EXAMPLES OF AREA-MINIMIZING SURFACES IN THE SUBRIEMANNIAN HEISENBERG GROUP $\mathbb{H}^1$ WITH LOW REGULARITY

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ABSTRACT. We give new examples of entire area-minimizing t-graphs in the subriemannian Heisenberg group $\mathbb{H}^1$. They are locally lipschitz in Euclidean sense. Some regular examples have prescribed singular set consisting of either a horizontal line or a finite number of horizontal halflines extending from a given point. Amongst them, a large family of area-minimizing cones is obtained.

1. INTRODUCTION

Variational problems related to the subriemannian area in the Heisenberg group $\mathbb{H}^1$ have received great attention recently. A major question in this theory is the regularity of minimizers. A related one is the construction of examples with low regularity properties. The study of minimal surfaces in subriemannian geometry was initiated in the paper by Garofalo and Nhieu [24]. Later Pauls [29] constructed minimal surfaces in $\mathbb{H}^1$ as limits of minimal surfaces in Nil manifolds, the riemannian Heisenberg groups. Cheng, Hwang and Yang [11] have studied the weak solutions of the minimal surface equation for t-graphs and have proven existence and uniqueness results. Regularity of minimal surfaces, assuming that they are least C^1, has been treated in the papers by Pauls [30] and Cheng, Hwang and Yang [12]. We would like also to mention the recently distributed notes by Bigolin and Serra Cassano [5], where they obtain regularity properties of an $\mathbb{H}$-regular surface from regularity properties of its horizontal unit normal. Interesting examples of minimal surfaces which are not area-minimizing are obtained in [13]. See also [14]. Smoothness of lipschitz intrinsic graphs which are viscosity solutions of the minimal surface equation in the Heisenberg groups $\mathbb{H}^n$, for $n > 1$ and $n = 1$, has been recently obtained by Capogna, Citti and Manfredini [7], [6].

Characterization in $\mathbb{H}^1$ of solutions of the Bernstein problem for C^2 surfaces has been obtained by Cheng, Hwang, Malchiodi and Yang [10], and Ritoré and Rosales [31] for t-graphs, and by Barone Adessi, Serra Cassano and Vittone [4] and Garofalo and Pauls [25] for vertical graphs.

Additional contributions concerning variational problems related to the subriemannian area in the Heisenberg groups include [28], [2], [10], [9], [11], [12], [23], [23], [22], [21],

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[20], [19], [18], [27], [31]. The recent monograph by Capogna, Danielli, Pauls and Tyson [8] gives a recent overview of the subject with an exhaustive list of references. We would like to stress that, in $\mathbb{H}^1$, the condition $H \equiv 0$ is not enough to guarantee that a given surface of class C^2 is even a stationary point for the area functional, see Ritoré and Rosales [31], and Cheng, Hwang and Yang [11] for minimizing t-graphs.

The aim of this paper is to provide new examples in $\mathbb{H}^1$ of Euclidean locally lipschitz area-minimizing entire graphs over the xy-plane. Previous examples of Euclidean locally lipschitz area-minimizing surfaces in $\mathbb{H}^1$ were constructed in the paper by Cheng, Hwang and Yang [11]. See also the examples in Pauls' paper [29, Thm. 4.2].

We have organized this paper into three more sections. The next one contains background material. In section 3 we construct the basic examples. We start from a given horizontal line L, and a monotone angle function $\alpha : L \to (0, \pi)$ over this line. For each $p \in L$, we consider the two horizontal halflines extending from p making an angle $\pm\alpha(p)$ with L. We prove that in this way we always obtain an entire graph over the xy-plane which is Euclidean locally lipschitz and area-minimizing. The angle function α is only assumed to be continuous and monotone. Of course, further regularity on α yields more regularity on the graph. In case α is at least C^2 we get that the associated surface is $C^{1,1}$.

The surfaces in section 3 are the building blocks for our next construction in section 4. We fix a point $p \in \mathbb{H}^1$, and a family of counter-clockwise oriented horizontal halflines R_1, …, R_n extending from p. We choose the bisector L_i of the wedge determined by R_{i-1} and R_i, and we consider angle functions $\alpha_i : L_i \to (0, \pi)$ which are continuous, nonincreasing as a function of the distance to p, and such that $\alpha_i(p)$ is equal to the angle between L_i and R_i. For every $q \in L_i$, we consider the halflines extending from q with angles $\pm\alpha_i(q)$. In this way we also a family of area-minimizing t-graphs which are Euclidean locally lipschitz. In case the obtained surface is regular enough we have that the singular set is precisely $\bigcup_{i=1}^n L_i$. If the angle functions α_i are constant, then we obtain area-minimizing cones (the original motivation of this paper), which are Euclidean locally $C^{1,1}$ minimizers, and C^∞ outside the singular set $\bigcup_{i=1}^n L_i$. For a single halfline L extending from the origin and an angle function $\alpha : L \to (0, \pi)$, continuous and nonincreasing as a function of the distance to 0, we patch the graph obtained over a wedge of the xy-plane with the plane $t = 0$ along the halflines extending from 0 making an angle $\alpha(0)$ with L. When α is constant we get again an area-minimizing cone which is Euclidean locally lipschitz. These cones are a generalization of the one obtained by Cheng, Hwang and Yang [11, Ex. 7.2].

An interesting consequence of this construction is that we get a large number of Euclidean locally $C^{1,1}$ area-minimizing cones with prescribed singular set consisting on either a horizontal line or a finite number of horizontal halflines extending from a given point. It is an open question to decide if these examples are the only area-minimizing cones, together with vertical halfspaces and the example by Cheng, Hwang and Yang [11, Ex. 7.2] with a singular halfline and its generalizations in the last section. The importance of tangent cones has been recently stressed in [1].

A final remark has been included considering discontinuous angle functions. These examples were suggested by the referee, to which the author is grateful by the careful reading of the manuscript and the valuable suggestions.

2. Preliminaries

The *Heisenberg group* $\mathbb{H}^1$ is the Lie group $(\mathbb{R}^3, *)$, where the product $*$ is defined, for any pair of points $[z,t]$, $[z',t'] \in \mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, as

$$[z,t] * [z',t'] := [z+z', t+t' + \operatorname{Im}(z\bar{z}')], \qquad (z = x+iy).$$

For $p \in \mathbb{H}^1$, the *left translation* by p is the diffeomorphism $L_p(q) = p * q$. A basis of left invariant vector fields (i.e., invariant by any left translation) is given by

$$X := \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \qquad Y := \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \qquad T := \frac{\partial}{\partial t}.$$

The *horizontal distribution* $\mathcal{H}$ in $\mathbb{H}^1$ is the smooth planar one generated by X and Y. The *horizontal projection* of a vector U onto $\mathcal{H}$ will be denoted by U_H. A vector field U is called *horizontal* if $U = U_H$. A *horizontal curve* is a C^1 curve whose tangent vector lies in the horizontal distribution.

We denote by $[U,V]$ the Lie bracket of two C^1 vector fields U, V on $\mathbb{H}^1$. Note that $[X,T] = [Y,T] = 0$, while $[X,Y] = -2T$. The last equality implies that $\mathcal{H}$ is a bracket generating distribution. Moreover, by Frobenius Theorem we have that $\mathcal{H}$ is nonintegrable. The vector fields X and Y generate the kernel of the (contact) 1-form $\omega := -y\,dx + x\,dy + dt$.

We shall consider on $\mathbb{H}^1$ the (left invariant) Riemannian metric $g = \langle\cdot,\cdot\rangle$ so that $\{X,Y,T\}$ is an orthonormal basis at every point, and the associated Levi-Civitá connection D. The modulus of a vector field U will be denoted by $|U|$.

Let $\gamma : I \to \mathbb{H}^1$ be a piecewise C^1 curve defined on a compact interval $I \subset \mathbb{R}$. The *length* of γ is the usual Riemannian length $L(\gamma) := \int_I |\dot{\gamma}|$, where $\dot{\gamma}$ is the tangent vector of γ. For two given points in $\mathbb{H}^1$ we can find, by Chow's connectivity Theorem [26, p. 95], a horizontal curve joining these points. The *Carnot-Carathédory distance* d_{cc} between two points in $\mathbb{H}^1$ is defined as the infimum of the length of horizontal curves joining the given points. A *geodesic* $\gamma : \mathbb{H}^1 \to \mathbb{R}$ is a horizontal curve which is a critical point of length under variations by horizontal curves. They satisfy the equation

$$D_{\dot{\gamma}}\dot{\gamma} + 2\lambda\, J(\dot{\gamma}) = 0, \tag{2.1}$$

where $\lambda \in \mathbb{R}$ is the *curvature* of the geodesic, and J is the $\pi/2$-degrees oriented rotation in the horizontal distribution. Geodesics in $\mathbb{H}^1$ with $\lambda = 0$ are horizontal straight lines. The reader is referred to the section on geodesics in [31] for further details.

The *volume* $|\Omega|$ of a Borel set $\Omega \subseteq \mathbb{H}^1$ is the Riemannian volume of the left invariant metric g, which coincides with the Lebesgue measure in $\mathbb{R}^3$. We shall denote this volume element by dv_g. The *perimeter* of $E \subset \mathbb{H}^1$ in an open subset $\Omega \subset \mathbb{H}^1$ is defined as

$$|\partial E|(\Omega) := \sup\left\{ \int_\Omega \operatorname{div} U\, dv_g : U \text{ horizontal and } C^1, |U| \leqslant 1, \operatorname{supp}(U) \subset\subset \Omega \right\}, \tag{2.2}$$

where $\operatorname{supp}(U)$ is the support of U. A set $E \subset \mathbb{H}^1$ is of *locally finite perimeter* if $|\partial E|(\Omega) < +\infty$ for any bounded open set $\Omega \subset \mathbb{H}^1$. A set of locally finite perimeter has a measurable *horizontal unit normal* ν_E, that satisfies the following divergence theorem [19, Corollary 7.6]: if U is a horizontal vector field with compact support, then

$$\int_E \operatorname{div} U \, dv_g = \int_{\mathbb{H}^1} \langle U, \nu_E \rangle \, d|\partial E|.$$

If $E \subset \mathbb{H}^1$ has Euclidean lipschitz boundary, then [19, Corollary 7.7]

$$|\partial E|(\Omega) = \int_{\partial E \cap \Omega} |N_H| \, d\mathcal{H}^2, \tag{2.3}$$

where N is the outer unit normal to ∂E, defined $\mathcal{H}^2$-almost everywhere. Here $\mathcal{H}^2$ is the 2-dimensional riemannian Hausdorff measure.

Let $\Omega \subset \mathbb{H}^1$ be an open set. We say that $E \subset \mathbb{H}^1$ of locally finite perimeter is *area-minimizing* in Ω if, for any set F such that $E = F$ outside Ω we have

$$|\partial E|(\Omega) \leqslant |\partial F|(\Omega).$$

The following extension of the divergence theorem will be needed to prove the area-minimizing property of sets of locally finite perimeter

Theorem 2.1. *Let $E \subset \mathbb{H}^1$ be a set of locally finite perimeter, $B \subset \mathbb{H}^1$ a set with piecewise smooth boundary, and U a C^1 horizontal vector field in* $\operatorname{int}(B)$ *that extends continuously to the boundary of B. Then*

$$\int_{E \cap B} \operatorname{div} U \, dv_g = \int_B \langle U, \nu_E \rangle \, d|\partial E| + \int_E \langle U, \nu_B \rangle \, d|\partial B|. \tag{2.4}$$

Proof. The proof is modelled on [17, § 5.7]. Let s denote the riemannian distance function to $\mathbb{H}^1 - B$. For $\varepsilon > 0$, define

$$h_\varepsilon(p) := \begin{cases} 1, & \varepsilon \leqslant s(p), \\ s(p)/\varepsilon, & 0 \leqslant s(p) \leqslant \varepsilon, \end{cases}$$

Then h_ε is a lipschitz function (in riemannian sense). For any smooth h with compact support in B we have $\operatorname{div}(hU) = h \operatorname{div}(U) + \langle \nabla h, U \rangle$. By applying the divergence theorem for sets of locally finite perimeter [19] we get

$$\int_{\mathbb{H}^1} h \, \langle U, \nu_E \rangle \, d|\partial E| = \int_E h \, \operatorname{div}(U) + \int_E \langle \nabla h, U \rangle.$$

By approximation, this formula is also valid for h_ε. Taking limits when $\varepsilon \to 0$ we have $h_\varepsilon \to \chi_B$. By the coarea formula for lipschitz functions

$$\frac{1}{\varepsilon} \int_{\{0 \leqslant s \leqslant \varepsilon\}} \chi_E \, \langle \nabla s, U \rangle = \frac{1}{\varepsilon} \int_0^\varepsilon \left\{ \int_{\{s=r\}} \chi_E \, \langle \nabla s, U \rangle \, d\mathcal{H}^2 \right\} dr,$$

and, taking again limits when $\varepsilon \to 0$ and calling N_B to the riemannian outer unit normal to ∂B (defined except on a small set), we have

$$\lim_{\varepsilon \to 0} \int_E \langle \nabla h_\varepsilon, U \rangle = \int_{\partial B} \chi_E \langle N_B, U \rangle \, d\mathcal{H}^2 = \int_E \langle \nu_B, U \rangle \, d|\partial B|.$$

Hence (2.4) is proved. $\square$

For a C^1 surface $\Sigma \subset \mathbb{H}^1$ the *singular set* Σ_0 consists of those points $p \in \Sigma$ for which the tangent plane $T_p\Sigma$ coincides with the horizontal distribution. As Σ_0 is closed and has empty interior in Σ, the *regular set* $\Sigma - \Sigma_0$ of Σ is open and dense in Σ. It was proved in [16, Lemme 1], see also [3, Theorem 1.2], that, for a C^2 surface, the Hausdorff dimension with respect to the Riemannian distance on $\mathbb{H}^1$ of Σ_0 is less than two.

If Σ is a C^1 oriented surface with unit normal vector N, then we can describe the singular set $\Sigma_0 \subset \Sigma$, in terms of N_H, as $\Sigma_0 = \{p \in \Sigma : N_H(p) = 0\}$. In the regular part $\Sigma - \Sigma_0$, we can define the *horizontal unit normal vector* ν_H, as in [15], [32] and [25] by

$$\nu_H := \frac{N_H}{|N_H|}. \tag{2.5}$$

Consider the *characteristic vector field* Z on $\Sigma - \Sigma_0$ given by

$$Z := J(\nu_H). \tag{2.6}$$

As Z is horizontal and orthogonal to ν_H, we conclude that Z is tangent to Σ. Hence Z_p generates the intersection of $T_p\Sigma$ with the horizontal distribution. The integral curves of Z in $\Sigma - \Sigma_0$ will be called *characteristic curves* of Σ. They are both tangent to Σ and horizontal. Note that these curves depend on the unit normal N to Σ. If we define

$$S := \langle N, T \rangle \, \nu_H - |N_H| \, T, \tag{2.7}$$

then $\{Z_p, S_p\}$ is an orthonormal basis of $T_p\Sigma$ whenever $p \in \Sigma - \Sigma_0$.

In the Heisenberg group $\mathbb{H}^1$ there is a one-parameter group of *dilations* $\{\varphi_s\}_{s\in\mathbb{R}}$ generated by the vector field

$$W := xX + yY + 2tT. \tag{2.8}$$

We may compute φ_s in coordinates to obtain

$$\varphi_s(x_0, y_0, t_0) = (e^s x_0, e^s y_0, e^{2s} t_0). \tag{2.9}$$

Conjugating with left translations we get the one-parameter family of dilations $\varphi_{p,s} := L_p \circ \varphi_s \circ L_p^{-1}$ with center at any point $p \in \mathbb{H}^1$. A set $E \subset \mathbb{H}^1$ is a *cone of center* p if $\varphi_{p,s}(E) \subset E$ for all $s \in \mathbb{R}$.

Any isometry of $(\mathbb{H}^1, g)$ leaving invariant the horizontal distribution preserves the area of surfaces in $\mathbb{H}^1$. Examples of such isometries are left translations, which act transitively on $\mathbb{H}^1$. The Euclidean rotation of angle θ about the t-axis given by

$$(x, y, t) \mapsto r_\theta(x, y, t) = (\cos\theta \, x - \sin\theta \, y, \sin\theta \, x + \cos\theta \, y, t),$$

is also an area-preserving isometry in $(\mathbb{H}^1, g)$ since it transforms the orthonormal basis $\{X, Y, T\}$ at the point p into the orthonormal basis $\{\cos\theta \, X + \sin\theta \, Y, -\sin\theta \, X + \cos\theta \, Y, T\}$ at the point $r_\theta(p)$.

3. Examples with one singular line

Consider the x-axis in $\mathbb{H}^1 = \mathbb{R}^3$parametrized by $\Gamma(v) := (v, 0, 0)$. Take a non-increasing continuous function $\alpha : \mathbb{R} \to (0, \pi)$. For every $v \in \mathbb{R}$, consider two horizontal halflines

L_v^+, L_v^- extending from $\Gamma(v)$ with angles $\alpha(v)$ and $-\alpha(v)$, respectively. The tangent vectors to these curves at $\Gamma(v)$ are given by $\cos\alpha(v)\,X_{\Gamma(v)}+\sin\alpha(v)\,Y_{\Gamma(v)}$ and $\cos\alpha(v)\,X_{\Gamma(v)}-\sin\alpha(v)\,Y_{\Gamma(v)}$, respectively.

The parametric equations of this surface are given by

$$(v,w)\mapsto\begin{cases}(v+w\cos\alpha(v),w\sin\alpha(v),-vw\sin\alpha(v)), & w\geqslant 0,\\ (v+|w|\cos\alpha(v),-|w|\sin\alpha(v),v|w|\sin\alpha(v)), & w\leqslant 0,\end{cases}\tag{3.1}$$

One can eliminate the parameters v, w to get the implicit equation

$$t+xy-y|y|\,\cot\alpha\left(-\frac{t}{y}\right)=0.$$

Letting $\beta:=\cot(\alpha)$, we get that β is a continuous non-decreasing function, and that the surface Σ_β defined by the parametric equations (3.1) is given by the implicit equation

$$0=f_\beta(x,y,t):=t+xy-y|y|\,\beta\left(-\frac{t}{y}\right).\tag{3.2}$$

Observe that, because of the monotonicity condition on α, the projection of relative interiors of the open horizontal halflines to the xy-plane together with the planar x-axis L_x produce a partition of the plane. Since Σ_β is the union of the horizonal lifting of these planar halflines and the x-axis to $\mathbb{H}^1$, it is the graph of a continuous function $u_\beta:\mathbb{R}^2\to\mathbb{R}$. For $(x,y)\in\mathbb{R}^2$, the only point in the intersection of Σ_β with the vertical line passing through (x,y) is precisely $(x,y,u_\beta(x,y))$. Obviously

$$f_\beta(x,y,u_\beta(x,y))=0.\tag{3.3}$$

For any $(x,y)\in\mathbb{R}^2$, denote by $\xi_\beta(x,y)$ the only value $v\in\mathbb{R}$ so that either $\Gamma(v)=(x,y,0)$, or $(x,y,u_\beta(x,y))$ is contained in one of the two above described halflines leaving $\Gamma(v)$. Trivially $\xi_\beta(x,0)=x$. Using (3.1) one checks that

$$\xi_\beta(x,y)=-\frac{u_\beta(x,y)}{y},\qquad y\neq 0.\tag{3.4}$$

Recalling that $\alpha=\cot^{-1}(\beta)$, we see that the mapping

$$(v,w)\mapsto\begin{cases}(v+w\cos\alpha(v),w\sin\alpha(v)), & w\geqslant 0,\\ (v+|w|\cos\alpha(v),-|w|\sin\alpha(v)), & w\leqslant 0,\end{cases}$$

is an homeomorphism of $\mathbb{R}^2$ whose inverse is given by

$$(x,y)\mapsto(\xi_\beta(x,y),\operatorname{sgn}(y)\,|(x-\xi_\beta(x,y),y)|),$$

where $\operatorname{sgn}(y):=y/|y|$ for $y\neq 0$. Hence $\xi_\beta:\mathbb{R}^2\to\mathbb{R}$ is a continuous function. By (3.4), the function $u_\beta(x,y)/y$ admits a continuous extension to $\mathbb{R}^2$.

Let us analyze first the properties of u_β for regular β

Lemma 3.1. *Let $\beta\in C^k(\mathbb{R})$, $k\geqslant 2$, be a non-decreasing function. Then*

(i) *u_β is a C^k function in $\mathbb{R}^2-L_x$,*
(ii) *u_β is merely $C^{1,1}$ near the x-axis when $\beta\neq 0$,*

(iii) *u_β is C^∞ in $\xi^{-1}(I)$ when $\beta \equiv 0$ on any open set $I \subset \mathbb{R}$, and*
(iv) *Σ_β is area-minimizing.*
(v) *The projection of the singular set of Σ_β to the xy-plane is L_x.*

Proof. The proof of (i) is just an application of the Implicit Function Theorem since f_β is a C^k function for $y \neq 0$ when β is C^k.

To prove (ii) we compute the partial derivatives of u_β for $y \neq 0$. They are given by

$$(u_\beta)_x(x,y) = \frac{-y}{1+|y|\,\beta'\big(\xi_\beta(x,y)\big)}, \tag{3.5}$$

$$(u_\beta)_y(x,y) = \frac{-x+|y|\,\big(2\beta\big(\xi_\beta(x,y)\big)-\beta'\big(\xi_\beta(x,y)\big)\,\xi_\beta(x,y)\big)}{1+|y|\,\beta'\big(\xi_\beta(x,y)\big)}. \tag{3.6}$$

Since $u_\beta(x,0)=0$ for all $x \in \mathbb{R}$ we get $(u_\beta)_x(x,0)=0$. On the other hand

$$(u_\beta)_y(x,0) = \lim_{y\to 0}\frac{u_\beta(x,y)}{y} = -\lim_{y\to 0}\xi_\beta(x,y) = -\xi_\beta(x,0) = -x.$$

The limits, when $y \to 0$, of (3.5) and (3.6) can be computed using (3.4). We conclude that the first derivatives of u_β are continuous functions and so u_β is a C^1 function on $\mathbb{R}^2$. To see that u_β is merely lipschitz, we get from (3.6) and (3.4)

$$\begin{aligned}
(u_\beta)_{yy}(x,0) &= \lim_{y\to 0^\pm}\frac{(u_\beta)_y(x,y)+x}{y}\\
&= \lim_{y\to 0^\pm}\frac{|y|\,\big(2\beta\big(\xi_\beta(x,y)\big)-\beta'\big(\xi_\beta(x,y)\big)\,\xi_\beta(x,y)+x\beta'\big(\xi_\beta(x,y)\big)\big)}{y\,\big(1+|y|\,\beta'\big(\xi_\beta(x,y)\big)\big)}\\
&= \pm 2\beta(x).
\end{aligned}$$

Hence side derivatives exist, but they do not coincide unless $\beta(x)=0$.

As $u_\beta\big|_{\xi^{-1}(I)} = -xy$, (iii) follows easily .

To prove (iv) we use a calibration argument. We shall drop the subscript β to simplify the notation. Let E be the subgraph of u. Let $F \subset \mathbb{H}^1$ such that $F = E$ outside a Euclidean ball B centered at the origin. Let $H^1 := \{(x,y,t) : y \geqslant 0\}$, $H^2 := \{(x,y,t) : y \leqslant 0\}$, $\Pi := \{(x,y,t) : y = 0\}$. Vertical translations of the horizontal unit normal ν_E, defined outside Π, provide two vector fields U^1 on H^1, and U^2 on H^2. They are C^2 in the interior of the halfspaces and extend continuously to the boundary plane Π. As in the proof of Theorem 5.3 in [31], we see that

$$\operatorname{div} U^i = 0, \qquad i = 1,2,$$

in the interior of the halfspaces. Here $\operatorname{div} U$ is the riemannian divergence of the vector field U. Observe that the vector field Y is the riemannian unit normal, and also the horizontal unit normal, to the plane Π. We may apply the divergence theorem to get

$$0 = \int_{E\cap\operatorname{int}(H^i)\cap B} \operatorname{div} U^i = \int_E \langle U^i, \nu_{\operatorname{int}(H^i)\cap B}\rangle \, d|\partial(\operatorname{int}(H^i)\cap B)| + \int_{\operatorname{int}(H^i)\cap B} \langle U^i, \nu_E\rangle \, d|\partial E|.$$

Let $D := \Pi \cap \overline{B}$. Then, for every $p \in D$, we have $\nu_{\operatorname{int}(H^1)\cap B} = -Y$, $\nu_{\operatorname{int}(H^2)\cap B} = Y$, and $U^1 = J(v)$, $U^2 = J(w)$, where $v - w$ is proportional to Y, by the construction of Σ_β. Hence

$$\langle U^1, \nu_{\operatorname{int}(H^1)\cap B}\rangle + \langle U^2, \nu_{\operatorname{int}(H^2)\cap B}\rangle = \langle v - w, J(Y)\rangle = 0, \qquad p \in D.$$

Adding the above integrals we obtain

$$0 = \sum_{i=1,2} \int_E \langle U^i, \nu_B\rangle \, d|\partial B| + \sum_{i=1,2} \int_{B\cap\operatorname{int}(H^i)} \langle U^i, \nu_E\rangle \, d|\partial E|.$$

We apply the same arguments to the set F and, since $E = F$ on ∂B we conclude

$$\sum_{i=1,2} \int_{B\cap\operatorname{int}(H^i)} \langle U^i, \nu_E\rangle \, d|\partial E| = \sum_{i=1,2} \int_{B\cap\operatorname{int}(H^i)} \langle U^i, \nu_F\rangle \, d|\partial F|. \tag{3.7}$$

As E is the subgraph of a lipschitz function, $|\partial E|(\Pi) = 0$ and so

$$|\partial E|(B) = \sum_{i=1,2} \int_{B\cap\operatorname{int}(H^i)} \langle U^i, \nu_B\rangle \, d|\partial E|.$$

Cauchy-Schwarz inequality and the fact that $|\partial F|$ is a positive measure imply

$$\sum_{i=1,2} \int_{B\cap\operatorname{int}(H^i)} \langle U^i, \nu_F\rangle \, d|\partial F| \leqslant |\partial F|(B),$$

which implies (iv).

To prove (v) simply take into account that the projection of the singular set of Σ_β to the xy-plane is composed of those points (x, y) such that $(u_\beta)_x - y = (u_\beta)_y + x = 0$. From (3.5) we get that $(u_\beta)_x - y = 0$ if and only if

$$y\,(2 + |y|\,\beta'(\xi_\beta(x, y))) = 0,$$

i.e, when $y = 0$. In this case, from (3.6), we see that equation $(u_\beta)_y + x = 0$ is trivially satisfied. □

We now prove the general properties of Σ_β from Lemma 3.1

Proposition 3.2. *Let $\beta : \mathbb{R} \to \mathbb{R}$ be a continuous non-decreasing function. Let u_β be the only solution of equation* (3.3)*, Σ_β the graph of u_β, and E_β the subgraph of u_β. Then*

(i) *u_β is locally lipschitz in Euclidean sense,*
(ii) *E_β is a set of locally finite perimeter in $\mathbb{H}^1$, and*
(iii) *Σ_β is area-minimizing in $\mathbb{H}^1$.*

Proof. Let

$$\beta_\varepsilon(x) := \int_{\mathbb{R}} \beta(y)\,\eta_\varepsilon(x-y)\,dy$$

the usual convolution, where η is a Dirac function and $\eta_\varepsilon(x) := \varepsilon^{-1}\eta(x/\varepsilon)$, see [17, § 4.2.1]. Then β_ε is a C^∞ non-decreasing function, and β_ε converges uniformly, on compact subsets of $\mathbb{R}$, to β. Let $u = u_\beta$, $u_\varepsilon = u_{\beta_\varepsilon}$, $f = f_\beta$, $f_\varepsilon = f_{\beta_\varepsilon}$.

Let $D \subset \mathbb{R}^2$ be a bounded subset. To check that u is lipschitz on D it is enough to prove that the first derivatives of u_ε are uniformly bounded on D.

From (3.3) we get

$$\xi(x,y) + |y|\,\beta\big(\xi(x,y)\big) = x, \qquad y \neq 0.$$

For y fixed, define the continuous strictly increasing function

$$\rho_y(x) := x + |y|\,\beta(x).$$

Hence we get

$$\xi(x,y) = \rho_y^{-1}(x). \tag{3.8}$$

We can also define $(\rho_\varepsilon)_y(x) := x + |y|\beta_\varepsilon(x)$. Equation (3.8) holds replacing u, β by u_ε, β_ε.

Since $\rho_y^{-1}(x) = \xi(x,y)$, we conclude that ρ_y^{-1} is a continuous function that depends continuously on y.

Let us estimate

$$|(\rho_\varepsilon)_y^{-1}(x) - \rho_y^{-1}(x)|.$$

Let $z_\varepsilon := (\rho_\varepsilon)_y^{-1}(x)$, $z = \rho_y^{-1}(x)$. Then $x = (\rho_\varepsilon)_y(z_\varepsilon) = \rho_y(z)$ and we have, assuming $z_\varepsilon \geqslant z$.

$$\begin{aligned}
0 &= (\rho_\varepsilon)_y(z_\varepsilon) - \rho_y(z) = z_\varepsilon + |y|\beta_\varepsilon(z_\varepsilon) - \big(z + |y|\beta(z)\big)\\
&= (z_\varepsilon - z) + |y|\,\big(\beta_\varepsilon(z_\varepsilon) - \beta_\varepsilon(z)\big) + |y|\,\big(\beta_\varepsilon(z) - \beta(z)\big)\\
&\geqslant (z_\varepsilon - z) + |y|\,\big(\beta_\varepsilon(z) - \beta(z)\big).
\end{aligned}$$

A similar computation can be performed for $z_\varepsilon \leqslant z$. The consequence is that

$$|z_\varepsilon - z| \leqslant |y|\,|\beta_\varepsilon(z) - \beta(z)|,$$

or, equivalently,

$$|(\rho_\varepsilon)_y^{-1}(x) - \rho_y^{-1}(x)| \leqslant |y|\,|\beta_\varepsilon(\rho_y^{-1}(x)) - \beta(\rho_y^{-1}(x))|.$$

As $\beta_\varepsilon \to \beta$ uniformly on compact subsets of $\mathbb{R}$, we have uniform convergence of $(\rho_\varepsilon)_y^{-1}(x)$ to $\rho_y^{-1}(x)$ on compact subsets of $\mathbb{R}^2$. This also implies the uniform convergence of $\xi_\varepsilon(x,y)$ to $\xi(x,y)$ on compact subsets. Hence also $u_\varepsilon(x,y)$ converges uniformly to $u(x,y)$ on compact subsets of $\mathbb{R}^2$.

From (3.5) and (3.6) we have

$$\begin{aligned}
&|(u_\varepsilon)_x(x,y)| \leqslant |y|,\\
&|(u_\varepsilon)_y(x,y)| \leqslant |x| + 2\,|y|\,\big|\beta_\varepsilon\big(\xi_\varepsilon(x,y)\big)\big| + \big|\xi_\varepsilon(x,y)\big|.
\end{aligned}$$

As $\beta_\varepsilon \to \beta$ and $\xi_\varepsilon(x,y) \to \xi(x,y)$ uniformly on compact subsets, we have that the first derivatives of u_ε are uniformly bounded on compact subsets. Hence u is locally lipschitz.

The subgraph of u_β is a set of locally finite perimeter in $\mathbb{H}^1$ since its boundary is locally lipschitz by 1. This follows from [19] and proves 2.

To prove 3 we use approximation and the calibration argument. Let $F \subset \mathbb{H}^1$ so that $F = E$ outside a Euclidean ball B centered at the origin. For the functions β_ε, consider the vector fields U^i_ε obtained by translating vertically the horizontal unit normal to the surface Σ_ε. We repeat the arguments on the proof of (iv) in Lemma 3.1 to conclude as in (3.7) that

$$\sum_{i=1,2} \int_{B\cap \mathrm{int}(H^i)} \langle U^i_\varepsilon, \nu_E \rangle \, d|\partial E| = \sum_{i=1,2} \int_{B\cap \mathrm{int}(H^i)} \langle U^i_\varepsilon, \nu_F \rangle \, d|\partial F|.$$

Trivially we have

$$\sum_{i=1,2} \int_{B\cap \mathrm{int}(H^i)} \langle U^i_\varepsilon, \nu_F \rangle \, d|\partial F| \leqslant |\partial F|(B).$$

On the other hand, U^i_ε converges uniformly, on compact subsets, to U^i by Lemma 3.3. Passing to the limit when $\varepsilon \to 0$ and taking into account that $U^i = \nu_E$ we conclude

$$|\partial E|(B) \leqslant |\partial F|(B),$$

as desired. □

Lemma 3.3. *Let β be a continuous non-decreasing function. Then the horizontal unit normal of Σ_β is given, in $\{X, Y\}$-coordinates, by*

$$\nu_\beta(x, y, u_\beta(x, y)) = \left(\frac{\mathrm{sgn}(y)}{(1+\beta^2)^{1/2}}, \frac{-\beta}{(1+\beta^2)^{1/2}} \right) (\xi_\beta(x, y)), \qquad y \neq 0. \tag{3.9}$$

Moreover, ν_β admits continuous extensions to $y = 0$ from both sides of this line.

Proof. The characteristic vector field is given, in $\{X, Y\}$-coordinates by

$$Z_\beta(x, y, u_\beta(x, y)) = \left(\frac{\beta}{(1+\beta^2)^{1/2}}, \frac{\mathrm{sgn}(y)}{(1+\beta^2)^{1/2}} \right) (\xi_\beta(x, y)),$$

and the horizontal unit normal by

$$\nu_\beta(x, y, u_\beta(x, y)) = J(-Z_\beta(x, y, u_\beta(x, y))) = \left(\frac{\mathrm{sgn}(y)}{(1+\beta^2)^{1/2}}, \frac{-\beta}{(1+\beta^2)^{1/2}} \right),$$

which yields (3.9). □

Example 3.4. Taking $\beta(x) := x$ we get

$$u_\beta(x, y) = -\frac{xy}{1+|y|},$$

which is a Euclidean $C^{1,1}$ graph.

Another family of interesting examples are the minimal cones obtained by taking the constant function $\beta(x) := \beta_0$. In this case we get

$$u_\beta(x, y) = -xy + \beta_0 \, y|y|.$$

In this case Σ_β is a $C^{1,1}$ surface which is invariant by the dilations centered at any point of the singular line.

Take now

$$\beta(x) := \begin{cases} 0, & x \leqslant 0, \\ x, & x \geqslant 0. \end{cases}$$

In this case we obtain the graph

$$u_\beta(x,y) := \begin{cases} -xy, & x \leqslant 0, \\ -\frac{xy}{1+|y|}, & x \geqslant 0, \end{cases}$$

which is simply locally Lipschitz.

This example was mentioned to me by Scott Pauls. Consider now a continuous nondecreasing function $\beta : \mathbb{R} \to \mathbb{R}$, locally constant outside the Cantor set $C \subset [0,1]$ with $\beta(0) = 0$, $\beta(1) = 1$. Then the associated surface Σ_β is an area-minimizing surface in $\mathbb{H}^1$.

4. EXAMPLES WITH SEVERAL SINGULAR HALFLINES MEETING AT A POINT

Let $\alpha_1^0, \dots, \alpha_k^0$, be a family of positive angles so that

$$\sum_{i=1}^{k} \alpha_i^0 = \pi.$$

Let r_β be the rotation of angle β around the origin in $\mathbb{R}^2$. Consider a family of closed halflines $L_i \subset \mathbb{R}^2$, $i \in \mathbb{Z}_k$, extending from the origin, so that $r_{\alpha_i^0+\alpha_{i+1}^0}(L_i) = L_{i+1}$. Finally, define $R_i := r_{\alpha_i^0}(L_i)$. (An alternative way of defining this configuration is to start from a family of counter-clockwise oriented halflines $R_i \subset \mathbb{R}^2$, $i \in \mathbb{Z}_k$, choosing L_i, $i \in \mathbb{Z}_k$, as the bisector of the angle determined by R_{i-1} and R_i, and defining α_i^0 as the angle between L_i and R_i). Define W_i as the closed wedge, containing L_i, bordered by R_{i-1} and R_i.

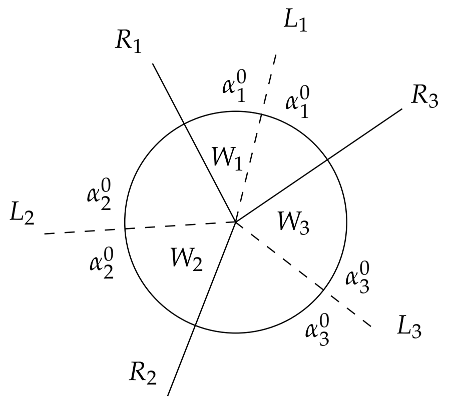

FIGURE 1. The initial configuration with three halflines L_1, L_2, L_3.

For every $i \in \mathbb{Z}_k$, let $\alpha_i : [0,\infty) \to (0,\pi)$ be a continuous nonincreasing function so that $\alpha_i(0) = \alpha_i^0$, and define, as in the previous section, $\beta_i := \cot(\alpha_i)$. Let $v_i \in \mathbb{S}^1$, $i \in \mathbb{Z}_k$, be such that $L_i = \{sv_i : s \geqslant 0\}$. For every $i \in \mathbb{Z}_k$ and $s \geqslant 0$, we take the two closed halflines

$L^{\pm}_{s,i}$ in $\mathbb{R}^2$ extending from the point sv_i with tangent vectors $(\cos\alpha_i(s), \pm\sin\alpha_i(s))$. In this way we cover all of $\mathbb{R}^2$. We shall define $\alpha := (\alpha_1, \ldots, \alpha_k)$.

Lift $L_1, \ldots, L_k$ to horizontal halflines $L'_1, \ldots, L'_k$ in $\mathbb{H}^1$ from the origin, and $L^{\pm}_{s,i}$ to horizontal halflines in $\mathbb{H}^1$ extending from the unique point in L'_i projecting onto sv_i. In this way we obtain a continuous function $u_\alpha : \mathbb{R}^2 \to \mathbb{R}$. The graph Σ_α of u_α is a topological surface in $\mathbb{H}^1$.

Obviously the angle functions $\alpha_i(s)$ can be extended continuously and preserving the monotonicity, to an angle function $\widetilde{\alpha}_i : \widetilde{L}_i \to (0, \pi)$, where $\widetilde{L}_i$ is the straight line containing the halfline L_i. The graph of u_α restricted to W_i coincides with the Euclidean locally lipschitz area-minimizing surface $u_{\widetilde{\beta}_i}$, for $\widetilde{\beta}_i := \cot\widetilde{\alpha}_i$, constructed in the previous section. So the examples in this section can be seen as pieces of the examples of the previous one patched together.

Theorem 4.1. *Under the above conditions*

(i) *The function u_α is locally lipschitz in the Euclidean sense.*
(ii) *The surface Σ_α is area-minimizing.*

Proof. It is immediate that u_α is a graph which is locally lipschitz in Euclidean sense: choose a disk $D \subset \mathbb{R}^2$. Let p, $q \in D$. Assume first that (p, q) intersects the halflines $R_1, \ldots, R_k$ transversally at the points $x_1, \ldots, x_n$. Then $[p, x_1]$, $[x_1, x_2], \ldots, [x_n, p]$ are contained in wedges and hence

$$\begin{aligned} |u_\alpha(p) - u_\alpha(q)| &\leqslant |u_\alpha(p) - u_\alpha(x_1)| + \cdots + |u_\alpha(x_n) - u_\alpha(q)| \\ &\leqslant C\left(|p - x_1| + \cdots |x_n - q|\right) = C\,|p - q|, \end{aligned}$$

where C is the supremum of the Lipschitz constants of $u_{\widetilde{\beta}_i}$ restricted to D. The general case is then obtained by approximating p and q by points in the condition of the assumption.

To prove that u_α is area minimizing we first approximate α_i by smooth angle functions $(\alpha_i)_\varepsilon$ with $(\alpha_i)_\varepsilon(0) = \alpha_i(0)$. In this way we obtain a calibrating vector field which is continuous along the vertical planes passing through R_i by Lemma 3.3. This allows us to apply the calibration argument to prove the area-minimizing property of Σ_α. $\square$

Example 4.2 (Minimizing cones)**.** Let $\alpha_i(s) = \alpha_i^0$ be a constant for all i. Then the subgraph of Σ_α is a minimizing cone with center at 0. Restricted to the interior of the wedges W_i, the surface Σ_α is $C^{1,1}$. An easy computation shows that, taking $\beta(s) := \beta_0$ in the construction of the first section, a Riemannian unit normal to Σ_β along the halflines $\beta_0|y| = x$, $x \geqslant 0$ (that make angle $\pm\cot^{-1}(\beta_0)$ with the positive x-axis) is given by

$$N = \frac{-2y\,X + 2\beta_0|y|\,Y - T}{\sqrt{1 + 4y^2 + 4\beta_0^2 y^2}} = \frac{-2y\,X + 2x\,Y - T}{\sqrt{1 + 4x^2 + 4y^2}}.$$

This vector field is invariant by rotations around the vertical axis. Hence in our construction, the normal vector field to Σ_α is continuous. It is straightforward to show that it is locally lipschitz in Euclidean sense.

Example 4.3 (Area-minimizing surfaces with a singular halfline)**.** These examples are inspired by [11, Example 7.2]. We consider a halfline L extending from the origin, and an angle function $\alpha : L \to (0, \pi)$ continuous and nonincreasing as a function of the distance to the origin. We consider the union of the halflines $L^+_{\alpha(q)}$, $L^-_{\alpha(q)}$ extending from $q \in L$ with angles $\alpha(q)$, $-\alpha(q)$, respectively. We patch the area-minimizing surface defined by α in the wedge delimited by the halflines $L^+_{\alpha(0)}$, $L^-_{\alpha(0)}$, with the plane $t = 0$. In this way we get an entire area-minimizing t-graph, with lipschitz regularity. In case the angle function α is constant, we get an area-minimizing cone with center 0, which is defined by the equation

$$u(x,y) := \begin{cases} -xy + \beta_0\, y|y|, & -xy + \beta_0\, y|y| \geqslant 0, \\ 0, & -xy + \beta_0\, y|y| \leqslant 0. \end{cases}$$

This surface is composed of two smooth pieces patched together along the halflines $x = \beta_0|y|$.

5. A final remark

Remark 5.1 (Examples with discontinuous monotone angle functions)**.** These examples were suggested by the referee. Consider $\alpha : \mathbb{R} \to (0, \pi)$ a nonincreasing discontinuous angle function. Let $D \subset \mathbb{R}$ the set of discontinuity points of α. We construct a t-graph by foliating with horizontal halflines in the following way: if $q \notin D$ then consider two horizontal halflines of initial velocity $\cos\alpha(q)\, X \pm \sin\alpha(q)\, Y$ starting from $(q, 0, 0)$. If $q \in D$ consider all halflines extending with initial velocity $\cos\omega\, X \pm \sin\omega\, Y$, for any ω such that

$$\lim_{p \to q^-} \alpha(p) \leqslant \omega \leqslant \lim_{p \to q^+} \alpha(p).$$

Computations similar to the ones of the first section show that the associated graph turns out to be Euclidean locally lipschitz, and area-minimizing by an approximation argument.

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