AREA-STATIONARY SURFACES IN THE HEISENBERG GROUP $\mathbb{H}^1$

MANUEL RITORÉ AND CÉSAR ROSALES

Abstract. We consider area-stationary surfaces, perhaps with a volume constraint, in the Heisenberg group $\mathbb{H}^1$ endowed with its Carnot-Carathéodory distance. By analyzing the first variation of area, we characterize C^2 area-stationary surfaces as those with mean curvature zero (or constant if a volume-preserving condition is assumed) and such that the characteristic curves meet orthogonally the singular curves. Moreover, a Minkowski-type formula relating the area, the mean curvature, and the volume is obtained for volume-preserving area-stationary surfaces enclosing a given region.

As a consequence of the characterization of area-stationary surfaces, we refine the Bernstein type theorem given in [CHMY] and [GP] to describe all C^2 entire area-stationary graphs over the xy-plane in $\mathbb{H}^1$. A calibration argument shows that these graphs are globally area-minimizing.

Finally, by using the description of the singular set in [CHMY], the characterization of area-stationary surfaces, and the ruling property of constant mean curvature surfaces, we prove our main results where we classify area-stationary surfaces in $\mathbb{H}^1$, with or without a volume constraint, and non-empty singular set. In particular, we deduce the following counterpart to Alexandrov uniqueness theorem in Euclidean space: any compact, connected, C^2 surface in $\mathbb{H}^1$, area-stationary under a volume constraint, must be congruent to a rotationally symmetric sphere obtained as the union of all the geodesics of the same curvature joining two points. As a consequence, we solve the isoperimetric problem in $\mathbb{H}^1$ assuming C^2 smoothness of the solutions.

1. **Introduction**

In the last years the study of variational questions in sub-Riemannian geometry has received an increasing interest. In particular, the desire to achieve a better understanding of global variational questions involving the area, such as the *Plateau problem* or the *isoperimetric problem*, has motivated the recent development of a theory of *constant mean curvature surfaces* in the *Heisenberg group* $\mathbb{H}^1$ endowed with its *Carnot-Carathéodory distance.*

It is well known that constant mean curvature surfaces arise as critical points of the area for variations preserving the volume enclosed by the surface. In this paper, we are interested in surfaces immersed in the Heisenberg group which are *stationary points* of the sub-Riemannian area, with or without a *volume constraint.*

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In order to precise the situation and state our results we recall some facts about the Heisenberg group, that will be treated in more detail in Section 2.

We denote by $\mathbb{H}^1$ the 3-dimensional *Heisenberg group*, which we identify with the Lie group $\mathbb{C}\times\mathbb{R}$, where the product is given by

$$[z,t]*[z',t']=[z+z',t+t'+\operatorname{Im}\big(z\overline{z}'\big)].$$

The Lie algebra of $\mathbb{H}^1$ is generated by three left invariant vector fields $\{X,Y,T\}$ with one non-trivial bracket relation given by $[X,Y]=-2T$. The 2-dimensional distribution generated by $\{X,Y\}$ is called the *horizontal distribution* in $\mathbb{H}^1$. Usually $\mathbb{H}^1$ is endowed with a structure of sub-Riemannian manifold by considering the Riemannian metric on the horizontal distribution so that the basis $\{X,Y\}$ is orthonormal. This metric allows to measure the length of horizontal curves and to define the *Carnot-Carathéodory distance* between two points as the infimum of length of horizontal curves joining both points, see [Gr2]. It is known that the Carnot-Carathéodory distance can be approximated by the distance functions associated to a family of dilated Riemannian metrics, see [Gr1], [K1], [K2], [P3] and [M, §1.10]. With respect to the relevance of the Heisenberg group in the context of complex analysis, it is appropriate to mention the works by Folland and Stein [FS1], [FS2], and Korányi and Reimann [KR]. The Heisenberg group $\mathbb{H}^1$ is also a pseudo-hermitian manifold. It is the simplest one and can be seen as a blow-up of general pseudo-hermitian manifolds ([CHMY, Appendix]). In addition, $\mathbb{H}^1$ is also a *Carnot group* since its Lie algebra is stratified and 2-nilpotent, see [DGN1].

One can consider on $\mathbb{H}^1$ its Haar measure, which turns out to coincide with the Lebesgue measure in $\mathbb{R}^3$. From the notions of distance and measure one can also define the Minkowski content and the sub-Riemannian perimeter of a set, and the spherical Hausdorff measure of a surface, so that different surface measures may be given on $\mathbb{H}^1$. As it is shown in [MoSC] and [FSSC], all these notions of "area" coincide for a C^2 surface.

In this paper we introduce the notions of volume and area in $\mathbb{H}^1$ as follows. We consider the left invariant Riemannian metric $g=\big\langle\cdot\,,\cdot\big\rangle$ on $\mathbb{H}^1$ so that $\{X,Y,T\}$ is an orthonormal basis at every point. We define the volume $V(\Omega)$ of a Borel set $\Omega\subseteq\mathbb{H}^1$ as the Riemannian measure of the set. The area of an immersed C^1 surface Σ in $\mathbb{H}^1$ is defined as the integral

$$A(\Sigma)=\int_\Sigma |N_H|\,d\Sigma,$$

where N is a unit vector normal to the surface, N_H denotes the orthogonal projection onto the horizontal distribution, and $d\Sigma$ is the Riemannian area element induced on Σ by the metric g. This definition of area agrees for C^2 surfaces with the ones mentioned above.

With these notions of volume and area, we study in Section 4 surfaces in $\mathbb{H}^1$ which are *stationary* points of the area either for arbitrary variations, or for variations preserving the volume enclosed by the surface. As in Riemannian geometry, one may expect that some geometric quantity defined on such a surface vanishes or remains constant. By using the first variation of area in Lemma 4.3 we will see that any C^2 area-stationary surface under a volume constraint must have *constant mean curvature*. The mean curvature H of a surface Σ is defined in (4.8) as the Riemannian divergence relative to Σ of the *horizontal unit normal vector* to Σ given by $\nu_H = N_H/|N_H|$. We remark that a notion of mean curvature in $\mathbb{H}^1$ for graphs over the xy-plane was previously introduced by S. Pauls [Pa1]. A more general definition of mean curvature was proposed by J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang [CHMY], and by N. Garofalo and S. Pauls [GP]. As was shown in [RR] our definition agrees with all the previous ones.

The analysis of the *singular set* plays an important role in the study of area-stationary surfaces in $\mathbb{H}^1$. Given a surface Σ immersed in $\mathbb{H}^1$, the singular set Σ_0 of Σ is the set of points where Σ is tangent to the horizontal distribution. Its structure has been determined for surfaces with bounded mean curvature in [CHMY], where it is proved that Σ_0 consists of isolated points and C^1 curves, see Theorem 4.15 for a more detailed description. The *regular part* $\Sigma - \Sigma_0$ of Σ is foliated by horizontal curves called the *characteristic curves*. As pointed out in [CHMY], when the surface Σ has constant mean curvature H, any of these curves is part of a *geodesic* in $\mathbb{H}^1$ of curvature H. In particular, any surface in $\mathbb{H}^1$ with $H \equiv 0$ is foliated, up to the singular set, by horizontal straight lines.

The recent study of surfaces with constant mean curvature in $\mathbb{H}^1$ has mainly focused on minimal surfaces (those with $H \equiv 0$). In fact, many interesting questions of the classical theory of minimal surfaces in $\mathbb{R}^3$, such as the Plateau problem, the Bernstein problem, or the global behavior of properly embedded surfaces, have been treated in $\mathbb{H}^1$, see [Pa1], [CHMY], [GP], [CH], and [Pa2]. These works also provide a rich variety of examples of minimal surfaces. However, in spite of the last advances, very little is known about non-minimal constant mean curvature surfaces in $\mathbb{H}^1$. It is easy to check that a graph $t = u(x, y)$ of class C^2 in $\mathbb{H}^1$ with constant mean curvature H satisfies the following degenerate (elliptic and hyperbolic) PDE

$$(u_y+x)^2 u_{xx} - 2\,(u_y+x)(u_x-y)\,u_{xy} + (u_x-y)^2 u_{yy} = -2H\,((u_x-y)^2+(u_y+x)^2)^{3/2}.$$

In [CHMY] some relevant properties concerning the above equation, such as the uniqueness of solutions for the Dirichlet problem or the structure of the singular set, are studied. As to the examples, the only known complete surfaces with non-vanishing constant mean curvature are the compact spherical ones described in [P1], [P2], [Mo2] and [LM], and the complete surfaces of revolution that we classified in [RR].

Now we briefly describe the organization and the results obtained in this paper. After the preliminaries Section 2, we recall some facts about sub-Riemannian geodesics and we study *Jacobi fields* in Section 3. In Section 4 we look at the first variation of area and prove a Minkowski-type formula for an area-stationary surface under a volume constraint relating area, volume and the mean curvature, Theorem 4.12. Then, a detailed analysis of the first variation of area, together with the aforementioned description of the singular set in Theorem 4.15, leads us to prove in Theorem 4.17 that an immersed surface is area-stationary if and only if its mean curvature is zero (or constant under a volume constraint) and the characteristic curves meet orthogonally the singular curves. This result allows us to refine in Section 5 the Bernstein-type theorems given in [CHMY] and [GP] for C^2 minimal graphs in $\mathbb{H}^1$. We classify all entire area-stationary graphs in $\mathbb{H}^1$ over the xy-plane in Theorem 5.1, and show that they are globally area-minimizing in Theorem 5.3. In Section 6, we prove our main results, where we completely describe immersed area-stationary surfaces in $\mathbb{H}^1$ with non-empty singular set, Theorems 6.1, 6.11, and 6.15. As a consequence we deduce an Alexandrov uniqueness type theorem for compact surfaces, Theorem 6.10, and we solve the isoperimetric problem in $\mathbb{H}^1$ assuming C^2 regularity of the solutions in Theorem 7.2.

Now we describe our results in more detail.

A classical formula by Minkowski in Euclidean space involving the integral of the support function over a compact surface without boundary in $\mathbb{R}^3$ yields, in the particular case of constant mean curvature, the relation $A = 3HV$, where A is the area of the surface, V is the volume enclosed, and H is the mean curvature of the surface. Our analysis of the first variation of the sub-Riemannian area and the existence in $\mathbb{H}^1$ of a one-parameter group of dilations provide a Minkowski-type formula for a surface Σ which is area-stationary under a volume constraint in $\mathbb{H}^1$. Such a formula reads

$$3A = 8HV,$$

where A is the sub-Riemannian area of Σ, H the mean curvature of Σ, and V the volume enclosed.

From previous works, as [CHMY], [DGN1], [GP], and [RR], it was already known that a necessary condition for a surface Σ to be area-stationary is that the mean curvature of Σ must be zero (or constant if the surface is area-stationary under a volume constraint). In Theorem 4.17 we show that such a condition is not sufficient. To obtain a stationary point for the area we must require in addition that the *characteristic curves meet orthogonally the singular curves*. We prove this result by obtaining an expression for the first variation of area for arbitrary variations of the surface Σ, not only for those fixing the singular set. Observe that the situation is different from the one in Riemannian geometry, where stationary surfaces are precisely those with vanishing mean curvature.

As a consequence of this analysis, we show that most of the entire graphs obtained in [CHMY] and [GP] with mean curvature zero are not area-stationary. We refine their result to prove that the only entire area-stationary graphs over the xy-plane in $\mathbb{H}^1$ are the Euclidean planes and vertical rotations of the graphs

$$u(x, y) = xy + (ay + b),$$

where a, $b \in \mathbb{R}$. Geometrically, the latter surfaces can be described as the union of all horizontal lines in $\mathbb{H}^1$ which are orthogonal to a given horizontal line (the singular curve). By using a calibration argument, we can prove that they are globally area-minimizing. This result is similar to the Euclidean one, where planes, the only entire minimal graphs in $\mathbb{R}^3$, are area-minimizing. In [CHMY, §6], also by a calibration argument, it was proved that a compact portion of the regular part of a graph over the xy-plane with mean curvature zero is area-minimizing.

It was already known that the regular part of a surface Σ immersed in $\mathbb{H}^1$ with constant mean curvature H is foliated by horizontal geodesics of curvature H. We derive in Section 3 an intrinsic equation for such geodesics and for Jacobi fields, and show in Theorem 4.8 that the characteristic curves of the surface are geodesics of curvature H. This is the starting point, together with the local description of the singular set in Theorem 4.15, to construct new examples and to classify surfaces of constant mean curvature in $\mathbb{H}^1$.

In Section 6 we use this idea to describe any complete, volume-preserving area-stationary surface Σ in $\mathbb{H}^1$ with non-vanishing mean curvature and non-empty singular set. We prove in Theorem 6.1 that if Σ has at least one isolated singular point then it must be congruent to one of the compact spherical examples $\mathbb{S}_\lambda$ obtained as the union of all the geodesics of curvature $\lambda > 0$ joining two given points (Example 3.3). Then, we introduce in Proposition 6.3 a procedure to construct examples of complete surfaces with non-vanishing constant mean curvature λ. Geometrically these surfaces consist of a horizontal curve Γ in $\mathbb{H}^1$, from which geodesics of curvature λ leave (or enter) orthogonally. An analysis of the variational vector field associated to this family of geodesics is necessary to understand the behavior of the geodesics far away from Γ. It follows that the resulting surface has two singular curves apart from Γ. Moreover, the family of geodesics meets both curves orthogonally if and only if they are equidistant to Γ. This geometric property allows to conclude in Theorem 6.8 the strong restriction that *the singular curves of any volume-preserving area-stationary surface in* $\mathbb{H}^1$ *with* $H \neq 0$ *are geodesics of* $\mathbb{H}^1$. This is the key ingredient to classify in Theorem 6.11 all surfaces of this kind. It follows that they must be congruent either to the cylindrical embedded surfaces in Example 6.6 or to the helicoidal immersed surfaces in Example 6.7.

This technique can also be used to describe complete area-stationary surfaces with non-empty singular set. It was proved in [CH, Proposition 2.1] and [GP, Lemma 8.2] that Euclidean planes are the only complete minimal surfaces in $\mathbb{H}^1$ with at least one isolated singular point. In Theorem 6.15 we give the classification of complete area-stationary surfaces with non-empty singular set: they are either Euclidean planes, or congruent to the hyperbolic paraboloid $t = xy$, or congruent to the helicoidal surfaces in Example 6.14.

Alexandrov uniqueness theorem in Euclidean space states that the only embedded compact surfaces with constant mean curvature in $\mathbb{R}^3$ are round spheres. This result is not true for immersed surfaces as illustrated by the toroidal examples in [W]. In pseudo-hermitian geometry, an interesting restriction on the topology of an immersed compact surface with bounded mean curvature inside a 3-spherical pseudo-hermitian manifold was given in [CHMY], where it was proved that such a surface is homeomorphic either to a sphere or to a torus. As shown in [CHMY], this bound on the genus is optimal on the standard pseudo-hermitian 3-sphere, where examples of constant mean curvature spheres and tori may be given. This estimate on the genus is also valid in $\mathbb{H}^1$ since the proof is based on the local description of the singular set (Theorem 4.15) and on the Hopf Index Theorem. In Theorem 6.10 we prove the following counterpart in $\mathbb{H}^1$ to Alexandrov uniqueness theorem in $\mathbb{R}^3$: any compact, connected, C^2 immersed volume-preserving area-stationary surface Σ in $\mathbb{H}^1$ is congruent to a sphere $\mathbb{S}_\lambda$. In particular we deduce the non-existence of volume-preserving area-stationary immersed tori in $\mathbb{H}^1$.

Finally in Section 7 we study the *isoperimetric problem* in $\mathbb{H}^1$. This problem consists on finding sets in $\mathbb{H}^1$ minimizing the sub-Riemannian perimeter under a volume constraint. It was proved by G. P. Leonardi and S. Rigot [LR] that the solutions to this problem exist and they are bounded, connected, open sets. This information is clearly far from characterizing isoperimetric sets. In the last years many authors have tried to adapt to the Heisenberg group different proofs of the classical isoperimetric inequality in Euclidean space. In [Mo1], [Mo2], and [LM] it was shown that there is no a direct counterpart in $\mathbb{H}^1$ to the Brunn-Minkowski inequality in Euclidean space, with the consequence that the Carnot-Carathéodory balls in $\mathbb{H}^1$ cannot be the solutions. Recently, interest has focused on solving the isoperimetric problem restricted to certain sets with additional symmetries. It was proved by D. Danielli, N. Garofalo and D.-M. Nhieu that the sets bounded by the spherical surfaces $\mathbb{S}_\lambda$ are the unique solutions in the class of sets bounded by two C^1 radial graphs over the xy-plane enclosing the same volume above and below the xy-plane [DGN1, Theorem 14.6]. In [RR] we pointed out that assuming C^2 smoothness and rotationally symmetry of isoperimetric regions, these must be congruent to the spheres $\mathbb{S}_\lambda$. We finish this work by showing in Theorem 7.2 that the spherical surfaces $\mathbb{S}_\lambda$ are the unique isoperimetric regions in $\mathbb{H}^1$ assuming C^2 regularity of the solutions, solving a conjecture by P. Pansu [P2, p. 172]. The study of the regularity of area-stationary surfaces in the Heisenberg groups is a hard problem. Some interesting contributions to the subject are [Pa2], [CHY2], [BSC], [CCM1], and [CCM2]. Examples with low regularity are given in [R2].

After the distribution of this paper, we have noticed some related works. In [CHY1], interesting results for graphs in the Heisenberg group $\mathbb{H}^n$ have been established. In particular, the authors show in [CHY1, p. 285] that C^2 minimal graphs over the xy-plane in $\mathbb{H}^1$ are area-minimizing if and only if the characteristic curves meet orthogonally the singular curves. In [MoR] it is proved that the "ball sets" bounded by the spheres $\mathbb{S}_\lambda$ solve the isoperimetric problem in the class of sets of $\mathbb{H}^1$ which are convex in the Euclidean sense. In [DGN2] it is obtained that the "ball sets" are the unique isoperimetric regions in $\mathbb{H}^n$ in the class of sets bounded by the union of two non-negative C^2 graphs over a round ball of the $t = 0$ hyperplane enclosing the same volume above and below this hyperplane. This result is improved in [R1] for the more general setting of sets of finite perimeter inside a right vertical cylinder containing a horizontal section of the cylinder. The result in [R1] has been recently used in [Mo3] to solve the isoperimetric problem in $\mathbb{H}^1$ for rotationally invariant sets. In [BoC] the mean curvature flow of a C^2 convex surface in $\mathbb{H}^1$, described as the union of two radial graphs, is proved to converge to a sphere $\mathbb{S}_\lambda$. In [DGN3] the authors show that there exists a family of entire *intrinsic minimal graphs* in $\mathbb{H}^1$ that are not area-minimizing. In [BSCV], a general calibration method is used to study the Bernstein problem for entire regular intrinsic minimal graphs in the Heisenberg group $\mathbb{H}^n$. In [DGNP], vertical Euclidean planes are characterized as the only C^2 area-minimizing entire graphs in $\mathbb{H}^1$ with empty singular set. Finally we mention the interesting survey [CDPT], where the authors give a broad overview of the isoperimetric problem in $\mathbb{H}^n$.

The techniques of this paper have been used by A. Hurtado and C. Rosales [HR] to study area-stationary surfaces in the sub-Riemannian 3-sphere.

2. **Preliminaries**

The *Heisenberg group* $\mathbb{H}^1$ is the Lie group $(\mathbb{R}^3, *)$, where the product $*$ is defined, for any pair of points $[z,t]$, $[z',t'] \in \mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, as

$$[z,t] * [z',t'] := [z + z', t + t' + \operatorname{Im}(z\overline{z}')], \qquad (z = x + iy).$$

For $p \in \mathbb{H}^1$, the *left translation* by p is the diffeomorphism $L_p(q) = p * q$. A basis of left invariant vector fields (i.e., invariant by any left translation) is given by

$$X := \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \qquad Y := \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \qquad T := \frac{\partial}{\partial t}.$$

The *horizontal distribution* $\mathcal{H}$ in $\mathbb{H}^1$ is the smooth planar one generated by X and Y. The *horizontal projection* of a vector U onto $\mathcal{H}$ will be denoted by U_H. A vector field U is called *horizontal* if $U = U_H$. A *horizontal curve* is a C^1 curve whose tangent vector lies in the horizontal distribution.

We denote by $[U,V]$ the Lie bracket of two C^1 vector fields U, V on $\mathbb{H}^1$. Note that $[X,T] = [Y,T] = 0$, while $[X,Y] = -2T$. The last equality implies that $\mathcal{H}$ is a bracket generating distribution. Moreover, by Frobenius Theorem we have that $\mathcal{H}$ is nonintegrable. The vector fields X and Y generate the kernel of the (contact) 1-form $\omega := -y\,dx + x\,dy + dt$.

We shall consider on $\mathbb{H}^1$ the (left invariant) Riemannian metric $g = \big\langle\cdot\,,\cdot\big\rangle$ so that $\{X,Y,T\}$ is an orthonormal basis at every point, and the associated Levi-Civitá connection D. The modulus of a vector field U will be denoted by $|U|$. The following derivatives can be easily computed

$$\begin{aligned} &D_X X = 0, && D_Y Y = 0, && D_T T = 0, \\ &D_X Y = -T, && D_X T = Y, && D_Y T = -X, \\ &D_Y X = T, && D_T X = Y, && D_T Y = -X. \end{aligned} \tag{2.1}$$

For any vector field U on $\mathbb{H}^1$ we define $J(U) := D_U T$. Then we have $J(X) = Y$, $J(Y) = -X$ and $J(T) = 0$, so that $J^2 = -$Identity when restricted to the horizontal distribution. It is also clear that

$$\big\langle J(U),V\big\rangle + \big\langle U,J(V)\big\rangle = 0, \tag{2.2}$$

for any pair of vector fields U and V. The endomorphism J restricted to the horizontal distribution is an involution of $\mathcal{H}$ that, together with the contact 1-form $\omega = -y\,dx + x\,dy + dt$, provides a pseudo-hermitian structure on $\mathbb{H}^1$, as stated in the Appendix in [CHMY].

Let $\gamma : I \to \mathbb{H}^1$ be a piecewise C^1 curve defined on a compact interval $I \subset \mathbb{R}$. The *length* of γ is the usual Riemannian length $L(\gamma) := \int_I |\dot\gamma|$, where $\dot\gamma$ is the tangent vector of γ. For two given points in $\mathbb{H}^1$ we can find, by Chow's connectivity theorem [Gr2, p. 95], a horizontal curve joining these points. The *Carnot-Carathéodory distance* d_{cc} between two points in $\mathbb{H}^1$ is defined as the infimum of the length of horizontal curves joining the given points.

Now we introduce notions of volume and area in $\mathbb{H}^1$. The volume $V(\Omega)$ of a Borel set $\Omega \subseteq \mathbb{H}^1$ is the Riemannian volume of the left invariant metric g, which coincides with the Lebesgue measure in $\mathbb{R}^3$. Given a C^1 surface Σ immersed in $\mathbb{H}^1$, and a unit vector field N normal to Σ, we define the area of Σ by

$$A(\Sigma) := \int_\Sigma |N_H|\,d\Sigma, \tag{2.3}$$

where $N_H = N - \big\langle N,T\big\rangle\,T$, and $d\Sigma$ is the Riemannian area element on Σ. If Σ is a C^1 surface enclosing a bounded set Ω then $A(\Sigma)$ coincides with the $\mathbb{H}^1$-perimeter of Ω, as defined in [CDG], [FSSC, Prop. 2.14] and [RR]. The area of Σ also coincides with the Minkowski content in $(\mathbb{H}^1, d_{cc})$ of a set $\Omega \subset \mathbb{H}^1$ bounded by a C^2 surface Σ, as proved in [MoSC, Theorem 5.1], and with the 3-dimensional spherical Hausdorff measure in $(\mathbb{H}^1, d_{cc})$ of Σ, see [FSSC, Corollary 7.7].

For a C^1 surface $\Sigma \subset \mathbb{H}^1$ the *singular set* Σ_0 consists of those points $p \in \Sigma$ for which the tangent plane $T_p\Sigma$ coincides with the horizontal distribution. As Σ_0 is closed and has empty interior in Σ, the *regular set* $\Sigma - \Sigma_0$ of Σ is open and dense in Σ. It was proved in [De, Lemme 1], see also [Ba, Theorem 1.2], that, for a C^2 surface, the Hausdorff dimension with respect to the Riemannian distance on $\mathbb{H}^1$ of Σ_0 is less than or equal to one.

If Σ is a C^1 oriented surface with unit normal vector N, then we can describe the singular set $\Sigma_0 \subset \Sigma$, in terms of N_H, as $\Sigma_0 = \{p \in \Sigma : N_H(p) = 0\}$. In the regular part $\Sigma - \Sigma_0$, we can define the *horizontal unit normal vector* ν_H, as in [DGN1], [RR] and [GP] by

$$\nu_H := \frac{N_H}{|N_H|}. \tag{2.4}$$

Consider the *characteristic vector field* Z on $\Sigma - \Sigma_0$ given by

$$Z := J(\nu_H). \tag{2.5}$$

As Z is horizontal and orthogonal to ν_H, we conclude that Z is tangent to Σ. Hence Z_p generates the intersection of $T_p\Sigma$ with the horizontal distribution. The integral curves of Z in $\Sigma - \Sigma_0$ will be called *characteristic curves* of Σ. They are both tangent to Σ and horizontal. Note that these curves depend on the unit normal N to Σ. If we define

$$S := \langle N, T\rangle \, \nu_H - |N_H| \, T, \tag{2.6}$$

then $\{Z_p, S_p\}$ is an orthonormal basis of $T_p\Sigma$ whenever $p \in \Sigma - \Sigma_0$.

In the Heisenberg group $\mathbb{H}^1$ there is a one-parameter group of *dilations* $\{\varphi_s\}_{s\in\mathbb{R}}$ generated by the vector field

$$W := xX + yY + 2tT. \tag{2.7}$$

From the Christoffel symbols (2.1), it can be easily proved that $\operatorname{div} W = 4$, where $\operatorname{div} W$ is the Riemannian divergence of the vector field W. We may compute φ_s in coordinates to obtain

$$\varphi_s(x_0, y_0, t_0) = (e^s x_0, e^s y_0, e^{2s} t_0). \tag{2.8}$$

From this expression we get, for fixed s and $p \in \mathbb{H}^1$, that $(d\varphi_s)_p(X_p) = e^s X_{\varphi_s(p)}$, $(d\varphi_s)_p(Y_p) = e^s Y_{\varphi_s(p)}$, and $(d\varphi_s)_p(T_p) = e^{2s} T_{\varphi_s(p)}$.

Any isometry of $(\mathbb{H}^1, g)$ leaving invariant the horizontal distribution preserves the area of surfaces in $\mathbb{H}^1$. Examples of such isometries are left translations, which act transitively on $\mathbb{H}^1$. The Euclidean rotation of angle θ about the t-axis given by

$$(x, y, t) \mapsto r_\theta(x, y, t) := (\cos\theta\, x - \sin\theta\, y, \sin\theta\, x + \cos\theta\, y, t),$$

is also an isometry in $(\mathbb{H}^1, g)$ preserving the horizontal distribution since it transforms the orthonormal basis $\{X, Y, T\}$ at the point p into the orthonormal basis $\{\cos\theta\, X + \sin\theta\, Y, -\sin\theta\, X + \cos\theta\, Y, T\}$ at the point $r_\theta(p)$.

3. Geodesics and Jacobi fields in the Heisenberg group $\mathbb{H}^1$

Usually, geodesics in $\mathbb{H}^1$ are defined as horizontal curves whose length coincides with the Carnot-Carathéodory distance between its endpoints. It is known that geodesics in $\mathbb{H}^1$ are curves of class C^∞, [K1] (see also [St] and [Mo1, Lemma 2.5]). We are interested in computing the equations of geodesics in terms of geometric data of the left invariant metric g in $\mathbb{H}^1$. For that we shall think of a geodesic in $\mathbb{H}^1$ as a smooth horizontal curve that is a critical point of length under any variation by horizontal curves with fixed endpoints. In this section we will obtain an *intrinsic* equation for the geodesics in terms of the left invariant metric g. Another explicit derivation of geodesics, which makes use of the Riemannian approximation scheme, can be found in [K1].

Let $\gamma : I \to \mathbb{H}^1$ be a C^2 horizontal curve defined on a compact interval $I \subset \mathbb{R}$. A variation of γ is a C^2 map $F : I \times J \to \mathbb{H}^1$, where J is an open interval around the origin, such that $F(s,0) = \gamma(s)$. We denote $\gamma_\varepsilon(s) = F(s,\varepsilon)$. Let $V_\varepsilon(s)$ be the vector field along γ_ε given by $(\partial F/\partial\varepsilon)(s,\varepsilon)$. Trivially $[V_\varepsilon, \dot\gamma_\varepsilon] = 0$. Let $V = V_0$. We say that the variation is *admissible* if the curves γ_ε are horizontal and have fixed boundary points. For such a variation it is clear that V vanishes at the endpoints of γ. Moreover, we have $\langle \dot\gamma_\varepsilon, T\rangle = 0$. As a consequence

$$\begin{aligned}
0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \langle \dot\gamma_\varepsilon, T\rangle &= \langle D_V \dot\gamma_\varepsilon, T\rangle + \langle \dot\gamma, D_V T\rangle \\
&= \langle D_{\dot\gamma} V, T\rangle + \langle \dot\gamma, J(V)\rangle \\
&= \dot\gamma\big(\langle V, T\rangle\big) - \langle V, D_{\dot\gamma} T\rangle + \langle \dot\gamma, J(V_H)\rangle \\
&= \dot\gamma\big(\langle V, T\rangle\big) - \langle V_H, J(\dot\gamma)\rangle + \langle \dot\gamma, J(V_H)\rangle \\
&= \dot\gamma\big(\langle V, T\rangle\big) - 2\,\langle V_H, J(\dot\gamma)\rangle,
\end{aligned}$$

where in the last equality we have used (2.2).

Conversely, if V is a C^1 vector field along γ vanishing at the endpoints and satisfying the equation

$$\dot\gamma\big(\langle V, T\rangle\big) = 2\,\langle V_H, J(\dot\gamma)\rangle, \tag{3.1}$$

then it is easy to check that there is an admissible variation of γ so that the associated vector field coincides with V. Indeed, since $V = f\dot\gamma + V_0$, with $V_0 \perp \dot\gamma$, we may assume that V is orthogonal to γ. Define, for $s \in I$ and ε small, $F(s,\varepsilon) := \exp_{\gamma(s)}(\varepsilon\, V(s))$, where exp is the exponential map associated to the Riemannian metric g in $\mathbb{H}^1$. If V is horizontal in some interval of γ then, by (3.1), we have $V = V_H = \lambda\dot\gamma$, so that V vanishes. If $V(s_0)$ is not horizontal, F defines locally a surface which is transversal to the horizontal distribution. This surface is foliated by horizontal curves. So there is a C^2 function $f(s,\varepsilon)$ such that $\gamma_\varepsilon(s) := \exp_{\gamma(s)}(f(s,\varepsilon)\, V(s))$ is a horizontal curve. We may take f so that $(\partial f/\partial\varepsilon)(s_0,0) = 1$. The vector field V_1 associated to the variation by horizontal curves γ_ε is given by $(\partial f/\partial\varepsilon)(s,0)\, V(s)$, and satisfies equation (3.1). Since V also satisfies this equation we obtain that $(\partial^2 f/\partial s\,\partial\varepsilon)(s,0) = 0$, and $(\partial f/\partial\varepsilon)(s,0)$ is constant. As $(\partial f/\partial\varepsilon)(s_0,0) = 1$ we conclude that $V_1(s) = V(s)$.

Proposition 3.1. *Let $\gamma : I \to \mathbb{H}^1$ be a C^2 horizontal curve parameterized by arc-length. Then γ is a critical point of length for any admissible variation if and only* *if there is $\lambda \in \mathbb{R}$ such that γ satisfies the second order ordinary differential equation*

$$D_{\dot\gamma}\dot\gamma + 2\lambda\, J(\dot\gamma) = 0. \tag{3.2}$$

Proof. Let V be the vector field of an admissible variation γ_ε of γ. Since γ is parameterized by arc-length, by the first variation of length [ChE, §1,(1.3)], we know that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L(\gamma_\varepsilon) = -\int_I \big\langle D_{\dot\gamma}\dot\gamma, V\big\rangle. \tag{3.3}$$

Suppose that γ is a critical point of length for any admissible variation. As $|\dot\gamma| = 1$ we deduce that $\big\langle D_{\dot\gamma}\dot\gamma, \dot\gamma\big\rangle = 0$. On the other hand, as γ is a horizontal curve, we have $\big\langle D_{\dot\gamma}\dot\gamma, T\big\rangle = 0$. So $D_{\dot\gamma}\dot\gamma$ is proportional to $J(\dot\gamma)$ at any point of γ. Assume, without loss of generality, that $I = [0, a]$. Consider a C^1 function $f : I \to \mathbb{R}$ vanishing at the endpoints and such that $\int_I f = 0$. Let V be the vector field on γ so that $V_H = f\, J(\dot\gamma)$ and $\big\langle V, T\big\rangle(s) = 2\int_0^s f$. As V satisfies (3.1), inserting it in the first variation of length (3.3), we obtain

$$\int_I f\,\big\langle D_{\dot\gamma}\dot\gamma, J(\dot\gamma)\big\rangle = 0.$$

As f is an arbitrary C^1 mean zero function we conclude that $\big\langle D_{\dot\gamma}\dot\gamma, J(\dot\gamma)\big\rangle$ is constant. Hence we have found $\lambda \in \mathbb{R}$ so that γ satisfies equation (3.2). The proof of the converse follows taking into account (3.3) and (3.1). $\square$

We will say that a C^2 horizontal curve γ is a *geodesic of curvature* λ if it is parameterized by arc-length and satisfies equation (3.2). Observe that the parameter λ in (3.2) changes to $-\lambda$ for the reversed curve $\gamma(-s)$.

Given a point $p \in \mathbb{H}^1$, a unit horizontal vector $v \in T_p\mathbb{H}^1$, and $\lambda \in \mathbb{R}$, we denote by $\gamma^\lambda_{p,v}$ the unique solution to (3.2) with initial conditions $\gamma(0) = p$, $\dot\gamma(0) = v$. Note that $\gamma^\lambda_{p,v}$ is a geodesic since it is horizontal and parameterized by arc-length (the functions $\big\langle\dot\gamma, T\big\rangle$ and $|\dot\gamma|^2$ are constant along any solution of (3.2)).

Consider a C^2 curve $\gamma(s) = (x(s), y(s), t(s))$ parameterized by arc-length. Let $(x_0, y_0, t_0) = (x(0), y(0), t(0))$, $(A, B) = (\dot x(0), \dot y(0))$. If γ is a geodesic, a straightforward computation from equation (3.2) gives, for curvature $\lambda \neq 0$,

$$\begin{aligned}
x(s) &= x_0 + A\left(\frac{\sin(2\lambda s)}{2\lambda}\right) + B\left(\frac{1-\cos(2\lambda s)}{2\lambda}\right),\\
y(s) &= y_0 - A\left(\frac{1-\cos(2\lambda s)}{2\lambda}\right) + B\left(\frac{\sin(2\lambda s)}{2\lambda}\right),\\
t(s) &= t_0 + \frac{1}{2\lambda}\left(s - \frac{\sin(2\lambda s)}{2\lambda}\right)\\
&\qquad + (Ax_0 + By_0)\left(\frac{1-\cos(2\lambda\, s)}{2\lambda}\right) - (Bx_0 - Ay_0)\left(\frac{\sin(2\lambda\, s)}{2\lambda}\right),
\end{aligned} \tag{3.4}$$

which are the parametric equations of Euclidean helices with vertical axis. For curvature $\lambda = 0$, we get

$$\begin{aligned} x(s) &= x_0 + As, \\ y(s) &= y_0 + Bs, \\ t(s) &= t_0 + (Ay_0 - Bx_0)\, s, \end{aligned}$$

which are Euclidean horizontal lines. Similar expressions for the geodesics in $\mathbb{H}^1$ can be found in numerous references, [K1], [Be, p. 28], [M, §1], [Mo1, p. 160] and [Ha, Ex. 8.5], amongst others. We conclude as in [M, Prop. 1.7] that complete geodesics in $\mathbb{H}^1$ are horizontal lifts of curves with constant geodesic curvature in the Euclidean xy-plane (circles or straight lines).

Remark 3.2. 1. Any isometry in $(\mathbb{H}^1, g)$ preserving the horizontal distribution transforms geodesics in geodesics since it respects the Levi-Civitá connection and commutes with J.

2. A dilation $\varphi_s(x, y, t) = (e^s x, e^s y, e^{2s} t)$ takes geodesics of curvature λ to geodesics of curvature $e^{-s}\lambda$.

3. If we consider the geodesic $\gamma^\lambda_{0,v}$, where v is a horizontal unit vector in $T_0\mathbb{H}^1$ and $\lambda \neq 0$, then the coordinate $t(s)$ in (3.4) is monotone increasing and unbounded. It follows that $\gamma^\lambda_{0,v}$ leaves every compact set in finite time. The same is true for any other horizontal geodesic, since it can be transformed into $\gamma^\lambda_{0,v}$ by a left translation followed by a rotation.

Example 3.3 (Spheres in $\mathbb{H}^1$)**.** Given $\lambda > 0$, we define $\mathbb{S}_\lambda$ as the union of all geodesics $\gamma^\lambda_{0,v}$ restricted to the interval $[0, \pi/\lambda]$. It is not difficult to see that $\mathbb{S}_\lambda$ is a compact embedded surface of revolution homeomorphic to a sphere, see Figure 1. These surfaces were first described by P. Pansu in [P1], [P2]. Any $\mathbb{S}_\lambda$ has two singular points at the *poles* $(0, 0, 0)$ and $(0, 0, \pi/(2\lambda^2))$. Alternatively, it was proved in [LM, Proof of Theorem 3.3] that $\mathbb{S}_\lambda$ can be described as the union of the following radial graphs over the xy-plane

$$t = \frac{\pi}{4\lambda^2} \pm \frac{1}{2\lambda^2} \left(\lambda\rho\sqrt{1 - \lambda^2\rho^2} + \arccos(\lambda\rho)\right), \qquad \rho = \sqrt{x^2 + y^2} \leqslant \frac{1}{\lambda}. \tag{3.5}$$

From (3.5) we can see that $\mathbb{S}_\lambda$ is C^2 but not C^3 around the poles. This was also observed in [DGN1, Proposition 14.11].

Now, we prove an analytic property for the vector field associated to the variation of a geodesic.

Lemma 3.4. *Let $\gamma : I \to \mathbb{H}^1$ be a geodesic of curvature λ, and V the C^1 vector field associated to a variation of γ by horizontal curves parameterized by arc-length. Then the function*

$$\lambda \left\langle V, T \right\rangle + \left\langle V, \dot\gamma \right\rangle$$

is constant along γ.

Proof. First note that

$$\dot\gamma\left(\left\langle V, T\right\rangle\right) = \left\langle D_{\dot\gamma} V, T\right\rangle + \left\langle V, J(\dot\gamma)\right\rangle = \left\langle D_V \dot\gamma, T\right\rangle - \left\langle \dot\gamma, J(V)\right\rangle = -2\left\langle \dot\gamma, J(V)\right\rangle,$$

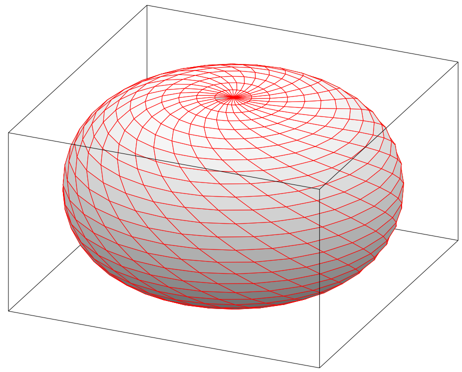

FIGURE 1. A spherical surface $\mathbb{S}_\lambda$ given by the union of all the geodesics of curvature λ joining the poles.

where we have used $[V,\dot\gamma]=0$, equality (2.2), and that γ is a horizontal curve. On the other hand, we have

$$\dot\gamma\,(\langle V,\dot\gamma\rangle)=\langle D_{\dot\gamma}V,\dot\gamma\rangle+\langle V,-2\lambda\,J(\dot\gamma)\rangle=\langle D_V\dot\gamma,\dot\gamma\rangle+2\lambda\,\langle\dot\gamma,J(V)\rangle=2\lambda\,\langle\dot\gamma,J(V)\rangle,$$

since γ is parameterized by arc-length and satisfies (3.2). From the two equations above the result follows. $\square$

As in Riemannian geometry we may expect that the vector field associated to a variation of a given geodesic by geodesics of the same curvature satisfies a certain second order differential equation. In fact, we have

Lemma 3.5. *Let γ_ε be a variation of γ by geodesics of the same curvature λ. Assume that the associated vector field V is C^2. Then V satisfies*

$$\ddot V+\mathrm{R}(V,\dot\gamma)\dot\gamma+2\lambda\,(J(\dot V)-\langle V,\dot\gamma\rangle\,T)=0, \tag{3.6}$$

where R *denotes the Riemannian curvature tensor in* $(\mathbb{H}^1,g)$.

Proof. As any γ_ε is a geodesic of curvature λ, we have

$$D_{\dot\gamma_\varepsilon}\dot\gamma_\varepsilon+2\lambda\,J(\dot\gamma_\varepsilon)=0.$$

Thus, if we derive with respect to V and we take into account that $D_VD_{\dot\gamma}\dot\gamma=\mathrm{R}(V,\dot\gamma)\dot\gamma+D_{\dot\gamma}D_V\dot\gamma+D_{[V,\dot\gamma]}\dot\gamma$ and that $[V,\dot\gamma]=0$, we deduce

$$\ddot V+\mathrm{R}(V,\dot\gamma)\dot\gamma+2\lambda\,D_VJ(\dot\gamma)=0.$$

Finally, it is not difficult to see that

$$D_VJ(\dot\gamma)=J(D_V\dot\gamma)-\langle V,\dot\gamma\rangle\,T=J(\dot V)-\langle V,\dot\gamma\rangle\,T,$$

and the proof follows. $\square$

We call (3.6) the *Jacobi equation* for geodesics in $\mathbb{H}^1$ of curvature λ. It is clearly a linear equation. Any solution of (3.6) is a *Jacobi field* along γ. It is easy to check that $V=f\dot\gamma$ is a Jacobi field if and only if $\ddot f\dot\gamma-2\lambda\dot fJ(\dot\gamma)=0$. Thus, any tangent Jacobi field to γ is of the form $(as+b)\,\dot\gamma$, with $a=0$ when $\lambda\neq0$.

4. Area-stationary surfaces. A Minkowski-type formula in $\mathbb{H}^1$

In this section we shall consider critical surfaces for the area functional (2.3) with or without a volume constraint. Let Σ be an oriented immersed surface of class C^2 in $\mathbb{H}^1$. Consider a C^2 vector field U with compact support on Σ. Denote by Σ_t, for t small, the immersed surface $\{\exp_p(tU_p); p \in \Sigma\}$, where $\exp_p$ is the exponential map of $(\mathbb{H}^1, g)$ at the point p. The family $\{\Sigma_t\}$, for t small, is the *variation* of Σ induced by U. We remark that our variations can move the singular set Σ_0 of Σ. Define $A(t) := A(\Sigma_t)$. In case Σ is an embedded compact surface, it encloses a region Ω so that $\Sigma = \partial\Omega$. Let Ω_t be the region enclosed by Σ_t and define $V(t) := V(\Omega_t)$. We say that the variation is *volume-preserving* if $V(t)$ is constant for t small enough. We say that Σ is *area-stationary* if $A'(0) = 0$ for any variation of Σ. In case that Σ encloses a bounded region, we say that Σ is *area-stationary under a volume constraint* or *volume-preserving area-stationary* if $A'(0) = 0$ for any volume-preserving variation of Σ.

Suppose that Ω is the set bounded by a C^2 embedded compact surface $\Sigma = \partial\Omega$. We shall always choose the unit *inner* normal N to Σ. The computation of $V'(0)$ is well known since the volume is the one associated to a Riemannian metric, and we have ([S, §9])

$$V'(0) = \int_\Omega \operatorname{div} U \, dv = -\int_\Sigma u \, d\Sigma, \tag{4.1}$$

where $u = \langle U, N\rangle$, and dv is the Riemannian volume element. It follows that u has mean zero whenever the variation is volume-preserving. Conversely, it was proven in [BdCE, Lemma 2.2] that, given a C^1 function $u : \Sigma \to \mathbb{R}$ with mean zero, a volume-preserving variation of Ω can be constructed so that the normal component of the associated vector field equals u.

Remark 4.1. Let Σ be a C^1 compact immersed surface in $\mathbb{H}^1$. Observe that the vector field W defined in (2.7) satisfies $\operatorname{div} W = 4$, so that if Σ is embedded, the divergence theorem yields

$$\text{volume enclosed by } \Sigma = -\frac{1}{4}\int_\Sigma \langle W, N\rangle \, d\Sigma, \tag{4.2}$$

where N is the inner unit normal to Σ. Formula (4.2) can be taken as a definition for the volume "enclosed" by an oriented compact immersed surface in $\mathbb{H}^1$. The first variation for this volume functional is given by (4.1). Also the *variation* of enclosed volume can be defined for a noncompact surface. We refer the reader to [BdCE] for details.

Now we will compute the first variation of area. We need the following lemma.

Lemma 4.2. *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 surface and N a unit vector normal to Σ. Consider a point $p \in \Sigma - \Sigma_0$, the horizontal normal ν_H defined in* (2.4)*, and $Z = J(\nu_H)$. Then, for any $u \in T_p\mathbb{H}^1$ we have*

$$D_u N_H = (D_u N)_H - \langle N, T\rangle \, J(u) - \langle N, J(u)\rangle \, T, \tag{4.3}$$

$$u\,(|N_H|) = \langle D_u N, \nu_H\rangle - \langle N, T\rangle \, \langle J(u), \nu_H\rangle, \tag{4.4}$$

$$D_u \nu_H = |N_H|^{-1} \left(\langle D_u N, Z\rangle - \langle N, T\rangle \, \langle J(u), Z\rangle\right) Z + \langle Z, u\rangle \, T. \tag{4.5}$$

Proof. Equalities (4.3) and (4.4) are easily obtained since $N_H = N - \langle N, T\rangle T$. Let us prove (4.5). As $|\nu_H| = 1$ and $\{(\nu_H)_p, Z_p, T_p\}$ is an orthonormal basis of $T_p\mathbb{H}^1$, we get

$$D_u\nu_H = \langle D_u\nu_H, Z\rangle Z + \langle D_u\nu_H, T\rangle T.$$

Note that $\langle D_u\nu_H, T\rangle = -\langle \nu_H, J(u)\rangle = \langle Z, u\rangle$ by (2.2). On the other hand, by using (4.3) and the fact that Z is tangent and horizontal, we deduce

$$\langle D_u\nu_H, Z\rangle = |N_H|^{-1}\langle D_uN_H, Z\rangle = |N_H|^{-1}\big(\langle D_uN, Z\rangle - \langle N, T\rangle\langle J(u), Z\rangle\big). \qquad \square$$

For a C^1 vector field U defined on a surface Σ, we denote by $U^\top$ and $U^\perp$ the tangent and orthogonal projections, respectively. We shall also denote by $\operatorname{div}_\Sigma U$ the Riemannian divergence of U relative to Σ, which is given by $\operatorname{div}_\Sigma U(p) := \sum_{1=1}^2 \langle D_{e_i}U, e_i\rangle$ for any orthonormal basis $\{e_1, e_2\}$ of $T_p\Sigma$. Let $L^1_{loc}(\Sigma)$ be the space of locally integrable functions with respect to the Riemannian measure on Σ. Now, we can prove

Lemma 4.3. *Let $\Sigma \subset \mathbb{H}^1$ be an oriented C^2 immersed surface. Suppose that U is a C^2 vector field with compact support on Σ and normal component $u = \langle U, N\rangle$. Then the first derivative at $t = 0$ of the area functional $A(t)$ associated to U is given by*

$$A'(0) = \int_\Sigma u\,\big(\operatorname{div}_\Sigma \nu_H\big)\,d\Sigma - \int_\Sigma \operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big)\,d\Sigma, \tag{4.6}$$

provided $\operatorname{div}_\Sigma \nu_H \in L^1_{loc}(\Sigma)$.

Moreover, if Σ is area-stationary (*resp. volume-preserving area-stationary*) *then*

$$A'(0) = \int_\Sigma u\,(\operatorname{div}_\Sigma \nu_H)\,d\Sigma. \tag{4.7}$$

Proof. First we remark that the Riemannian area of the singular set Σ_0 of Σ vanishes, as was proved in [De, Lemme 1] and [Ba, Theorem 1.2]. Thus functions defined on the regular set $\Sigma - \Sigma_0$ are measurable when considered on Σ .

Let $\{\Sigma_t\}$ be the variation of Σ associated to U, and let $d\Sigma_t$ be the Riemannian area element on Σ_t. Consider a C^1 vector field N whose restriction to Σ_t coincides with a unit vector normal to Σ_t. By using (2.3) and the coarea formula, we have

$$A(t) = \int_{\Sigma_t} |N_H|\,d\Sigma_t = \int_\Sigma (|N_H|\circ\varphi_t)\,|\operatorname{Jac}\varphi_t|\,d\Sigma = \int_{\Sigma-\Sigma_0} (|N_H|\circ\varphi_t)\,|\operatorname{Jac}\varphi_t|\,d\Sigma,$$

where $\varphi_t(p) = \exp_p(tU_p)$ and $\operatorname{Jac}\varphi_t$ is the Jacobian determinant of the map $\varphi_t : \Sigma \to \Sigma_t$. Now, we differentiate with respect to t, and we use the known fact that $(d/dt)|_{t=0}\,|\operatorname{Jac}\varphi_t| = \operatorname{div}_\Sigma U$ ([S, §9]), to get

$$\begin{aligned} A'(0) &= \int_{\Sigma-\Sigma_0} \{U(|N_H|) + |N_H|\,\operatorname{div}_\Sigma U\}\,d\Sigma \\ &= \int_{\Sigma-\Sigma_0} \{U^\perp(|N_H|) + \operatorname{div}_\Sigma(|N_H|\,U)\}\,d\Sigma \\ &= \int_{\Sigma-\Sigma_0} \{\operatorname{div}_\Sigma(|N_H|\,U^\top) + U^\perp(|N_H|) + |N_H|\,\operatorname{div}_\Sigma U^\perp\}\,d\Sigma \\ &= \int_{\Sigma-\Sigma_0} \{U^\perp(|N_H|) + |N_H|\,\operatorname{div}_\Sigma U^\perp\}\,d\Sigma. \end{aligned}$$

To obtain the last equality we have used the Riemannian divergence theorem to get that the integral of the divergence of the Lipschitz vector field $|N_H|\,U^\top$ over Σ vanishes (the modulus of a C^1 vector field in a Riemannian manifold is a Lipschitz function). We observe that the function $U^\perp(|N_H|) + |N_H|\,\operatorname{div}_\Sigma U^\perp$ is bounded in $\Sigma - \Sigma_0$ and so it lies in $L^1_{loc}(\Sigma)$.

On the other hand, we can use (4.4) to obtain

$$U^\perp(|N_H|) = \big\langle D_{U^\perp}N, \nu_H\big\rangle - \big\langle N, T\big\rangle\,\big\langle J(U^\perp), \nu_H\big\rangle = -\big\langle \nabla_\Sigma u, \nu_H\big\rangle,$$

since $J(U^\perp)$ is orthogonal to ν_H and $D_{U^\perp}N = -\nabla_\Sigma u$. Here $\nabla_\Sigma u$ represents the gradient of u relative to Σ. Then, we get in $\Sigma - \Sigma_0$

$$\begin{aligned} U^\perp(|N_H|) + |N_H|\,\operatorname{div}_\Sigma U^\perp &= -(\nu_H)^\top(u) + u\,|N_H|\,\operatorname{div}_\Sigma N \\ &= -\operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big) + u\,\operatorname{div}_\Sigma\big((\nu_H)^\top\big) \\ &\quad + u\,\operatorname{div}_\Sigma(|N_H|\,N) \\ &= -\operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big) + u\,\operatorname{div}_\Sigma \nu_H. \end{aligned}$$

As a consequence, we conclude that

$$\int_\Sigma \{U^\perp(|N_H|) + |N_H|\,\operatorname{div}_\Sigma U^\perp\}\,d\Sigma = \int_\Sigma u\,\big(\operatorname{div}_\Sigma \nu_H\big)\,d\Sigma - \int_\Sigma \operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big)\,d\Sigma.$$

Since we are assuming that $\operatorname{div}_\Sigma \nu_H \in L^1_{loc}(\Sigma)$ we conclude that $\operatorname{div}_\Sigma(u\,(\nu_H)^\top) \in L^1_{loc}(\Sigma)$ and so we have

$$A'(0) = \int_\Sigma u\,\big(\operatorname{div}_\Sigma \nu_H\big)\,d\Sigma - \int_\Sigma \operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big)\,d\Sigma.$$

Note that the second integral above vanishes by virtue of the Riemannian divergence theorem whenever u has compact support disjoint from the singular set Σ_0.

Now we shall prove (4.7) for area-stationary surfaces under a volume constraint. The proof for area-stationary ones follows with the obvious modifications. Inserting in (4.6) mean zero functions of class C^1 with compact support inside the regular set $\Sigma - \Sigma_0$, we get that $\operatorname{div}_\Sigma \nu_H$ is a constant function on $\Sigma - \Sigma_0$. If $u : \Sigma \to \mathbb{R}$ is any function, then we consider $v : \Sigma \to \mathbb{R}$ with support in $\Sigma - \Sigma_0$ such that $\int_\Sigma (u+v)\,d\Sigma = 0$. Inserting the mean zero function $u+v$ in (4.6), taking into account that $\operatorname{div}_\Sigma \nu_H$ is constant, and using the divergence theorem, we deduce that $\int_\Sigma \operatorname{div}_\Sigma(u\,(\nu_H)^\top)\,d\Sigma = 0$, and (4.7) is proved. □

Remark 4.4. The first variation of area (4.7) holds for any C^2 surface whenever the support of the vector field U is disjoint from the singular set, see also [RR, Lemma 3.2]. For area-stationary surfaces we have shown that (4.7) is also valid for vector fields moving the singular set.

For a C^2 immersed surface Σ in $\mathbb{H}^1$ with a C^1 unit normal vector N we define, as in [RR], the *mean curvature* H of Σ by the equality

$$-2H(p) := (\operatorname{div}_\Sigma \nu_H)(p), \qquad p \in \Sigma - \Sigma_0. \tag{4.8}$$

For any point in $\Sigma - \Sigma_0$ we consider the orthonormal basis of the tangent space to Σ given by the vector fields Z and S defined in (2.5) and (2.6). Then we have

$$-2H = \big\langle D_Z \nu_H, Z\big\rangle + \big\langle D_S \nu_H, S\big\rangle.$$

From (4.5) in Lemma 4.2 we get $\big\langle D_S\nu_H, S\big\rangle = 0$, and we conclude that

$$-2H = \big\langle D_Z\nu_H, Z\big\rangle = |N_H|^{-1}\,\big\langle D_Z N, Z\big\rangle. \tag{4.9}$$

By using variations supported in the regular set of a surface immersed in $\mathbb{H}^1$, the first variation of area (4.6), and the first variation of volume (4.1), we get

Corollary 4.5. *Let Σ be a C^2 oriented immersed surface in $\mathbb{H}^1$. Then*

(i) *If Σ is area-stationary then the mean curvature of $\Sigma - \Sigma_0$ vanishes.*
(ii) *If Σ is area-stationary under a volume constraint then the mean curvature of $\Sigma - \Sigma_0$ is constant.*

Remark 4.6. The first derivative of area for variations with compact support in the regular set, and the notion of mean curvature, were given by S. Pauls [Pa1] for graphs over the xy-plane in $\mathbb{H}^1$, and later extended by J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang [CHMY] for any surface inside a 3-dimensional pseudo-hermitian manifold. The case of the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ has been treated in [DGN1], [RR] and [BoC]. In [HP], R. Hladky and S. Pauls extend the notion of mean curvature and Corollary 4.5 for stationary surfaces inside vertically rigid sub-Riemannian manifolds. In the recent paper [CHY1] the first variation of area for graphs over $\mathbb{R}^{2n}$ has been computed for some more general variations moving the singular set. A definition of mean curvature by using Riemannian approximations to the Carnot-Carathéodory distance in $\mathbb{H}^1$ can be found in [Ni, p. 562] and [CDPT, §3].

Example 4.7. 1. According to our definition, the graph of a C^2 function $u(x, y)$ has constant mean curvature H if and only if satisfies the equation

$$(u_y+x)^2u_{xx} - 2\,(u_y+x)(u_x-y)\,u_{xy} + (u_x-y)^2u_{yy} = -2H\,((u_x-y)^2+(u_y+x)^2)^{3/2}$$

outside the singular set.

2. The spherical surface $\mathbb{S}_\lambda$ in Example 3.3 has constant mean curvature λ with respect to the inner normal vector. This can be seen by using the equation for constant mean curvature graphs above and (3.5). It was proved in [RR, Theorem 5.4] that $\mathbb{S}_\lambda$ is, up to a vertical translation, the unique C^2 compact surface of revolution around the t-axis with constant mean curvature λ.

The ruling property of constant mean curvature surfaces in $\mathbb{H}^1$, already observed in [CHMY, (2.1), (2.24)], [GP, Corollary 5.3] and [HP, Corollaries 4.5 and 6.10], follows immediately from the expression (4.9) for the mean curvature and the equation of geodesics (3.2).

Theorem 4.8. *Let Σ be an oriented immersed surface in $\mathbb{H}^1$ of class C^2 with constant mean curvature H outside the singular set. Then any characteristic curve of Σ coincides with an open arc of a geodesic of curvature H. As a consequence, the regular set of Σ is foliated by geodesics of curvature H.*

Proof. A characteristic curve γ is parameterized by arc-length since the tangent to γ is the characteristic vector field Z defined in (2.5). We must see that γ satisfies equation (3.2) for $\lambda = H$. For any point of this curve, the vector fields Z, ν_H and

T provide an orthonormal basis of the tangent space to $\mathbb{H}^1$. Thus, we have

$$\begin{aligned} D_{\dot\gamma}\dot\gamma = D_Z Z &= \big\langle D_Z Z, \nu_H\big\rangle\, \nu_H + \big\langle D_Z Z, T\big\rangle\, T \\ &= -\big\langle Z, D_Z \nu_H\big\rangle\, \nu_H - \big\langle Z, J(Z)\big\rangle\, T \\ &= 2H\nu_H = -2HJ(Z) = -2HJ(\dot\gamma), \end{aligned}$$

where in the last equalities we have used (4.9) and that $J(Z) = -\nu_H$. □

Remark 4.9. The ruling property is satisfied by minimal t-graphs whose horizontal unit normal lies in the Sobolev space $W^{1,1}$, see [Pa2, Theorem A], and by C^1 weak solutions of the constant mean curvature equation [CHY2, Theorem A]. See also the recently posted paper by L. Capogna, G. Citti and M. Manfredini [CCM2, Corollary 1.6].

Remark 4.10. Let Σ be a C^2 surface in $\mathbb{H}^1$ and φ_s the dilation of $\mathbb{H}^1$ defined in (2.8). The ruling property in Theorem 4.8 and the behavior of geodesics under φ_s (Remark 3.2) imply that Σ has constant mean curvature λ if and only if the dilated surface $\varphi_s(\Sigma)$ has constant mean curvature $e^{-s}\lambda$.

Now, we will prove a counterpart in $\mathbb{H}^1$ of the Minkowski formula for compact surfaces with constant mean curvature in $\mathbb{R}^3$. We need the following consequence of (4.7), Corollary 4.5 and the definition of the mean curvature

Corollary 4.11. *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 surface enclosing a bounded region Ω. Then Σ is volume-preserving area-stationary if and only if there is a real constant H such that Σ is a critical point of the functional $A - 2HV$ for any given variation.*

This corollary and the existence in $\mathbb{H}^1$ of a one-parameter group of dilations allow us to prove the following Minkowski-type formula for volume-preserving area-stationary surfaces enclosing a bounded region in $\mathbb{H}^1$. The result also holds for oriented compact immersed surfaces in $\mathbb{H}^1$ when the volume is given by (4.2).

Theorem 4.12 (Minkowski formula in $\mathbb{H}^1$)**.** *Let $\Sigma \subset \mathbb{H}^1$ be a volume-preserving area-stationary C^2 surface enclosing a bounded region Ω. Then we have*

$$3A(\Sigma) = 8HV(\Omega), \tag{4.10}$$

where H is the mean curvature of Σ with respect to the inner normal vector.

Proof. We take the vector field W in (2.7) and the one-parameter group of dilations $\{\varphi_s\}_{s\in\mathbb{R}}$ in (2.8). Let $\Omega_s = \varphi_s(\Omega)$ and $\Sigma_s = \partial\Omega_s$. Denote $V(s) := V(\Omega_s)$ and $A(s) := A(\Sigma_s)$. From the Christoffel symbols (2.1), it can be easily proved that $\operatorname{div} W = 4$, where $\operatorname{div} W$ is the Riemannian divergence of W. By the first variation formula of volume (4.1) we have

$$V'(0) = \int_\Omega \operatorname{div} W = 4\,V(\Omega),$$

and so $V(s) = e^{4s}V(\Omega)$.

Let us calculate now the variation of area $A'(0)$. Recall that for fixed s and $p \in \mathbb{H}^1$, we have $(d\varphi_s)_p(X_p) = e^s X_{\varphi_s(p)}$, $(d\varphi_s)_p(Y_p) = e^s Y_{\varphi_s(p)}$, and $(d\varphi_s)_p(T_p) = e^{2s}T_{\varphi_s(p)}$. Let N be the inner unit normal to Σ, and $p \in \Sigma$. From the calculus of $(d\varphi_s)_p$ we see that φ_s preserves the horizontal distribution, so that p lies in the regular part of Σ if and only if $\varphi_s(p)$ lies in the regular part of Σ_s. Assume p is a regular point of Σ. Then we can choose α, $\beta\in\mathbb{R}$ so that $\{e_1,e_2\}$, with $e_1=\cos\alpha\,X_p+\sin\alpha\,Y_p$, and $e_2=\cos\beta\,(-\sin\alpha\,X_p+\cos\alpha\,Y_p)+\sin\beta\,T_p$, is an orthonormal basis of $T_p\Sigma$. For the normal N we have $\pm N_p=-\sin\beta\,(-\sin\alpha\,X_p+\cos\alpha\,Y_p)+\cos\beta\,T_p$, and so $|N_H|_p=|\sin\beta|$. We have $(d\varphi_s)_p(e_1)=e^s\,(\cos\alpha\,X_{\varphi_s(p)}+\sin\alpha\,Y_{\varphi_s(p)})$, and $(d\varphi_s)_p(e_2)=e^s\cos\beta\,(-\sin\alpha\,X_{\varphi_s(p)}+\cos\alpha\,Y_{\varphi_s(p)})+e^{2s}\sin\beta\,T_{\varphi_s(p)}$, and so $|\text{Jac}(\varphi_s)|_p=e^{2s}(\cos^2\beta+e^{2s}\sin^2\beta)^{1/2}$. Hence the relation $(d\Sigma_s)_{\varphi_s(p)}=e^{2s}(\cos^2\beta+e^{2s}\sin^2\beta)^{1/2}(d\Sigma)_p$ holds between the area elements of Σ_s and Σ. For the unit normal N' of Σ_s at $\varphi_s(p)$ we have

$$\begin{aligned}\pm N'_{\varphi_s(p)}&=e^{-s}(\cos^2\beta+e^{2s}\sin^2\beta)^{-1/2}\\&\quad\times[-e^{2s}\sin\beta\,(-\sin\alpha\,X_{\varphi_s(p)}+\cos\alpha\,Y_{\varphi_s(p)})+e^s\cos\beta\,T_{\varphi_s(p)}],\end{aligned}$$

and so $|N'_H|_{\varphi_s(p)}=e^s|\sin\beta|\,(\cos^2\beta+e^{2s}\sin^2\beta)^{-1/2}$. Hence

$$|N'_H|_{\varphi_s(p)}\,(d\Sigma_s)_{\varphi_s(p)}=e^{3s}|N_H|\,(d\Sigma)_p.$$

Since p is an arbitrary regular point of Σ, integrating the above displayed formula over $\Sigma-\Sigma_0$ and using the area formula we have $A(s)=e^{3s}A(\Sigma)$, and so

$$A'(0)=3\,A(\Sigma).$$

Finally, as Σ is volume-preserving area-stationary, we deduce from Corollary 4.11 that $A'(0)=2HV'(0)$, and equality (4.10) follows. □

Corollary 4.13. *Let $\Sigma\subset\mathbb{H}^1$ be a volume-preserving area-stationary C^2 surface enclosing a bounded region Ω. Then the constant mean curvature of the regular part of Σ with respect to the inner normal is positive. In particular, there are no compact area-stationary C^2 surfaces in $\mathbb{H}^1$.*

Remark 4.14. The generalization of (4.10) to the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ is immediate. By using the first variation formula in [RR, Lemma 3.2] and the arguments in this section we get that, for a C^2 volume-preserving area-stationary hypersurface $\Sigma\subset\mathbb{H}^n$ enclosing a bounded region Ω, we have

$$(2n+1)\,A(\Sigma)=4n(n+1)H\,V(\Omega).$$

We finish this section with a characterization of area-stationary surfaces in terms of geometric conditions. For that, we need additional information on the singular set Σ_0 of a constant mean curvature surface $\Sigma\subset\mathbb{H}^1$. The set Σ_0 has been recently studied by J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang [CHMY]. Their results are local and also valid when the mean curvature is bounded on the regular set $\Sigma-\Sigma_0$. By Theorem 4.8 we can replace "characteristic curves" in their statement by "geodesics of the same curvature". We summarize their results in the following

Theorem 4.15 ([CHMY, Theorem B])**.** *Let $\Sigma\subset\mathbb{H}^1$ be a C^2 oriented immersed surface with constant mean curvature H. Then the singular set Σ_0 consists of isolated points and C^1 curves with non-vanishing tangent vector. Moreover, we have*

(i) ([CHMY, Theorem 3.10]) *If $p\in\Sigma_0$ is isolated then there are $r>0$ and $\lambda\in\mathbb{R}$ with $|\lambda|=|H|$ such that the set described as*

$$D_r(p)=\{\gamma^\lambda_{p,v}(s);v\in T_p\Sigma,\,|v|=1,\,s\in[0,r)\},$$

is an open neighborhood of p in Σ.

(ii) ([CHMY, Proposition 3.5 and Corollary 3.6]) *If p is contained in a C^1 curve $\Gamma \subset \Sigma_0$ then there is a neighborhood B of p in Σ such that $B - \Gamma$ is the union of two disjoint connected open sets B^+ and B^- contained in $\Sigma - \Sigma_0$, and ν_H extends continuously to Γ from both sides of $B - \Gamma$, i.e., the limits*

$$\nu_H^+(q) = \lim_{x \to q,\ x \in B^+} \nu_H(x), \qquad \nu_H^-(q) = \lim_{x \to q,\ x \in B^-} \nu_H(x)$$

exist for any $q \in \Gamma \cap B$. These extensions satisfy $\nu_H^+(q) = -\nu_H^-(q)$. Moreover, there are exactly two geodesics $\gamma_1^\lambda \subset B^+$ and $\gamma_2^\lambda \subset B^-$ starting from q and meeting transversally Γ at q with initial velocities

$$(\gamma_1^\lambda)'(0) = -(\gamma_2^\lambda)'(0).$$

The curvature λ does not depend on q and satisfies $|\lambda| = |H|$.

Remark 4.16. The relation between λ and H depends on the value of the normal N in the singular point p. If $N_p = T$ then $\lambda = H$, while we have $\lambda = -H$ whenever $N_p = -T$. In case $\lambda = H$ the geodesics γ^λ in Theorem 4.15 are characteristic curves of Σ.

In Euclidean space it is equivalent for a surface to be area-stationary (resp. volume-preserving area-stationary) and to have zero (resp. constant) mean curvature. For a surface Σ is $\mathbb{H}^1$ this also holds if the singular set Σ_0 consists only of isolated points. In the general case, we have the following

Theorem 4.17. *Let $\Sigma \subset \mathbb{H}^1$ be an oriented C^2 immersed surface. Assume that Σ is an area-stationary surface (resp., a volume-preserving area-stationary compact surface enclosing a region Ω). Then the mean curvature of $\Sigma - \Sigma_0$ is zero (resp., constant). In both cases, the characteristic curves meet the singular curves, if they exist, orthogonally. The converse is also true.*

Proof. Suppose first that Σ is area-stationary. That the mean curvature is zero or constant on $\Sigma - \Sigma_0$ follows from Corollary 4.5. Assume Γ is a singular curve and let $p \in \Gamma$. By Theorem 4.15(ii) the curve Γ is C^1 and we can take a neighborhood B of p in Σ such that $B - \Gamma$ consists of the union of two open connected sets B^+ and B^- contained in $\Sigma - \Sigma_0$. Let ξ be the unit normal to Γ in Σ pointing into B^+. Let $f : \Gamma \to \mathbb{R}$ be any C^1 function supported on $\Gamma \cap B$. Extend f to a C^1 function $u : B \to \mathbb{R}$ with compact support in B and mean zero. Since Σ is area-stationary, by (4.6) and the divergence theorem we have

$$\begin{aligned} 0 &= -\int_B \operatorname{div}_\Sigma \left(u\,(\nu_H)^\top\right) d\Sigma \\ &= -\int_{B^+} \operatorname{div}_\Sigma \left(u\,(\nu_H)^\top\right) d\Sigma - \int_{B^-} \operatorname{div}_\Sigma \left(u\,(\nu_H)^\top\right) d\Sigma \\ &= \int_\Gamma f \left\langle \xi, \nu_H^+ \right\rangle d\Gamma - \int_\Gamma f \left\langle \xi, \nu_H^- \right\rangle d\Gamma \\ &= 2\int_\Gamma f \left\langle \xi, \nu_H^+ \right\rangle d\Gamma, \end{aligned}$$

since the extensions ν_H^+, ν_H^- of ν_H given in Theorem 4.15 (ii) satisfy $\nu_H^+ = -\nu_H^-$. As f is an arbitrary function on $\Gamma \cap B$ we conclude that $\left\langle \xi, \nu_H^+ \right\rangle \equiv 0$ on $\Gamma \cap B$. This means that ν_H^+ is tangent to $\Gamma \cap B$ and so the two characteristic curves approaching p meet the singular curve Γ in an orthogonal way.

We will see the converse for constant mean curvature. Let U be a C^2 vector field inducing a volume-preserving variation of Σ. Let $u = \big\langle U, N \big\rangle$. By the first variation of volume (4.1) we have $\int_\Sigma u \, d\Sigma = 0$. By (4.6)

$$A'(0) = -\int_\Sigma \operatorname{div}_\Sigma \big(u \, (\nu_H)^\top \big) \, d\Sigma,$$

since u has mean zero and $\operatorname{div}_\Sigma \nu_H$ is a constant. To analyze the above integral, we consider disjoint open balls $B_\varepsilon(p_i)$ (for the Riemannian distance on Σ) of small radius $\varepsilon > 0$, centered at the isolated points $p_1, \ldots, p_k$ of the singular set Σ_0. By the divergence theorem in Σ, and the fact that the characteristic curves meet orthogonally the singular curves we have, for $\Sigma_\varepsilon = \Sigma - \bigcup_{i=1}^k B_\varepsilon(p_i)$,

$$-\int_{\Sigma_\varepsilon} \operatorname{div}_\Sigma \big(u \, (\nu_H)^\top \big) \, d\Sigma = \sum_{i=1}^k \int_{\partial B_\varepsilon(p_i)} u \, \big\langle \xi_i, \nu_H \big\rangle \, dl,$$

where ξ_i is the inner unit normal vector to $\partial B_\varepsilon(p_i)$ in Σ. Note also that

$$\left| \sum_{i=1}^k \int_{\partial B_\varepsilon(p_i)} u \, \big\langle \xi_i, \nu_H \big\rangle \, dl \right| \leqslant \big(\sup_\Sigma |u| \big) \sum_{i=1}^k L(\partial B_\varepsilon(p_i)),$$

where $L(\partial B_\varepsilon(p_i))$ is the Riemannian length of $\partial B_\varepsilon(p_i)$. Finally, as $|\operatorname{div}_\Sigma(u \, (\nu_H^\top))| \leqslant (\sup_\Sigma |u|) \, |\operatorname{div}_\Sigma \nu_H - |N_H| \operatorname{div}_\Sigma N| + |\nabla_\Sigma u| \in L^1(\Sigma)$, we can apply the dominated convergence theorem and the fact that $L(\partial B_\varepsilon(p_i)) \to 0$ when $\varepsilon \to 0$ to prove the claim. □

Remark 4.18. Recently, J.-H. Cheng, J.-F. Hwang and P. Yang [CHY1, Theorem 6.3 and (7.2)] have obtained Theorem 4.17 when Σ is a C^2 graph over a bounded set D of the xy-plane which is a weak solution of the equation $\operatorname{div}_\Sigma \nu_H = -2H$ ([CHY1, Equation (3.12)]). As it is proved in [CHY1, Theorem 3.3] such a weak solution minimizes the functional $A - 2HV$ amongst all graphs Σ' in the Sobolev space $W^{1,1}(D)$ with $\partial\Sigma' = \partial\Sigma$. This fact allows the authors to construct examples of area-minimizing graphs which are not C^2 smooth ($C^{0,1}$), see [CHY1, §7]. In fact, in [CHY1, (7.1)], it is shown that a t-graph of class C^1, which is composed of C^2 pieces with mean curvature $H = 0$ joining along the singular curves, is area-stationary if and only if the characteristic curves meet along the singular lines in such a way that the incident and the reflected angles are equal. When the graph is C^2 this condition turns out to be the orthogonality condition in Theorem 4.17. A large number of examples of Euclidean Lipschitz area-minimizing t-graphs have been obtained recently in [R2].

Example 4.19. Any sphere $\mathbb{S}_\lambda$ is a volume-preserving area-stationary surface by Theorem 4.17 since it has constant mean curvature in $\Sigma - \Sigma_0$ and Σ_0 consists of isolated points.

For a C^2 area-stationary surface we can use Theorem 4.17 to improve the C^1 regularity of the singular curves obtained in [CHMY, Theorem 3.3].

Proposition 4.20. *If Σ is a C^2 oriented immersed area-stationary surface (with or without a volume constraint) then any singular curve of Σ is a C^2 smooth curve.*

Proof. By Corollary 4.5 we know that $\Sigma - \Sigma_0$ has constant mean curvature H. Let Γ be a connected singular curve of Σ and $p_0 \in \Gamma$. By taking the opposite unit normal to Σ if necessary we can assume that $N = -T$ along Γ. By using Theorem 4.17 (ii) and the remark below, we can find a small neighborhood B of p_0 in Σ such that B^+ is foliated by geodesics of the same curvature $\lambda = H$ reaching $\Gamma \cap B$ at finite, positive time. These geodesics are characteristic curves of Σ and meet Γ orthogonally by Theorem 4.17.

Let Z be the characteristic vector field of Σ with respect to N. Take a point $q \in B^+$ such that $\gamma^\lambda_{q,Z(q)}(s(q)) = p_0$ for some $s(q) > 0$. We consider a C^2 curve $\mathcal{C} \subset B^+$ passing through q and meeting transversally the geodesics only at one point. We define the C^1 map $F : \mathcal{C} \times (0, +\infty) \to \mathbb{H}^1$ given by $F(x, s) = \gamma^\lambda_{x,Z(x)}(s)$. For any $x \in \mathcal{C}$ there is a first value $s(x) > 0$ such that $F(x, s(x)) \in \Gamma$. Moreover, by using the orthogonality condition in Theorem 4.17 we can choose the curve $\mathcal{C}$ so that the differential of F has rank two for any $(x, s(x))$ near to $(q, s(q))$. Thus, for some $\delta > 0$ we have that $\Sigma' = \{F(x, s); x \in [q-\delta, q+\delta],\ s \in [0, s(x)+\delta]\}$ is a C^1 extension of Σ beyond the singular curve Γ. In particular Σ and Σ' are tangent along Γ. The horizontal tangent vector to Σ' given by $Z' = (\partial F/\partial s)(x, s) = (\gamma^\lambda_{x,Z(x)})'(s)$ is a C^1 extension of Z. Finally the orthogonality condition implies that the restriction of $J(Z')$ is a unit C^1 tangent vector to Γ. We conclude that Γ is a C^2 smooth curve around p_0 and the proof follows. $\square$

5. Entire area-stationary graphs in $\mathbb{H}^1$

An *entire graph* over a plane is one defined over the whole plane. A classical theorem by Bernstein shows that the only entire minimal graphs in Euclidean space $\mathbb{R}^3$ are the planes. In [Pa1, Theorem D], S. Pauls observed the existence of entire graphs with $H = 0$ in $\mathbb{H}^1$ different from Euclidean planes. These are obtained by rotations about the t-axis of a graph of the form

$$t = xy + g(y), \qquad \text{where } \ g \in C^2(\mathbb{R}). \tag{5.1}$$

In [CHMY, Theorem A], J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang proved that Euclidean planes and vertical rotations of (5.1) are the unique C^2 graphs over the xy-plane with $H = 0$, see also [GP, Theorem D]. Here we show that according to Theorem 4.17 not all the graphs in (5.1) are area-stationary. In precise terms, we have

Theorem 5.1. *The unique entire C^2 area-stationary graphs over the xy-plane in $\mathbb{H}^1$ are Euclidean planes and vertical rotations of graphs of the form*

$$t = xy + (ay + b),$$

where a and b are real constants.

Proof. Let Σ be a C^2 entire area-stationary graph over the xy-plane in $\mathbb{H}^1$. By Theorem 4.17 we know that the mean curvature of $\Sigma - \Sigma_0$ vanishes and the intersection between characteristic lines and singular curves is orthogonal. By the classification in [CHMY, Theorem A] for entire graphs with $H = 0$ we have that Σ is a Euclidean plane or a vertical rotation of (5.1). That Euclidean planes are area-stationary follows from Theorem 4.17 since they only have isolated singularities. To prove the claim we suppose that Σ coincides with (5.1). The surface Σ has a connected curve Γ of singular points whose projection to the xy-plane is given by the equation $x=-g'(y)/2$. We can parameterize Γ by

$$\Gamma(s)=\left(-\frac{g'(s)}{2},s,g(s)-\frac{g'(s)\,s}{2}\right),\qquad s\in\mathbb{R},$$

and so, if $\Gamma(s_0)=p_0$, then $\dot\Gamma(s_0)=(-g''(s_0)/2)\,X_{p_0}+Y_{p_0}$. On the other hand, it is not difficult to check that for a fixed $y\in\mathbb{R}$, the straight line $t=xy+g(y)$ is a characteristic curve of Σ when removing the contact point with Γ. We parameterize this line as

$$S_y(s)=(s,y,sy+g(y)),\qquad s\in\mathbb{R},$$

so that if $S_y(s_1)=p_0$ then $\dot S_y(s_1)=X_{p_0}$. From these computations we see that, for $p_0=\Gamma(s_0)=S_y(s_1)$ we have

$$\big\langle\dot\Gamma(s_0),\dot S_y(s_1)\big\rangle=-\frac{g''(y)}{2}.$$

We conclude that the characteristic lines S_y meet orthogonally the singular curve Γ if and only if $g(y)=ay+b$ for some real constants a and b. $\square$

Remark 5.2. While Euclidean planes have only an isolated singular point, the entire area-stationary graphs obtained by rotations of $t=xy+(ay+b)$ have a straight line of singular points. From a geometric point of view, these second surfaces are constructed by taking a horizontal straight line R and attaching at any point of R the unique straight line which is both horizontal and orthogonal to R. The remaining surfaces defined by equation (5.1) have vanishing mean curvature outside the singular set, but they are not area-stationary.

We finish this section showing that the graphs obtained in Theorem 5.1 are globally area-minimizing. This is a counterpart in $\mathbb{H}^1$ of a well-known result for minimal graphs in $\mathbb{R}^3$.

We say that a surface $\Sigma\subset\mathbb{H}^1$ is *area-minimizing* if any region $M\subset\Sigma$ has less area than any other C^1 compact surface M' in $\mathbb{H}^1$ with $\partial M=\partial M'$. In [CHMY, Proposition 6.2] it was proved by using a calibration argument that any C^2 surface in $\mathbb{H}^1$ with vanishing mean curvature locally minimizes the area around any point in the regular set. Here, we adapt the calibration argument in order to deal with surfaces with singularities, and we obtain

Theorem 5.3. *Any entire C^2 area-stationary graph Σ over the xy-plane in $\mathbb{H}^1$ is area-minimizing.*

Proof. After a vertical rotation we may assume, by Theorem 5.1, that Σ coincides with a Euclidean plane or with a graph of the form $t=xy+ay+b$, for some $a,b\in\mathbb{R}$. Let Σ_t be area-stationary graph obtained by applying to Σ the left translation L_t by the vertical vector tT. The family $\{\Sigma_t\}_{t\in\mathbb{R}}$ is a foliation of $\mathbb{H}^1$ by area-stationary surfaces. Moreover, L_t preserves the horizontal distribution and hence $p\in\Sigma-\Sigma_0$ if and only if $L_t(p)\in\Sigma_t-(\Sigma_t)_0$. Therefore, the set $P=\bigcup_t(\Sigma_t)_0$ is either a vertical straight line if Σ is a plane or a vertical plane if Σ is a graph $t=xy+ay+b$. Consider a C^1 vector field N on $\mathbb{H}^1$ so that the restriction N_t of N to Σ_t is a unit normal vector to Σ_t. We denote $N_H/|N_H|$ by ν_H, and $Z=J(\nu_H)$, which are C^1 vector fields on $\mathbb{H}^1-P$.

Let us compute $\operatorname{div}\nu_H$. Take a point p in the regular set of Σ_t for some $t \in \mathbb{R}$. We have an orthonormal basis of $T_p\mathbb{H}^1$ given by $\{Z_p, (\nu_H)_p, T\}$. Denote by H_t the mean curvature of Σ_t with respect to N_t. By using equation (4.9) and that ν_H is a horizontal unit vector field, we get

$$\begin{aligned}\operatorname{div}\nu_H &= \big\langle D_Z\nu_H, Z\big\rangle + \big\langle D_{\nu_H}\nu_H, \nu_H\big\rangle + \big\langle D_T\nu_H, T\big\rangle \\ &= -2H_t - \big\langle \nu_H, D_T T\big\rangle = 0,\end{aligned}$$

where in the last equality we have used that $H_t \equiv 0$ since Σ_t is area-stationary (Corollary 4.5(i)), and that $D_T T = 0$.

Consider a region $M \subset \Sigma$ and a compact C^1 surface $M' \subset \mathbb{H}^1$ with $\partial M = \partial M'$. We denote by Ω the open set bounded by M and M'. The set Ω has finite perimeter in the Riemannian manifold $(\mathbb{H}^1, g)$ since it is bounded and the two-dimensional Riemannian Hausdorff measure of $\partial\Omega \cap C$ is finite for any compact set $C \subset \mathbb{H}^1$, see [EG, Theorem 1, p. 222]. For the following arguments we may assume Ω connected, and that $\partial\Omega = M \cup M'$. We fix the outward normal vector N to Σ, and the unit normal vector N' to M', to point into Ω. As a consequence, we can apply the Gauss-Green Theorem for sets of finite perimeter [EG, Theorem 1, p. 209] so that, for any C^1 vector field U on $\mathbb{H}^1$, we have

$$\int_\Omega \operatorname{div} U\, dv = \int_M \big\langle U, N\big\rangle\, dM - \int_{M'} \big\langle U, N'\big\rangle\, dM'. \tag{5.2}$$

In order to prove $A(M) \leqslant A(M')$ we distinguish two cases.

Case 1. If Σ is a Euclidean plane, then ν_H is defined in the closure of Ω outside a set contained in a straight line. Thus, we can apply (5.2) to deduce

$$\begin{aligned}0 = \int_\Omega \operatorname{div}\nu_H\, dv &= \int_M \big\langle \nu_H, N\big\rangle\, dM - \int_{M'} \big\langle \nu_H, N'\big\rangle\, dM' \\ &= \int_M |N_H|\, dM - \int_{M'} \big\langle \nu_H, N'_H\big\rangle\, dM' \\ &\geqslant A(M) - A(M').\end{aligned}$$

To obtain the last inequality we have used the Cauchy-Schwarz inequality and that $|\nu_H| = 1$. This proves the claim.

Case 2. If Σ is a graph of the form $t = xy + ay + b$, then ν_H is defined on $\Omega - P$, where P is a vertical Euclidean plane. Denote by P^+ and P^- the open half-planes determined by P. For any set $E \subset \mathbb{H}^1$, we let $E^+ = E \cap P^+$ and $E^- = E \cap P^-$. The sets Ω^+ and Ω^- has finite perimeter in $(\mathbb{H}^1, g)$. Moreover, by Theorem 4.15 (ii) the vector field ν_H extends continuously to P from Ω^+ and Ω^- . Therefore

$$\begin{aligned}0 &= \int_{\Omega^+} \operatorname{div}\nu_H\, dv = \int_{M^+} \big\langle \nu_H, N\big\rangle\, dM - \int_{(M')^+} \big\langle \nu_H, N'\big\rangle\, dM' - \int_{\Omega\cap P} \big\langle \nu_H^+, \xi\big\rangle\, dP \\ 0 &= \int_{\Omega^-} \operatorname{div}\nu_H\, dv = \int_{M^-} \big\langle \nu_H, N\big\rangle\, dM - \int_{(M')^-} \big\langle \nu_H, N'\big\rangle\, dM' + \int_{\Omega\cap P} \big\langle \nu_H^-, \xi\big\rangle\, dP,\end{aligned}$$

where ξ is the unit normal vector to P pointing into Ω^+. As $\nu_H^+ = -\nu_H^-$, by summing the previous equalities we deduce

$$\begin{aligned} 0 &= \int_M \langle \nu_H, N\rangle \, dM - \int_{M'} \langle \nu_H, N'\rangle \, dM' - 2\int_{\Omega\cap P} \langle \nu_H^+, \xi\rangle \, dP \\ &\geqslant A(M) - A(M') - 2\int_{\Omega\cap P} \langle \nu_H^+, \xi\rangle \, dP. \end{aligned}$$

Finally, the orthogonality condition between characteristic lines and singular curves in Theorem 4.17 implies that $\langle \nu_H^+, \xi\rangle = 0$ on $\Omega\cap P$. Thus, we get $A(M) \leqslant A(M')$. $\square$

Remark 5.4. If Σ is an area-stationary surface in $\mathbb{H}^1$, and there is a left invariant vector field V in $\mathbb{H}^1$ transverse to Σ, then we can produce a local foliation by area-stationary surfaces around Σ by using the flow associated to V. The arguments in the proof of Theorem 5.3 show that Σ is locally area-minimizing, i.e., bounded portions of Σ minimize area amongst surfaces with boundary on Σ and contained in the foliated neighborhood of Σ.

Remark 5.5. 1. It follows from [CHY1, Proposition 6.2 and Theorem 3.3] that a C^2 area-stationary graph over a bounded domain D of the xy-plane minimizes the area amongst all graphs Σ' in the Sobolev space $W^{1,1}(D)$ with $\partial\Sigma' = \partial\Sigma$. This has been recently improved in [BSCV, Example 2.7] where it is shown that such a graph is area-minimizing.

2. Theorem 5.3 does not hold for a graph over the xt-plane, see an example in [DGN3]. In [BSCV, Theorem 5.3] it is proved that the unique C^2 entire, area-minimizing *intrinsic graphs* over the xt-plane are vertical planes.

3. In [DGNP], vertical Euclidean planes are characterized as the only C^2 area-minimizing entire graphs with empty singular set.

6. Complete volume-preserving area-stationary surfaces in $\mathbb{H}^1$

An immersed surface $\Sigma \subset \mathbb{H}^1$ is *complete* if it is complete in the Riemannian manifold $(\mathbb{H}^1, g)$. Completeness for a constant mean curvature surface implies that characteristic curves in $\Sigma - \Sigma_0$ extend up to singular points of Σ.

In this section we obtain classification results for complete area-stationary surfaces under a volume constraint in $\mathbb{H}^1$. We say that a complete noncompact oriented C^2 surface in $\mathbb{H}^1$ is volume-preserving area-stationary if it has constant mean curvature outside the singular set and the characteristic curves meet orthogonally the singular curves. By Theorem 4.17 this implies that the surface is area-stationary for any variation with compact support of the surface such that the volume (4.2) of the perturbed region remains constant.

We begin with the description of constant mean curvature surfaces with isolated singularities. It was shown in [CHMY, Proof of Theorem A] (see also [CH, Proposition 2.1]) and [GP, Lemma 8.2] that any C^2 surface with vanishing mean curvature and an isolated singular point must coincide with a Euclidean plane. By using the local behavior of a constant mean curvature surface around a singular point (Theorem 4.15) we can prove the following

Theorem 6.1. *Let Σ be a complete, connected, C^2 oriented immersed surface in $\mathbb{H}^1$ with non-vanishing constant mean curvature. If Σ contains an isolated singular point then Σ is congruent to a sphere $\mathbb{S}_H$.*

Proof. We choose the unit normal N to Σ such that the mean curvature H is positive. Let p be an isolated singular point of Σ. By applying to Σ the left translation $(L_p)^{-1}$ we can assume that $p=0$ and the tangent plane $T_p\Sigma$ coincides with the xy-plane. Suppose that $N_p=T$. For any $r>0$ we consider the set

$$D_r=\{\gamma^H_{0,v}(s);\,|v|=1,\ s\in[0,r)\}.$$

It is clear that the union of D_r, for $r\in(0,\pi/H)$, coincides with the sphere $\mathbb{S}_H$ removing the north pole (see Example 3.3). By Theorem 4.15 (i) and Remark 4.16, we can find $r>0$ such that $D_r\subset\Sigma$. Let $R=\sup\{r>0\ ;\ D_r\subset\Sigma\}$. As Σ is complete and connected, and $\mathbb{S}_H$ is compact, to prove the claim it suffices to see that $R=\pi/H$.

Suppose that $R<\pi/H$. In this case we would have $\overline{D}_R\subset\Sigma$ and so, Σ and $\mathbb{S}_H$ would be tangent along the curve ∂D_R. In particular, this curve is contained in the regular set of Σ. By Theorem 4.8 the characteristic curve of Σ passing through any $q\in\partial D_R$ is an open arc of a geodesic of curvature H. By the uniqueness of the geodesics this would imply that we may extend any $\gamma^H_{0,v}$ inside Σ beyond ∂D_R, a contradiction with the definition of R. This proves $R\geqslant\pi/H$. On the other hand, $R>\pi/H$ would imply that Σ contains a neighborhood of a tangent point between two different spheres of the same curvature which is not possible since Σ is immersed.

Finally, if $N_p=-T$ we repeat the previous arguments by using geodesics of curvature $-H$ and we conclude that Σ coincides with a vertical translation of $\mathbb{S}_H$. $\square$

Theorem 6.1 does not provide information about non-vanishing constant mean curvature surfaces in $\mathbb{H}^1$ with at least one singular curve. We will treat this situation in the particular case of volume-preserving area-stationary surfaces, where we have by Theorem 4.17 the additional condition that the characteristic curves meet orthogonally the singular curves. We first study in more detail the behavior of the characteristic curves far away from a singular curve.

Let Γ be a C^2 horizontal curve in $\mathbb{H}^1$. We parameterize $\Gamma=(x,y,t)$ by arc-length $\varepsilon\in I$, where I is an open interval. The projection $\alpha=(x,y)$ is a plane curve with $|\dot\alpha|=1$. We denote by h the planar geodesic curvature of α with respect to the unit normal vector $(-\dot y,\dot x)$, that is $h=\dot x\ddot y-\ddot x\dot y$. As Γ is horizontal, we have $\dot t=\dot x y-x\dot y$. Fix $\lambda\neq 0$. For any $\varepsilon\in I$ let γ_ε be the unique geodesic of curvature λ with initial conditions $\gamma_\varepsilon(0)=\Gamma(\varepsilon)$ and $\dot\gamma_\varepsilon(0)=J(\dot\Gamma(\varepsilon))$. We consider the family of all these geodesics orthogonal to Γ parameterized by $F(\varepsilon,s)=\gamma_\varepsilon(s)=(x(\varepsilon,s),y(\varepsilon,s),t(\varepsilon,s))$, for $\varepsilon\in I$ and $s\in[0,\pi/|\lambda|]$. By equation

(3.4) we have

$$
\begin{aligned}
x(\varepsilon,s)&=x(\varepsilon)-\dot y(\varepsilon)\left(\frac{\sin(2\lambda s)}{2\lambda}\right)+\dot x(\varepsilon)\left(\frac{1-\cos(2\lambda s)}{2\lambda}\right),\\
y(\varepsilon,s)&=y(\varepsilon)+\dot y(\varepsilon)\left(\frac{1-\cos(2\lambda s)}{2\lambda}\right)+\dot x(\varepsilon)\left(\frac{\sin(2\lambda s)}{2\lambda}\right),\\
t(\varepsilon,s)&=t(\varepsilon)+\frac{1}{2\lambda}\left(s-\frac{\sin(2\lambda s)}{2\lambda}\right)-(x(\varepsilon)\,\dot x(\varepsilon)+y(\varepsilon)\,\dot y(\varepsilon))\left(\frac{\sin(2\lambda s)}{2\lambda}\right)\\
&\qquad+(\dot x(\varepsilon)\,y(\varepsilon)-x(\varepsilon)\,\dot y(\varepsilon))\left(\frac{1-\cos(2\lambda s)}{2\lambda}\right).
\end{aligned}
\tag{6.1}
$$

From the equations above we see that F is a C^1 map. Clearly $(\partial F/\partial s)(\varepsilon,s)=\dot\gamma_\varepsilon(s)$. We denote $V_\varepsilon(s):=(\partial F/\partial\varepsilon)(\varepsilon,s)$. In the next result we show some properties of V_ε.

Lemma 6.2. *In the situation above, V_ε is a Jacobi vector field along γ_ε with $V_\varepsilon(0)=\dot\Gamma(\varepsilon)$. For any $\varepsilon\in I$ there is a unique $s_\varepsilon\in(0,\pi/|\lambda|)$ such that $\big\langle V_\varepsilon(s_\varepsilon),T\big\rangle=0$. We have $\big\langle V_\varepsilon,T\big\rangle<0$ on $(0,s_\varepsilon)$ and $\big\langle V_\varepsilon,T\big\rangle>0$ on $(s_\varepsilon,\pi/|\lambda|)$. Moreover $V_\varepsilon(s_\varepsilon)=J(\dot\gamma_\varepsilon(s_\varepsilon))$.*

Proof. By the definition of V_ε we have $V_\varepsilon(0)=\dot\Gamma(\varepsilon)$ and

$$V_\varepsilon(s)=\frac{\partial x}{\partial\varepsilon}(\varepsilon,s)\;X+\frac{\partial y}{\partial\varepsilon}(\varepsilon,s)\;Y+\left(\frac{\partial t}{\partial\varepsilon}-y\;\frac{\partial x}{\partial\varepsilon}+x\;\frac{\partial y}{\partial\varepsilon}\right)(\varepsilon,s)\;T.$$

The Euclidean components of $V_\varepsilon(s)$ are easily computed from (6.1), so that we obtain

$$
\begin{aligned}
\frac{\partial x}{\partial\varepsilon}(\varepsilon,s)&=\dot x(\varepsilon)-\ddot y(\varepsilon)\left(\frac{\sin(2\lambda s)}{2\lambda}\right)+\ddot x(\varepsilon)\left(\frac{1-\cos(2\lambda s)}{2\lambda}\right),\\
\frac{\partial y}{\partial\varepsilon}(\varepsilon,s)&=\dot y(\varepsilon)+\ddot y(\varepsilon)\left(\frac{1-\cos(2\lambda s)}{2\lambda}\right)+\ddot x(\varepsilon)\left(\frac{\sin(2\lambda s)}{2\lambda}\right),\\
\frac{\partial t}{\partial\varepsilon}(\varepsilon,s)&=\dot t(\varepsilon)-(1+x(\varepsilon)\,\ddot x(\varepsilon)+y(\varepsilon)\,\ddot y(\varepsilon))\left(\frac{\sin(2\lambda s)}{2\lambda}\right)\\
&\qquad+(\ddot x(\varepsilon)\,y(\varepsilon)-x(\varepsilon)\,\ddot y(\varepsilon))\left(\frac{1-\cos(2\lambda s)}{2\lambda}\right).
\end{aligned}
$$

We deduce that V_ε is C^∞ vector field along γ_ε and

$$\big\langle V_\varepsilon(s),T\big\rangle=\frac{1}{\lambda}\left(\frac{1-\cos(2\lambda\,s)}{2\lambda}\;h(\varepsilon)-\sin(2\lambda s)\right),\qquad s\in[0,\pi/|\lambda|].$$

That V_ε is a Jacobi vector field along γ_ε follows from Lemma 3.5 since V_ε is associated to a variation of γ_ε by geodesics of the same curvature. On the other hand, the equation above implies that $\big\langle V_\varepsilon(s_\varepsilon),T\big\rangle=0$ for some $s_\varepsilon\in(0,\pi/|\lambda|)$ if and only if

$$h(\varepsilon)=\frac{2\lambda\,\sin(2\lambda s_\varepsilon)}{1-\cos(2\lambda s_\varepsilon)}.\tag{6.2}$$

The existence and uniqueness of s_ε, and the sign of $\big\langle V_\varepsilon,T\big\rangle$ are consequences of the fact that the function $f(x)=\sin(x)\,(1-\cos(x))^{-1}$ is periodic, decreasing on $(0,2\pi)$ and satisfies $\lim_{x\to0^+}f(x)=+\infty$ and $\lim_{x\to(2\pi)^-}f(x)=-\infty$.

Now we use Lemma 3.4 and the fact that $V_\varepsilon(0) = \dot\Gamma(\varepsilon)$ to deduce that the function $\lambda \langle V_\varepsilon, T\rangle + \langle V_\varepsilon, \dot\gamma_\varepsilon\rangle$ vanishes along γ_ε. In particular, $V_\varepsilon(s_\varepsilon)$ is a horizontal vector orthogonal to $\dot\gamma_\varepsilon(s_\varepsilon)$. Finally, we have, for $s \in [0, \pi/|\lambda|]$,

$$\big\langle V_\varepsilon(s), J(\dot\gamma_\varepsilon(s))\big\rangle = \bigg(-\frac{\partial x}{\partial \varepsilon}\frac{\partial y}{\partial s} + \frac{\partial y}{\partial \varepsilon}\frac{\partial x}{\partial s}\bigg)(\varepsilon, s) = \frac{\sin(2\lambda s)}{2\lambda}\, h(\varepsilon) - \cos(2\lambda s),$$

which is equal to 1 for $s = s_\varepsilon$ by (6.2). □

The following proposition provides a method to construct immersed surfaces with constant mean curvature in $\mathbb{H}^1$ bounded by two singular curves. Geometrically we only have to leave from a given horizontal curve by segments of orthogonal geodesics of the same curvature. The length of these segments depends on the *cut function* s_ε introduced in Lemma 6.2. We also characterize when the resulting surface is volume-preserving area-stationary.

Proposition 6.3. *Let Γ be a C^{k+1} $(k \geqslant 1)$ horizontal curve in $\mathbb{H}^1$ parameterized by arc-length $\varepsilon \in I$. Consider the map $F(\varepsilon, s) = \gamma_\varepsilon(s)$, where $\gamma_\varepsilon : [0, \pi/|\lambda|] \to \mathbb{H}^1$ is the geodesic of curvature $\lambda \neq 0$ with initial conditions $\Gamma(\varepsilon)$ and $J(\dot\Gamma(\varepsilon))$. Let s_ε be the function introduced in Lemma 6.2, and let $\Sigma_\lambda(\Gamma) = \{F(\varepsilon, s);\, \varepsilon \in I,\, s \in [0, s_\varepsilon]\}$. Then we have*

(i) $\Sigma_\lambda(\Gamma)$ *is an immersed surface of class C^k in $\mathbb{H}^1$.*
(ii) *The singular set of $\Sigma_\lambda(\Gamma)$ consists of two curves $\Gamma(\varepsilon)$ and $\Gamma_1(\varepsilon) = F(\varepsilon, s_\varepsilon)$.*
(iii) *There is a C^{k-1} unit normal vector N to $\Sigma_\lambda(\Gamma)$ such that $N = T$ on Γ and $N = -T$ on Γ_1.*
(iv) *Any $\gamma_\varepsilon : (0, s_\varepsilon) \to \mathbb{H}^1$ is a characteristic curve of $\Sigma_\lambda(\Gamma)$. In particular, if $k \geqslant 2$ then $\Sigma_\lambda(\Gamma)$ has constant mean curvature λ with respect to N.*
(v) *If Γ_1 is a C^2 smooth curve then the geodesics γ_ε meet orthogonally Γ_1 if and only if s_ε is constant along Γ. This is equivalent to that the xy-projection of Γ is either a line segment or a piece of a planar circle.*

Proof. As Γ is C^{k+1} and the geodesics γ_ε depend C^1 smoothly on the initial conditions we get that F is a map of class C^k. Let us consider the vector fields $(\partial F/\partial\varepsilon)(\varepsilon, s) = V_\varepsilon(s)$ and $(\partial F/\partial s)(\varepsilon, s) = \dot\gamma_\varepsilon(s)$. By using Lemma 6.2 we deduce that the differential of F has rank two for any $(s, \varepsilon) \in I \times [0, \pi/|\lambda|)$, and that the tangent plane to $\Sigma_\lambda(\Gamma)$ is horizontal only for the points in Γ and Γ_1. This proves (i) and (ii).

Now define the C^{k-1} unit normal vector to the immersion $F : I \times [0, \pi/|\lambda|) \to \mathbb{H}^1$ given by $N(\varepsilon, s) = |V_\varepsilon(s) \wedge \dot\gamma_\varepsilon(s)|^{-1}\,(V_\varepsilon(s) \wedge \dot\gamma_\varepsilon(s))$. To compute N along Γ and Γ_1 it suffices to use $v \wedge J(v) = T$ for any unit horizontal vector v together with the fact that $V_\varepsilon(0) = \dot\Gamma(\varepsilon)$ and $V_\varepsilon(s_\varepsilon) = J(\dot\gamma_\varepsilon(s_\varepsilon))$. It is easy to see that the characteristic vector field Z to the immersion is given by

$$Z(\varepsilon, s) = -\frac{\langle V_\varepsilon(s), T\rangle}{|\langle V_\varepsilon(s), T\rangle|}\, \dot\gamma_\varepsilon(s), \qquad \varepsilon \in I, \quad s \neq 0, s_\varepsilon.$$

By using Lemma 6.2 it follows that $Z(\varepsilon, s) = \dot\gamma_\varepsilon(s)$ whenever $s \in (0, s_\varepsilon)$. This fact and Theorem 4.8 prove (iv).

Finally, suppose that Γ_1 is a C^2 smooth curve (which is immediate if $k \geqslant 3$). The cut function $s(\varepsilon) = s_\varepsilon$ is C^1 since the graph $(\varepsilon, s(\varepsilon))$ coincides, up to the C^1

immersion F, with Γ_1. The tangent vector to Γ_1 is given by

$$\dot{\Gamma}_1(\varepsilon) = V_\varepsilon(s_\varepsilon) + \dot{s}(\varepsilon)\,\dot{\gamma}_\varepsilon(s_\varepsilon).$$

As $V_\varepsilon(s_\varepsilon) = J(\dot{\gamma}_\varepsilon(s_\varepsilon))$, we conclude that the geodesics γ_ε meet Γ_1 orthogonally if and only if $s(\varepsilon)$ is a constant function. As a consequence, we deduce from (6.2) that the planar geodesic curvature of the xy-projection of Γ is constant and so, this plane curve must coincide with a line segment or a piece of a Euclidean circle. $\square$

Remark 6.4. 1. In the proof above it is shown that if we extend $\Sigma_\lambda(\Gamma)$ by the geodesics γ_ε beyond the singular curve Γ_1 then the resulting surface has mean curvature $-\lambda$ beyond Γ_1. As indicated in Theorem 4.15 (ii), in order to get an extension of $\Sigma_\lambda(\Gamma)$ with constant mean curvature λ we must leave from Γ_1 by geodesics of curvature $-\lambda$ (we must arrive at Γ_1 by geodesics of curvature λ).

2. The singular curves Γ and Γ_1 of the surface $\Sigma_\lambda(\Gamma)$ could coincide. We will illustrate this situation in Example 6.7.

Remark 6.5. Let Γ be a C^{k+1} ($k \geqslant 1$) horizontal curve in $\mathbb{H}^1$ parameterized by arc-length $\varepsilon \in I$. We consider the family of geodesics $\widetilde{\gamma}_\varepsilon : [0, \pi/|\lambda|] \to \mathbb{H}^1$ with curvature $\lambda \neq 0$ and initial conditions $\Gamma(\varepsilon)$ and $-J(\dot{\Gamma}(\varepsilon))$. By following the arguments in Lemma 6.2 and Proposition 6.3 we can construct the surface

$$\widetilde{\Sigma}_\lambda(\Gamma) := \{\widetilde{\gamma}_\varepsilon(s)\ ;\ \varepsilon \in I,\ s \in [0, \widetilde{s}_\varepsilon]\},$$

which is bounded by two singular curves Γ and Γ_2. The cut function $\widetilde{s}_\varepsilon$ associated to Γ_2 is defined by the equality $\langle \widetilde{V}_\varepsilon(\widetilde{s}_\varepsilon), T\rangle = 0$, where $\widetilde{V}_\varepsilon$ is the Jacobi vector field associated to $\{\widetilde{\gamma}_\varepsilon\}$. It is easy to see that $\widetilde{s}_\varepsilon$ satisfies

$$h(\varepsilon) = \frac{-2\lambda \sin(2\lambda \widetilde{s}_\varepsilon)}{1 - \cos(2\lambda \widetilde{s}_\varepsilon)}.$$

From (6.2) it follows that $s_\varepsilon + \widetilde{s}_\varepsilon = \pi/|\lambda|$. The vector field $\widetilde{V}_\varepsilon$ coincides with $-J(\dot{\widetilde{\gamma}}_\varepsilon)$ for $s = \widetilde{s}_\varepsilon$. The unit normal $\widetilde{N}$ to $\widetilde{\Sigma}_\lambda(\Gamma)$ equals T on Γ and $-T$ on Γ_2. When $k \geqslant 2$, we deduce that the union of $\Sigma_\lambda(\Gamma)$ and $\widetilde{\Sigma}_\lambda(\Gamma)$ is an oriented immersed surface with constant mean curvature λ and at most three singular curves.

Now we shall use Proposition 6.3 and Remark 6.5 to obtain new examples of complete volume-preserving area-stationary surfaces in $\mathbb{H}^1$ with singular curves. We know by Proposition 6.3 (iv) that the xy-projection of the initial curve Γ must be either a straight line or a planar circle. We shall consider the two cases.

Example 6.6 (Cylindrical surfaces $\mathcal{S}_\lambda$)**.** Consider the x-axis in $\mathbb{R}^3$ parameterized by $\Gamma(\varepsilon) = (\varepsilon, 0, 0)$. For any $\lambda \neq 0$ we denote by $\mathcal{S}_\lambda$ the union of the surfaces $\Sigma_\lambda(\Gamma)$ and $\widetilde{\Sigma}_\lambda(\Gamma)$ constructed in Proposition 6.3 and Remark 6.5. The surface $\mathcal{S}_\lambda$ is C^∞ outside the singular curves and has constant mean curvature λ. The cut functions s_ε and $\widetilde{s}_\varepsilon$ can be computed from (6.2) and the relation $s_\varepsilon + \widetilde{s}_\varepsilon = \pi/|\lambda|$, so that, by using $h_\varepsilon \equiv 0$, we get $s_\varepsilon = \widetilde{s}_\varepsilon = \pi/|2\lambda|$. From (6.1) we see that the singular curves Γ_1 and Γ_2 are different parameterizations of the same curve, namely, the x-axis translated by the vertical vector $(\operatorname{sgn}(\lambda)\,\pi/(4\lambda^2))\,T$, where $\operatorname{sgn}(x)$ is the sign of $x \in \mathbb{R}$. A straightforward computation from (6.1) shows that $\mathcal{S}_\lambda$ is the union of the graphs of the functions f and g defined on the xy-strip $-1/|2\lambda| \leqslant y \leqslant 1/|2\lambda|$

by

$$f(x,y)=\frac{\operatorname{sgn}(y)}{2\lambda}\left(\frac{\arcsin(2\lambda\,y)}{2\lambda}-y\,\sqrt{1-4\lambda\,y^2}\right)-xy,$$

$$g(x,y)=\frac{1}{2\lambda}\left(\frac{\operatorname{sgn}(\lambda)\,\pi-\operatorname{sgn}(y)\,\arcsin(2\lambda\,y)}{2\lambda}+\operatorname{sgn}(y)\,y\,\sqrt{1-4\lambda^2\,y^2}\right)-xy.$$

Both functions coincide on the boundary of the strip. Moreover, it is easy to see that $\mathcal{S}_\lambda$ is C^2 smooth around Γ and $\Gamma_1=\Gamma_2$ but not C^3 since

$$\frac{\partial^3 f}{\partial y^3}\,(x,y)=-\frac{\partial^3 g}{\partial y^3}\,(x,y)=\operatorname{sgn}(y)\;\frac{8\lambda\,(1+2\lambda^2\,y^2)}{(1-4\lambda^2\,y^2)^{5/2}}.$$

Finally, an easy argument proves that $\operatorname{sgn}(\lambda)\,f(x,y)<\operatorname{sgn}(\lambda)\,g(x,y)$ for any (x,y) such that $-1/|2\lambda|<y<1/|2\lambda|$. We conclude that $\mathcal{S}_\lambda$ is a complete volume-preserving area-stationary embedded cylinder in $\mathbb{H}^1$ with two singular curves given by parallel straight lines, see Figures 2 and 3.

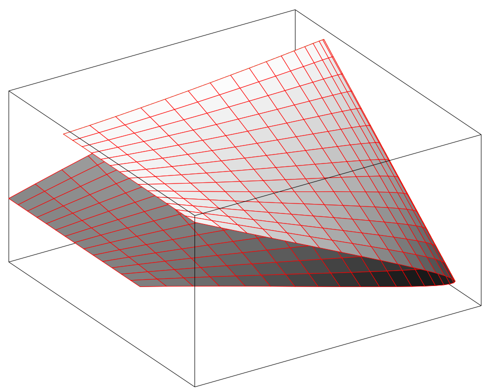

FIGURE 2. A portion of the surface $\mathcal{S}_\lambda$ composed of geodesics of curvature $\lambda>0$ joining two horizontal and parallel straight lines.

Example 6.7 (Helicoidal surfaces $\mathcal{L}_\lambda$)**.** Let Γ be the helix of radius $r>0$ and pitch $\pi/(2r^2)$ in $\mathbb{R}^3$ given by

$$\Gamma(\varepsilon)=\left(\frac{\sin(2r\varepsilon)}{2r},\frac{\cos(2r\varepsilon)-1}{2r},\frac{1}{2r}\left(\varepsilon-\frac{\sin(2r\varepsilon)}{2r}\right)\right).$$

The planar geodesic curvature of the xy-projection of Γ is $h(\varepsilon)=-2r$. For any $\lambda\neq 0$ we consider the union of the surfaces $\Sigma_\lambda(\Gamma)$ and $\widetilde{\Sigma}_\lambda(\Gamma)$ given in Proposition 6.3 and Remark 6.5, respectively. Easy computations from (6.1) show that the singular curves Γ_1 and Γ_2 are vertical translations of Γ by $c_1(\lambda)\,T$ and $c_2(\lambda)\,T$, where

$$c_1(\lambda)=\frac{s_\varepsilon}{2\lambda}+\frac{\operatorname{sgn}(\lambda)\,\pi-2\lambda s_\varepsilon}{4r^2}-\frac{(r^2+\lambda^2)\,\sin(2\lambda s_\varepsilon)}{4\lambda^2 r^2},$$

$$c_2(\lambda)=\frac{\operatorname{sgn}(\lambda)\,\pi}{2\lambda^2}-c_1(\lambda).$$

In the first equation above s_ε is the cut function associated to Γ_1. In general $\Gamma_1\neq\Gamma_2$ so that we can extend the surface by geodesics of the same curvature orthogonal to Γ_i. As indicated in Remark 6.4 and according with the value of $\dot{\Gamma}_i$, in order

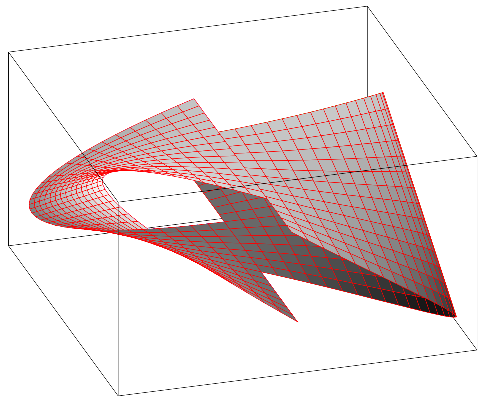

FIGURE 3. The complete surface $\mathcal{S}_\lambda$.

to preserve the constant mean curvature λ we must consider the surfaces $\widetilde{\Sigma}_{-\lambda}(\Gamma_1)$ and $\Sigma_{-\lambda}(\Gamma_2)$. Two new singular curves Γ_{12} and Γ_{22} are obtained. We repeat this process by induction so that at any step $k+1$ we leave from the singular curves Γ_{1k} and Γ_{2k} by the corresponding orthogonal geodesics of curvature $(-1)^k\lambda$. We denote by $\mathcal{L}_\lambda$ the union of all these surfaces. This is a C^2 immersed surface (in fact, it is C^∞ outside the singular curves) and, by construction, it is volume-preserving area-stationary with constant mean curvature λ. Any singular curve Γ_{ik} of $\mathcal{L}_\lambda$ is a vertical translation of the helix Γ by the vector $c_{ik}(\lambda)\,T$, where

$$\begin{aligned} c_{1k}(\lambda) &= k\,c_1(\lambda) - \operatorname{sgn}(\lambda)\left[\frac{k}{2}\right]\frac{\pi}{2\lambda^2}, \\ c_{2k}(\lambda) &= \frac{\operatorname{sgn}(\lambda)\,\pi}{2\lambda^2} - c_{1k}(\lambda), \end{aligned}$$

where $[x]$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$.

The singular curves Γ_{ik} could coincide depending on the values of λ. For example, an easy analytical argument shows that there is a discrete set of values of $\lambda \in (0,r)$ for which Γ_1 coincides with Γ (those for which $c_1(\lambda)$ is an integer multiple of $\pi/(2r^2)$). This situation is not possible when $\lambda^2 \geqslant r^2$. In fact, for the case $r=\lambda=1$ explicit computations from the equations above show that all the curves Γ_{ik} are different. So the resulting surface contains infinitely many singular helices. Also, it is not difficult to see that for a discrete set of values of $\lambda \in (0,r)$, we have $\Gamma_{1i}=\Gamma_{2i}$, so that we can obtain complete surfaces $\mathcal{L}_\lambda$ with any given even number of singular curves. In general, the surfaces $\mathcal{L}_\lambda$ are not embedded.

In Theorem 6.11 we will prove that any complete volume-preserving area-stationary surface Σ in $\mathbb{H}^1$ with singular curves and non-vanishing mean curvature is congruent to one of the surfaces $\mathcal{S}_\lambda$ or $\mathcal{L}_\lambda$ introduced above. We need the following strong restriction on the singular curves of Σ obtained as a consequence of Propositions 4.20 and 6.3 (iv).

Theorem 6.8. *Let Σ be a complete, oriented, C^2 immersed volume-preserving area-stationary surface in $\mathbb{H}^1$ with non-vanishing mean curvature. Then any connected singular curve of Σ is a complete geodesic of $\mathbb{H}^1$.*

Proof. Let $\mathcal{C}$ be a connected singular curve of Σ. By Proposition 4.20 we know that $\mathcal{C}$ is a C^2 smooth horizontal curve. We consider the unit normal N to Σ such that $N = T$ along $\mathcal{C}$. Let H be the mean curvature of Σ with respect to N. By using Theorem 4.15 (ii) and Remark 4.16, for any $p \in \mathcal{C}$ there is a small neighborhood of p in Σ foliated by geodesics of curvature H leaving from $\mathcal{C}$. By Theorem 4.17 these geodesics are characteristic curves of Σ and meet $\mathcal{C}$ orthogonally.

Let Γ be any closed arc of $\mathcal{C}$. We parameterize Γ by arc-length $\varepsilon \in [a, b]$. By compactness we can find a small $r > 0$ such that, for any $\varepsilon \in [a, b]$, the geodesic $\gamma_\varepsilon : [0, r) \to \mathbb{H}^1$ of curvature H with initial conditions $\Gamma(\varepsilon)$ and $J(\dot{\Gamma}(\varepsilon))$ is entirely contained in Σ. This implies that Σ and the surface $\Sigma_H(\Gamma)$ in Proposition 6.3 locally coincides at one side of Γ. Moreover, as Σ is complete we deduce that $\Sigma_H(\Gamma) \subset \Sigma$. In particular, Γ_1 is a piece of a singular curve of Σ and so it is C^2 smooth by Proposition 4.20. As Σ is volume-preserving area-stationary we deduce by Theorem 4.17 that the geodesics γ_ε meet Γ_1 orthogonally. This implies by Proposition 6.3 (iv) that the cut function s_ε is constant along Γ. As Γ is an arbitrary closed arc of $\mathcal{C}$, we have proved that the xy-projection of $\mathcal{C} = (x, y, t)$ is a straight line or a planar circle. Finally, by integrating the "horizontal" equation $\dot{t} = \dot{x}y - x\dot{y}$ (as was done in Section 3) we conclude that $\mathcal{C}$ is a complete geodesic of $\mathbb{H}^1$. $\square$

Now, we will see how to apply our previous results to describe all compact volume-preserving area-stationary surfaces in $\mathbb{H}^1$.

The first relevant results about compact surfaces with constant mean curvature in $\mathbb{H}^1$ were given in [CHMY, Theorem E], where it was obtained an interesting restriction on the topology of an immersed surface inside a spherical 3-dimensional pseudo-hermitian manifold under the weaker assumption that the mean curvature is bounded outside the singular set. The arguments in the proof use the local behavior of the singular set studied in Theorem 4.15 and Hopf Index Theorem for line fields. They also apply to $\mathbb{H}^1$ so that we get

Proposition 6.9 ([CHMY, Theorem E]). *Any compact, connected, C^2 immersed surface Σ in $\mathbb{H}^1$ with constant mean curvature is homeomorphic either to a sphere or to a torus.*

Moreover, in [CHMY, §7, Examples 1 and 2] we can find examples of constant mean curvature surfaces of spherical and toroidal type inside the standard pseudo-hermitian 3-sphere. In $\mathbb{H}^1$ we could expect, by analogy with the Euclidean space, the existence of immersed tori with constant mean curvature [W]. However, this is not possible as a consequence of our following result, that may be seen as a counterpart in $\mathbb{H}^1$ to Alexandrov uniqueness theorem for embedded surfaces in $\mathbb{R}^3$.

Theorem 6.10 (Alexandrov Theorem in $\mathbb{H}^1$). *Let Σ be a compact, connected, C^2 immersed volume-preserving area-stationary surface in $\mathbb{H}^1$. Then Σ is congruent to a sphere $\mathbb{S}_H$ of the same constant mean curvature.*

Proof. From the Minkowski formula (4.10) we have that the constant mean curvature H of Σ with respect to the inner normal must be positive. Observe that Σ must contain a singular point. Otherwise Theorem 4.8 would imply that Σ is foliated by complete geodesics, a contradiction since any geodesic of $\mathbb{H}^1$ leaves a compact set in finite time (Remark 3.2). On the other hand Σ cannot contain a singular curve since this curve would be a complete geodesic by Theorem 6.8, contradicting that Σ is compact. We conclude that Σ has an isolated singularity. We finally invoke Theorem 6.1 to deduce that Σ is congruent to a sphere $\mathbb{S}_H$ of the same mean curvature. $\square$

Now, we shall prove the following classification theorem

Theorem 6.11. *Let Σ be a complete, oriented, connected, C^2 immersed volume-preserving area-stationary surface in $\mathbb{H}^1$ with non-vanishing mean curvature. If Σ contains a singular curve then Σ is congruent either to the surface $\mathcal{S}_H$ in Example 6.6 or to the surface $\mathcal{L}_H$ in Example 6.7 of the same mean curvature as Σ.*

Proof. Let Γ be a connected singular curve of Σ. By Theorem 6.8 we know that Γ is a complete geodesic of $\mathbb{H}^1$. After applying a left translation and a vertical rotation we can suppose that Γ coincides either with the x-axis or with a helix passing through the origin. We can choose the unit normal N to Σ so that $N = T$ along Γ. By Theorem 4.15 (ii) and Remark 4.16 there is $r > 0$ such that the geodesics $\gamma_\varepsilon : [0, r] \to \mathbb{H}^1$ and $\widetilde{\gamma}_\varepsilon : [0, r] \to \mathbb{H}^1$ of curvature H with initial conditions $\Gamma(\varepsilon)$ and $J(\dot{\Gamma}(\varepsilon))$ (resp. $\Gamma(\varepsilon)$ and $-J(\dot{\Gamma}(\varepsilon))$) are contained in Σ. As Σ is complete and connected we can extend these geodesics until they meet a singular curve. This implies that the union of the surfaces $\Sigma_\lambda(\Gamma)$ and $\widetilde{\Sigma}_\lambda(\Gamma)$ constructed in Proposition 6.3 and Remark 6.5 is included in Σ. The proof then follows by using the description of the surfaces $\mathcal{S}_\lambda$ and $\mathcal{L}_\lambda$ in Examples 6.6 and 6.7 together with the completeness and the connectedness of Σ. $\square$

Remark 6.12. The previous result and Theorem 6.1 provide the description of complete C^2 immersed area-stationary surfaces under a volume constraint in $\mathbb{H}^1$ with non-empty singular set and non-vanishing mean curvature. Unduloids, cylinders and nodoids in $\mathbb{H}^1$ are examples of complete volume-preserving area-stationary surfaces in $\mathbb{H}^1$ with non-vanishing mean curvature and empty singular set, see [RR].

The arguments in this section can also be used to obtain the complete classification of C^2 complete area-stationary surfaces in $\mathbb{H}^1$ with singular curves.

Proposition 6.13. *Let $\Sigma \subset \mathbb{H}^1$ be a complete, connected, oriented, C^2 immersed area-stationary surface. Let $\Gamma \subset \Sigma$ be a connected singular curve. Then Γ is a complete geodesic of curvature λ and either*

(i) $\lambda = 0$ *and* $\Sigma_0 = \Gamma$, *or*
(ii) $\lambda \neq 0$ *and* Σ_0 *is the union of* Γ *and a second geodesic* Γ_1 *of the same curvature.*

Proof. We argue as in the proof of Lemma 6.2 and Theorem 6.8. Take an arc-length parameterization $\Gamma(\varepsilon) := (x(\varepsilon), y(\varepsilon), t(\varepsilon))$ of Γ. Consider the union of the horizontal geodesics $\gamma_\varepsilon(s)$ of curvature 0 extending from the point $\Gamma(\varepsilon)$ of Γ with tangent vector $J(\dot{\Gamma}(\varepsilon)) = -\dot{y}(\varepsilon)\, X_{\Gamma(\varepsilon)} + \dot{x}(\varepsilon)\, Y_{\Gamma(\varepsilon)}$. In parametric coordinates, this surface is given by

$$
\begin{aligned}
x(\varepsilon, s) &= x(\varepsilon) - \dot{y}(\varepsilon)\, s, \\
y(\varepsilon, s) &= y(\varepsilon) + \dot{x}(\varepsilon)\, s, \\
t(\varepsilon, s) &= t(\varepsilon) - (\dot{x}(\varepsilon)x(\varepsilon) + \dot{y}(\varepsilon)y(\varepsilon))\, s.
\end{aligned}
\tag{6.3}
$$

Consider the Jacobi field

$$V(\varepsilon, s) = \left(\frac{\partial x}{\partial \varepsilon}, \frac{\partial y}{\partial \varepsilon}, \frac{\partial t}{\partial \varepsilon}\right)(\varepsilon, s).$$

A direct computation using (6.3) shows that

$$\big\langle V, T\big\rangle(\varepsilon, s) = h(\varepsilon)\, s^2 - 2s, \tag{6.4}$$

where $h(\varepsilon) = (\dot{x}\ddot{y} - \ddot{x}\dot{y})(\varepsilon)$ is the geodesic curvature of the projection of Γ to the xy-plane. In case $h(\varepsilon) = 0$, equation (6.4) shows that $s = 0$ is the only zero of $\big\langle V, T\big\rangle$. In case $h(\varepsilon) \neq 0$, we have that $s = 0$ and $s(\varepsilon) := 2\, h(\varepsilon)^{-1}$ are the only solutions of $\big\langle V, T\big\rangle = 0$.

In case $h \equiv 0$, the curve Γ is a straight line. Assume $h(\varepsilon) \not\equiv 0$. Fix ε_0 so that $h(\varepsilon_0) \neq 0$. For ε close to ε_0, the geodesic segments $\gamma_\varepsilon([0, s(\varepsilon)])$ are contained in Σ and the curve $\Gamma_1(\varepsilon) := (x(\varepsilon, s(\varepsilon)), y(\varepsilon, s(\varepsilon)), t(\varepsilon, s(\varepsilon)))$ is composed of singular points by (6.4) and the definition of $s(\varepsilon)$. Observe that Γ_1 is C^2 by Proposition 4.20. The tangent vector to $\Gamma_1(\varepsilon)$ is given by

$$\dot{\Gamma}_1(\varepsilon) = V(\varepsilon, s_\varepsilon) + \dot{s}(\varepsilon)\, \dot{\gamma}_\varepsilon(s_\varepsilon).$$

As Σ is area-stationary, Theorem 4.17 implies that the curves $\gamma_\varepsilon(s)$ must meet the singular curve $\Gamma_1(\varepsilon)$ in an orthogonal way. As $\big\langle V(\varepsilon, s), \dot{\gamma}_\varepsilon(s)\big\rangle \equiv 0$ we get

$$0 = \big\langle \dot{\gamma}_\varepsilon(s(\varepsilon)), \dot{\Gamma}_1(\varepsilon)\big\rangle = \dot{s}(\varepsilon),$$

and so $s(\varepsilon)$ and $h(\varepsilon)$ are constant for ε close to ε_0. As the geodesic curvature $h(\varepsilon)$ is a continuous function of ε we conclude that it is a constant function. Anyway, we obtain that Γ is a complete geodesic of curvature $\lambda = -h(\varepsilon)/2$.

If $\lambda = 0$, equation (6.4) implies that there are no singular points along γ_ε different from $\gamma_\varepsilon(0) = \Gamma(\varepsilon)$.

If $\lambda \neq 0$, using (6.3) and the fact that $h(\varepsilon)$ is a non-zero constant, we easily deduce that Γ_1 is an arc-length parameterized geodesic of curvature λ. Using the connectedness and completeness of Σ we conclude that there are no more singular points on Σ. □

Example 6.14 (The helicoids $\mathcal{H}_\lambda$)**.** Starting from a geodesic of curvature $\lambda \neq 0$, area-stationary helicoidal surfaces in $\mathbb{H}^1$ can be obtained, see [Pa1, Theorem D]. For any $\lambda \neq 0$ we define the helicoid $\mathcal{H}_\lambda$ as the one given by the parameterization

$$F_\lambda(u, v) := \big(v \sin(2\lambda u), v \cos(2\lambda u), \frac{1}{2\lambda}\, u\big), \quad (u, v) \in \mathbb{R}^2.$$

The singular curves are $v = \pm 1/(2\lambda)$. The surfaces $\mathcal{H}_\lambda$ coincide with the classical minimal helicoids of $\mathbb{R}^3$. In particular, they are embedded surfaces.

Using Proposition 6.13 and arguments similar to those in the proof of Theorem 6.11, we can describe all the area-stationary surfaces with singular curves. This fact, together with the result in [CHMY] that any complete minimal surface with an isolated singularity must coincide with a Euclidean plane, provides the complete classification of complete area-stationary surfaces in $\mathbb{H}^1$ with non-empty singular set.

Theorem 6.15. *Let $\Sigma \subset \mathbb{H}^1$ be a complete, connected, oriented, C^2 immersed area-stationary surface with non-empty singular set. Then Σ is either a plane, or congruent to the hyperbolic paraboloid $t = xy$, or congruent to a helicoid $\mathcal{H}_\lambda$.*

It is difficult to get a complete classification of minimal or constant mean curvature surfaces without singular points in $\mathbb{H}^1$, see [CH].

We will say that a C^1 surface Σ is *vertical* if the vertical vector T is contained in $T_p\Sigma$ for any $p \in \Sigma$. A complete vertical surface Σ is foliated by vertical straight lines. Since a C^2 vertical surface has no singular points, to have constant mean curvature H implies that Σ is either area-stationary in case $H = 0$, or volume-preserving area-stationary in case $H \neq 0$. From Theorem 4.8 is easy to get the following, compare with [GP, Lemma 4.9].

Proposition 6.16. *Let Σ be a C^2 complete, connected, immersed, oriented, constant mean curvature surface in $\mathbb{H}^1$. If Σ is vertical then Σ is either a vertical plane, or a right circular cylinder.*

7. The isoperimetric problem in $\mathbb{H}^1$

The isoperimetric problem in $\mathbb{H}^1$ consists on finding global minimizers of the sub-Riemannian perimeter under a volume constraint. For any Borel set $\Omega \subseteq \mathbb{H}^1$ the *perimeter* of Ω is defined by

$$\mathcal{P}(\Omega) := \sup \left\{ \int_\Omega \operatorname{div}(U)\, dv;\ |U| \leqslant 1 \right\},$$

where the supremum is taken over C^1 *horizontal vector fields* with compact support on $\mathbb{H}^1$. In the definition above, dv and $\operatorname{div}(\cdot)$ are the Riemannian volume and divergence of the left invariant metric g, respectively. This notion of perimeter coincides with the $\mathbb{H}^1$-perimeter introduced in [CDG] and [FSSC]. For a set Ω bounded by a surface Σ of class C^2 we have $\mathcal{P}(\Omega) = A(\Sigma)$ by virtue of the Riemannian divergence theorem.

It is not difficult to prove that the perimeter is 3-homogeneous with respect to the family of dilations in (2.8), see for instance [MoSC, Lemma 4.5]. Precisely, for any Borel set $\Omega \subseteq \mathbb{H}^1$ and any $s \in \mathbb{R}$ we have

$$V(\varphi_s(\Omega)) = e^{4s}\, V(\Omega), \qquad \mathcal{P}(\varphi_s(\Omega)) = e^{3s}\, \mathcal{P}(\Omega).$$

This property leads us to the isoperimetric inequality

$$\mathcal{P}(\Omega)^4 \geqslant \alpha\, V(\Omega)^3, \tag{7.1}$$

that holds for any Borel set $\Omega \subseteq \mathbb{H}^1$. Inequality (7.1) was first obtained by P. Pansu [P1] for regular sets. Many other generalizations have been established but always without the sharp constant α, see [GN] and [DGN2].

An *isoperimetric region* in $\mathbb{H}^1$ is a set $\Omega \subset \mathbb{H}^1$ such that

$$\mathcal{P}(\Omega) \leqslant \mathcal{P}(\Omega')$$

amongst all sets $\Omega' \subset \mathbb{H}^1$ with $V(\Omega) = V(\Omega')$.

The existence of isoperimetric regions was proved by G. P. Leonardi and S. Rigot [LR, Theorem 2.5] in the more general context of Carnot groups, see also [DGN1, Theorem 13.7]. We summarize their results in the following theorem.

Theorem 7.1 ([LR]). *For any $V > 0$ there is an isoperimetric region Ω in $\mathbb{H}^1$ with $V(\Omega) = V$. The set Ω is, up to a set of measure zero, a bounded connected open set. Moreover, the boundary $\partial\Omega$ is Alhfors regular and verifies condition B.*

The condition B in the theorem above is a certain separation property. It means that there is a constant $\beta > 0$ such that for any Carnot-Carathéodory ball B centered on $\partial\Omega$ with radius $r \leqslant 1$ there exist two balls B_1 and B_2 with radius βr such that $B_1 \subset B \cap \Omega$ and $B_2 \subset B - \overline{\Omega}$.

The properties in Theorem 7.1 are not sufficient to describe the isoperimetric regions in $\mathbb{H}^1$. In 1983 P. Pansu made the following

Conjecture ([P2, p. 172]). In the Heisenberg group $\mathbb{H}^1$ any isoperimetric region bounded by a smooth surface is congruent to a sphere $\mathbb{S}_\lambda$.

In the last years many authors have tried to adapt to the Heisenberg setting different proofs of the classical isoperimetric inequality in Euclidean space. In [Mo1], [Mo2] and [LM] it was shown that there is not a direct counterpart in $\mathbb{H}^1$ to the Brunn-Minkowski inequality in Euclidean space, with the consequence that the Carnot-Carathéodory metric balls in $\mathbb{H}^1$ cannot be the solutions. Recently, expecting that symmetrization could work in $\mathbb{H}^1$, interest has focused on solving the isoperimetric problem restricted to certain sets with additional symmetries. It has been recently proved by D. Danielli, N. Garofalo and D.-M. Nhieu that the sets Ω_λ bounded by the spherical surfaces $\mathbb{S}_\lambda$ are the unique solutions in the class of sets bounded by two C^1 graphs over the xy-plane [DGN2, Theorem 1.1]. An intrinsic description of the solutions was given by G. P. Leonardi and S. Masnou [LM, Theorem 3.3], where it was proved that any sphere $\mathbb{S}_\lambda$ is the union of all the geodesics of curvature λ in $\mathbb{H}^1$ connecting the poles. In [RR] we pointed out that assuming C^2 smoothness and rotationally symmetry of isoperimetric regions, these must be congruent to the spheres $\mathbb{S}_\lambda$. We also mention the interesting recent work [BoC] in which it is proved that the flow by mean curvature of a C^2 convex surface in $\mathbb{H}^1$ described as the union of the radial graphs $t = \pm f(|z|)$, with $f' > 0$, converges to the spheres $\mathbb{S}_\lambda$.

The regularity of isoperimetric regions in $\mathbb{H}^1$ is still an open question. The regularity of the spheres $\mathbb{S}_\lambda$ and of the examples of complete volume-preserving area-stationary surfaces in Section 6 may suggest that the isoperimetric solutions in $\mathbb{H}^1$ are C^∞ away from the singular set and only C^2 around the singularities.

By assuming C^2 regularity of the solutions we can use the uniqueness of spheres in Theorem 6.10 to solve the isoperimetric problem in $\mathbb{H}^1$.

Theorem 7.2. *If Ω is an isoperimetric region in $\mathbb{H}^1$ bounded by a C^2 smooth surface Σ, then Ω is congruent to a set bounded by a sphere $\mathbb{S}_\lambda$.*

Proof. Let Ω be an isoperimetric region of class C^2 in $\mathbb{H}^1$. By using Theorem 7.1 we can assume that Ω is bounded and connected. The boundary $\Sigma = \partial\Omega$ is a C^2 compact surface with finitely many connected components. Let us see that Σ is connected. Otherwise we may find a bounded component Ω_0 of $\mathbb{H}^1 - \overline{\Omega}$. Consider the set $\Omega_1 = \Omega \cup \Omega_0$. It is clear that $V(\Omega_1) > V(\Omega)$ and $\mathcal{P}(\Omega_1) < \mathcal{P}(\Omega)$. Thus by applying an appropriated dilation to Ω_1 we would obtain a new set Ω' so that $V(\Omega') = V(\Omega)$ and $\mathcal{P}(\Omega') < \mathcal{P}(\Omega)$, a contradiction since Ω is isoperimetric. As Σ is a C^2 compact, connected, volume-preserving area-stationary surface in $\mathbb{H}^1$, we conclude by Alexandrov (Theorem 6.10) that Σ is congruent to a sphere $\mathbb{S}_\lambda$. □

Remark 7.3 (The isoperimetric constant in $\mathbb{H}^1$)**.** The area of the sphere $\mathbb{S}_\lambda$ can be easily computed from (3.5). Using polar coordinates and Fubini's theorem we get

$$A(\mathbb{S}_\lambda) = \frac{\pi^2}{\lambda^3}.$$

On the other hand, we can use Minkowski formula (4.10) to compute the volume of the set Ω_λ enclosed by $\mathbb{S}_\lambda$. We obtain

$$V(\Omega_\lambda) = \frac{3\pi^2}{8\lambda^4}.$$

In case the C^2 regularity of isoperimetric sets in $\mathbb{H}^1$ were established, we would deduce from Theorem 7.2 that the optimal isoperimetric constant in (7.1) would be given by

$$\alpha = \frac{A(\mathbb{S}_\lambda)^4}{V(\Omega_\lambda)^3} = \left(\frac{8}{3}\right)^3 \pi^2.$$

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Departamento de Geometría y Topología, Universidad de Granada, E–18071 Granada, España

E-mail address: `ritore@ugr.es`

Departamento de Geometría y Topología, Universidad de Granada, E–18071 Granada, España

E-mail address: `crosales@ugr.es`