew. Math. **377** (1987), 97–107. MR 887402
5. Hédy Attouch and Gerald Beer, *On the convergence of subdifferentials of convex functions*, Archiv der Mathematik **60** (1993), no. 4, 389–400.
6. Christophe Bavard and Pierre Pansu, *Sur le volume minimal de $\mathbf{R}^2$*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 4, 479–490. MR 875084 (88b:53048)
7. Vincent Bayle, *Propriétés de concavité du profil isopérimétrique et applications*, Ph.D. thesis, Institut Fourier, 2003.
8. Vincent Bayle and César Rosales, *Some isoperimetric comparison theorems for convex bodies in Riemannian manifolds*, Indiana Univ. Math. J. **54** (2005), no. 5, 1371–1394. MR 2177105 (2006f:53040)
9. Pierre Bérard and Daniel Meyer, *Inégalités isopérimétriques et applications*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 3, 513–541. MR 690651 (84h:58147)
10. S. G. Bobkov, *On isoperimetric constants for log-concave probability distributions*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 1910, Springer, Berlin, 2007, pp. 81–88. MR 2347041 (2008j:60047)
11. Jürgen Bokowski and Emanuel Sperner, Jr., *Zerlegung konvexer Körper durch minimale Trennflächen*, J. Reine Angew. Math. **311/312** (1979), 80–100. MR 549959 (81b:52010)
12. T. Bonnesen and W. Fenchel, *Theory of convex bodies*, BCS Associates, Moscow, ID, 1987, Translated from the German and edited by L. Boron, C. Christenson and B. Smith. MR 920366 (88j:52001)
13. J. B. Bostwick and P. H. Steen, *Stability of constrained capillary surfaces*, Annu. Rev. Fluid Mech. **47** (2015), 539–568.
14. Herm Jan Brascamp and Elliott H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis **22** (1976), no. 4, 366–389. MR 0450480
15. Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR MR1835418 (2002e:53053)
16. Yu. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 285, Springer-Verlag, Berlin, 1988, Translated from the Russian by A. B. Sosinskiĭ, Springer Series in Soviet Mathematics. MR 936419 (89b:52020)
17. Isaac Chavel, *Isoperimetric inequalities*, Cambridge Tracts in Mathematics, vol. 145, Cambridge University Press, Cambridge, 2001, Differential geometric and analytic perspectives. MR 1849187
18. ———, *Riemannian geometry*, second ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, Cambridge, 2006, A modern introduction. MR 2229062
19. Jeff Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis, Princeton University Press, 1969, pp. 195–199.

20. Jaigyoung Choe, Mohammad Ghomi, and Manuel Ritoré, *Total positive curvature of hyper-surfaces with convex boundary*, J. Differential Geom. **72** (2006), no. 1, 129–147. MR 2215458 (2007a:53076)

21. _____ , *The relative isoperimetric inequality outside convex domains in* $\mathbf{R}^n$, Calc. Var. Partial Differential Equations **29** (2007), no. 4, 421–429. MR 2329803 (2008k:58042)

22. Jaigyoung Choe and Manuel Ritoré, *The relative isoperimetric inequality in Cartan-Hadamard 3-manifolds*, J. Reine Angew. Math. **605** (2007), 179–191. MR 2338131 (2009c:53044)

23. F. H. Clarke, *Optimization and nonsmooth analysis*, second ed., Classics in Applied Mathematics, vol. 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990. MR 1058436

24. Thierry Coulhon and Laurent Saloff-Coste, *Isopérimétrie pour les groupes et les variétés*, Rev. Mat. Iberoamericana **9** (1993), no. 2, 293–314. MR 1232845 (94g:58263)

25. Martin Dyer and Alan Frieze, *Computing the volume of convex bodies: a case where randomness provably helps*, Probabilistic combinatorics and its applications (San Francisco, CA, 1991), Proc. Sympos. Appl. Math., vol. 44, Amer. Math. Soc., Providence, RI, 1991, pp. 123–169. MR 1141926 (93a:52004)

26. Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. MR 1158660 (93f:28001)

27. Mouhamed Moustapha Fall, *Area-minimizing regions with small volume in Riemannian manifolds with boundary*, Pacific J. Math. **244** (2010), no. 2, 235–260. MR 2587431 (2011b:53070)

28. F. Fiala, *Le problème des isopérimètres sur les surfaces ouvertes à courbure positive*, Comment. Math. Helv. **13** (1941), 293–346. MR 0006422

29. A. Figalli and E. Indrei, *A Sharp Stability Result for the Relative Isoperimetric Inequality Inside Convex Cones*, J. Geom. Anal. **23** (2013), no. 2, 938–969. MR 3023863

30. Robert Finn, *Equilibrium capillary surfaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 284, Springer-Verlag, New York, 1986. MR 816345

31. Matteo Galli and Manuel Ritoré, *Existence of isoperimetric regions in contact sub-riemannian manifolds*, arXiv:1011.0633v1, 2010.

32. Sylvestre Gallot, *Inégalités isopérimétriques et analytiques sur les variétés riemanniennes*, Astérisque (1988), no. 163-164, 5–6, 31–91, 281 (1989), On the geometry of differentiable manifolds (Rome, 1986). MR 999971 (90f:58173)

33. Enrico Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser Verlag, Basel, 1984. MR 775682 (87a:58041)

34. E. Gonzalez, U. Massari, and I. Tamanini, *On the regularity of boundaries of sets minimizing perimeter with a volume constraint*, Indiana Univ. Math. J. **32** (1983), no. 1, 25–37. MR 684753 (84d:49043)

35. Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, english ed., Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2007, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR 2307192 (2007k:53049)

36. Michael Grüter, *Boundary regularity for solutions of a partitioning problem*, Arch. Rational Mech. Anal. **97** (1987), no. 3, 261–270. MR 862549 (87k:49050)

37. John E. Hutchinson and Yoshihiro Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. Partial Differential Equations **10** (2000), no. 1, 49–84. MR 1803974

38. R. Kannan, L. Lovász, and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom. **13** (1995), no. 3-4, 541–559. MR 1318794 (96e:52018)

39. Edgar Krahn, *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Mathematische Annalen **94** (1925), no. 1, 97–100.

40. Ernst Kuwert, *Note on the isoperimetric profile of a convex body*, Geometric analysis and nonlinear partial differential equations, Springer, Berlin, 2003, pp. 195–200. MR 2008339 (2004g:49065)

41. G. P. Leonardi and S. Rigot, *Isoperimetric sets on Carnot groups*, Houston J. Math. **29** (2003), no. 3, 609–637 (electronic). MR MR2000099 (2004d:28008)

42. P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana **1** (1985), no. 1, 145–201. MR 834360

43. _____, *The concentration-compactness principle in the calculus of variations. The limit case. II*, Rev. Mat. Iberoamericana **1** (1985), no. 2, 45–121. MR 850686

44. Pierre-Louis Lions and Filomena Pacella, *Isoperimetric inequalities for convex cones*, Proc. Amer. Math. Soc. **109** (1990), no. 2, 477–485. MR 90i:52021

45. Francesco Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012. MR 2976521

46. D. Michael, *Meniscus stability*, Annu. Rev. Fluid Mech. **13** (1981), 189–215.

47. Emanuel Milman, *On the role of convexity in isoperimetry, spectral gap and concentration*, Invent. Math. **177** (2009), no. 1, 1–43. MR 2507637 (2010j:28004)

48. Luciano Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98** (1987), no. 2, 123–142. MR 866718

49. Andrea Mondino and Stefano Nardulli, *Existence of isoperimetric regions in non-compact Riemannian manifolds under Ricci or scalar curvature conditions*, Comm. Anal. Geom. **24** (2016), 79–102.

50. Frank Morgan, *Clusters minimizing area plus length of singular curves*, Math. Ann. **299** (1994), no. 4, 697–714. MR 1286892

51. _____, *In polytopes, small balls about some vertex minimize perimeter*, J. Geom. Anal. **17** (2007), no. 1, 97–106. MR 2302876 (2007k:49090)

52. _____, *The Levy-Gromov isoperimetric inequality in convex manifolds with boundary*, J. Geom. Anal. **18** (2008), no. 4, 1053–1057. MR 2438911 (2009m:53079)

53. _____, *Geometric measure theory*, fourth ed., Elsevier/Academic Press, Amsterdam, 2009, A beginner's guide. MR 2455580 (2009i:49001)

54. Frank Morgan and David L. Johnson, *Some sharp isoperimetric theorems for Riemannian manifolds*, Indiana Univ. Math. J. **49** (2000), no. 3, 1017–1041. MR 1803220 (2002e:53043)

55. Stefano Nardulli, *Generalized existence of isoperimetric regions in non-compact riemannian manifolds and applications to the isoperimetric profile*, Asian Journal of Mathematics **18** (2014), no. 1, 1–28.

56. Stefano Nardulli and Pierre Pansu, *A discontinuous isoperimetric profile for a complete Riemannian manifold*, arXiv:1506.04892, 16 Jun 2015.

57. Takao Ohta and Kyozi Kawasaki, *Equilibrium morphology of block copolymer melts*, Macromolecules **19** (1986), no. 10, 2621–2632.

58. Frank Pacard and Manuel Ritoré, *From constant mean curvature hypersurfaces to the gradient theory of phase transitions*, J. Differential Geom. **64** (2003), no. 3, 359–423. MR 2032110

59. L. E. Payne and H. F. Weinberger, *An optimal Poincaré inequality for convex domains*, Arch. Rational Mech. Anal. **5** (1960), 286–292 (1960). MR 0117419 (22 #8198)

60. Renato H. L. Pedrosa and Manuel Ritoré, *Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems*, Indiana Univ. Math. J. **48** (1999), no. 4, 1357–1394. MR 1757077 (2001k:53120)

61. Ch. Pittet, *The isoperimetric profile of homogeneous Riemannian manifolds*, J. Differential Geom. **54** (2000), no. 2, 255–302. MR 1818180 (2002g:53088)

62. Manuel Ritoré, *Applications of compactness results for harmonic maps to stable constant mean curvature surfaces*, Math. Z. **226** (1997), no. 3, 465–481. MR 1483543

63. _____, *Examples of constant mean curvature surfaces obtained from harmonic maps to the two sphere*, Math. Z. **226** (1997), no. 1, 127–146. MR 1472144

64. _____, *Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces*, Comm. Anal. Geom. **9** (2001), no. 5, 1093–1138. MR 1883725

65. _____, *The isoperimetric problem in complete surfaces of nonnegative curvature*, J. Geom. Anal. **11** (2001), no. 3, 509–517. MR 1857855 (2002f:53109)

66. _____, *Continuity of the isoperimetric profile of a complete Riemannian manifold under sectional curvature conditions*, Revista Matemática Iberoamericana (to appear). arXiv:1503.07014, 24 Mar 2015.

67. Manuel Ritoré and Antonio Ros, *Stable constant mean curvature tori and the isoperimetric problem in three space forms*, Comment. Math. Helv. **67** (1992), no. 2, 293–305. MR 1161286

68. _____, *The spaces of index one minimal surfaces and stable constant mean curvature surfaces embedded in flat three manifolds*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 391–410. MR 1322955

69. Manuel Ritoré and César Rosales, *Existence and characterization of regions minimizing perimeter under a volume constraint inside Euclidean cones*, Trans. Amer. Math. Soc. **356** (2004), no. 11, 4601–4622 (electronic). MR 2067135 (2005g:49076)

70. Manuel Ritoré and Efstratios Vernadakis, *Large isoperimetric regions in the product of a compact manifold with Euclidean space*, arXiv:1312.1581, 5 Dec 2013.

71. _____ , *Isoperimetric inequalities in convex cylinders and cylindrically bounded convex bodies*, Calc. Var. Partial Differential Equations **54** (2015), no. 1, 643–663. MR 3385175

72. _____ , *Isoperimetric inequalities in Euclidean convex bodies*, Trans. Amer. Math. Soc. **367** (2015), no. 7, 4983–5014. MR 3335407

73. _____ , *Isoperimetric inequalities in conically bounded convex bodies*, J. Geom. Anal. **26** (2016), no. 1, 474–498. MR 3441524

74. R. Tyrrell Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR 0274683 (43 #445)

75. Antonio Ros, *Isoperimetric inequalities in crystallography*, J. Amer. Math. Soc. **17** (2004), no. 2, 373–388 (electronic). MR 2051615 (2005a:53012)

76. _____ , *The isoperimetric problem*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 175–209. MR 2167260 (2006e:53023)

77. Antonio Ros and Enaldo Vergasta, *Stability for hypersurfaces of constant mean curvature with free boundary*, Geom. Dedicata **56** (1995), no. 1, 19–33. MR 1338315

78. César Rosales, *Isoperimetric regions in rotationally symmetric convex bodies*, Indiana Univ. Math. J. **52** (2003), no. 5, 1201–1214. MR 2010323 (2004h:58018)

79. Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993. MR MR1216521 (94d:52007)

80. P. Steinhagen, *Über die größte Kugel in einer konvexen Punktmenge.*, Hamb. Abh. **1** (1921), 15–26 (German).

81. Peter Sternberg, *The effect of a singular perturbation on nonconvex variational problems*, Arch. Rational Mech. Anal. **101** (1988), no. 3, 209–260. MR 930124

82. Peter Sternberg and Kevin Zumbrun, *On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint*, Comm. Anal. Geom. **7** (1999), no. 1, 199–220. MR 1674097 (2000d:49062)

83. Edward Stredulinsky and William P. Ziemer, *Area minimizing sets subject to a volume constraint in a convex set*, J. Geom. Anal. **7** (1997), no. 4, 653–677. MR 1669207 (99k:49089)

84. Giorgio Talenti, *Best constant in Sobolev inequality*, Annali di Matematica pura ed Applicata **110** (1976), no. 1, 353–372.

85. Edwin L. Thomas, David M. Anderson, Chris S. Henkee, and David Hoffman, *Periodic area-minimizing surfaces in block copolymers*, Nature **334** (1988), 598–601.

86. Thomas I. Vogel, *Stability of a liquid drop trapped between two parallel planes*, SIAM J. Appl. Math. **47** (1987), no. 3, 516–525. MR 889635