GEOMETRIC MEASURE THEORY AND THE PROOF OF THE DOUBLE BUBBLE CONJECTURE

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Abstract. These notes by Ritoré are based on a nine-hour course given by Morgan in June-July 2001 during the Clay Mathematics Institute Summer School at MSRI. They follow closely the text [27] and can be read as an introduction to it. These notes provide a very brief overview of geometric measure theory, with a few sketches of proofs and references to the appropiate sources.

1. Introduction

Geometric measure theory was developed in the second half of the 20th century to treat existence and regularity questions in the calculus of variations.

The lack of compactness properties of the spaces of submanifolds made it necessary to consider "generalized" submanifolds (currents, varifolds, finite perimeter sets ...), with appropriate topologies, to apply the direct method of the calculus of variations. To find a minimum of some functional, take a sequence approaching the infimum. If this sequence is compact in some topology then extract a convergent subsequence to a generalized submanifold realizing the minimum. Then prove a regularity result to recover a classical solution. In some sense, this theory is similar to Sobolev space theory, in which one enlarges the space of smooth functions to find a solution to a differential equation, and then proves regularity to recover a classical solution.

These notes provide a brief overview of geometric measure theory. A few sketches of proofs will be given, with references to appropriate sources. Recommended texts include Morgan [27], Simon [38], Giusti [15], Massari-Miranda [23], Ziemer [43] on finite perimeter sets and functions of bounded variation, and of course Federer's treatise [9] on *Geometric Measure Theory*.

2. Rectifiable sets

2.1. **Lebesgue and Hausdorff measures** [27, Ch. 2]**.** Lebesgue measure in $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n$. An important outer measure in $\mathbb{R}^n$, defined for any nonnegative integer m, is Hausdorff measure $\mathcal{H}^m$. Let $\alpha_m = \mathcal{L}^m(\mathbf{B}(0,1))$, and $A \subset \mathbb{R}^n$. Fix $\delta > 0$ and consider

$$\mathcal{H}^m_\delta(A) = \inf\left\{ \sum_j \alpha_m \, r_j^m : A \subset \bigcup_j S_j,\ \operatorname{diam} S_j = 2r_j \leqslant \delta \right\}.$$

Then define the m-dimensional Hausdorff measure of A as

$$\mathcal{H}^m(A) = \lim_{\delta \to 0} \mathcal{H}^m_\delta(A).$$

Note that this limit always exists, although it could be ∞. We recall that Hausdorff measure is just an outer measure, but restricted to the σ-algebra of measurable sets is a Borel-regular measure by Caratheodory's criterion. In $\mathbb{R}^n$, we have $\mathcal{L}^n = \mathcal{H}^n = \mathcal{H}^n_\delta$, for any $\delta > 0$. One may extend the definition of Hausdorff measure to any real dimension, by replacing α_m by the corresponding expression in terms of gamma functions ($\alpha_m = \pi^{m/2}/\Gamma(m/2+1)$). $\mathcal{H}^0$ is just counting measure.

If $m < p$ then $\mathcal{H}^m(A) < \infty$ implies $\mathcal{H}^p(A) = 0$. If $p < m$ then $\mathcal{H}^m(A) > 0$ implies $\mathcal{H}^p(A) = \infty$. The Hausdorff dimension of $A \subset \mathbb{R}^n$ is defined as the real number

$$\begin{aligned}\inf\{p : \mathcal{H}^p(A) < \infty\} &= \inf\{p : \mathcal{H}^p(A) = 0\} \\ &= \sup\{p : \mathcal{H}^p(A) > 0\} = \sup\{p : \mathcal{H}^p(A) = \infty\}.\end{aligned}$$

The m-dimensional Hausdorff measure of an m-dimensional submanifold of $\mathbb{R}^n$ coincides with the Riemannian volume of the submanifold.

2.2. **Densities** [27, 2.5]**.** If $A \subset \mathbb{R}^n$, one can define the m-dimensional density of A at a by

$$\Theta(A, a) = \lim_{r \to 0} \frac{\mathcal{H}^m(A \cap \mathbf{B}^n(a, r))}{\alpha_m r^m},$$

when this limit exists. In the formula, $\mathbf{B}^n(a, r)$ is the closed ball of center a and radius $r > 0$ in $\mathbb{R}^n$. Similarly, one can define the density of A with respect to a measure μ in $\mathbb{R}^n$ by replacing $\mathcal{H}^m$ by μ in the previous formula.

The m-dimensional density of any smooth m-dimensional submanifold of $\mathbb{R}^n$ equals 1 at any of its points. If A is the boundary of a cube in $\mathbb{R}^3$ then the 2-dimensional density is 1 at every point except at the vertices, where the density is 3/4.

2.3. **Lipschitz functions** [27, Ch. 3]**.** A Lipschitz function f between metric spaces is one which satisfies the inequality $d(f(x), f(y)) \leqslant C\, d(x, y)$ for some constant $C > 0$. The smallest such C is called the Lipschitz constant of f. We can say that Lipschitz functions are those with controlled slope. A Lipschitz function between open sets of Euclidean spaces is differentiable almost everywhere by Rademacher's Theorem [27, 3.2]. Hence the Jacobian of a Lipschitz function can be integrated at least locally and we have the following important formulae.

Theorem 2.1 (Area Formula)**.** *Consider a Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^n$ for $m \leqslant n$.*

(i) *If $A \subset \mathbb{R}^m$ is an $\mathcal{L}^m$-measurable set, then*

$$\int_A J(f)\, d\mathcal{H}^m = \int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^m(y).$$

(ii) *If $u : A \to \mathbb{R}$ is an $\mathcal{L}^m$-integrable function, then*

$$\int_A u\, J(f)\, d\mathcal{H}^m = \int_{\mathbb{R}^n} \sum_{x \in f^{-1}(y)} u(x)\, d\mathcal{H}^m(y).$$

We have denoted the Jacobian of f by $J(f)$. This formula shows that Hausdorff measure is compatible with classical mapping area.

Theorem 2.2 (Coarea Formula for Lipschitz maps)**.** *Consider a Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^n$ with $m > n$. If $A \subset \mathbb{R}^m$ is an $\mathcal{L}^m$-measurable set, then*

$$\int_A J(f)\, d\mathcal{H}^m = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(A \cap f^{-1}(y))\, d\mathcal{H}^n(y).$$

The coarea formula is very useful in geometric analysis since it allows slicing to recover important geometric quantities. As we shall show, it can be generalized to maps between rectifiable sets.

2.4. **Tangent cones** [27, 3.9]**.** For a set $A \subset \mathbb{R}^n$, one may consider the tangent cone of A at a, $\mathrm{Tan}(A, a)$, as the cone over the subset S of the unit sphere given by

$$S = \bigcap_{\varepsilon > 0} \mathrm{Cl}\left\{ \frac{x-a}{|x-a|} : x \in A, 0 < |x-a| < \varepsilon \right\}.$$

For a C^1 m-dimensional submanifold the tangent cone is precisely the tangent m-plane. We define a smaller approximate tangent cone by

$$\mathrm{Tan}^m(A, a) = \bigcap \{\mathrm{Tan}(S, a) : \Theta^m(A - S, a) = 0\},$$

which omits parts of $\mathrm{Tan}(A, a)$ with dimension less than m.

2.5. **Rectifiable sets** [27, 3.10]**,** [38, Ch. 3]**.** A set $A \subset \mathbb{R}^n$ is called $(\mathcal{H}^m, m)$-rectifiable set if $\mathcal{H}^m(A)$ is finite and $\mathcal{H}^m$ almost all of A is the union of the images of countably many Lipschitz maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. We shall often refer to such sets as m-rectifiable sets.

In some sense, a rectifiable set behaves like a C^1 submanifold of $\mathbb{R}^n$ since it has an approximate tangent plane almost everywhere.

Proposition 2.3. *If A is an m-rectifiable set in $\mathbb{R}^n$ then for $\mathcal{H}^m$-almost all points $a \in A$, we have $\Theta(A, a) = 1$ and $\mathrm{Tan}^m(A, a)$ is an m-dimensional plane.*

On rectifiable sets, one can define the notion of the gradient of a function, the divergence of a vector field, and the jacobian of a map. Hence there is a generalization of the coarea formula to such sets.

Theorem 2.4 (General coarea formula)**.** *Consider an m-dimensional rectifiable set $A \subset \mathbb{R}^n$, an μ-dimensional rectifiable set $B \subset \mathbb{R}^\nu$, $m \geqslant n \geqslant 1$, and $f : A \to B$ a Lipschitz function. Then*

$$\int_A J(f)\, d\mathcal{H}^m = \int_B \mathcal{H}^{m-\mu}(f^{-1}\{z\})\, d\mathcal{H}^\mu(z).$$

More generally, for any $\mathcal{H}^m$-integrable function g on A

$$\int_A g\, J(f)\, d\mathcal{H}^m = \int_z \int_{f^{-1}\{z\}} g\, d\mathcal{H}^{m-\mu}\, d\mathcal{H}^\mu(z).$$

In the above result, $J(f)$ is the Jacobian of the Lipschitz map f, which is defined almost everywhere. An important consequence of coarea formula is Crofton formula

Corollary 2.5 (Crofton's formula)**.** *If $A \subset \mathbb{R}^n$ is an m-rectifiable set, then*

$$\mathcal{H}^m(A) = \frac{1}{\beta(n, m)} \int_{P \in \mathcal{P}} \#\{P \cap A\},$$

where $\beta(n,m)$ is a constant depending only on n and m, and $\mathcal{P}$ is the set of $(n-m)$ planes.

Finally we state the celebrated Federer Structure Theorem, which allows the decomposition of any set in $\mathbb{R}^n$ as the union of a rectifiable set and a set with negligable projection on almost all m-dimensional subspaces.

Theorem 2.6 (Federer's Structure Theorem)**.** *Let $A \subset \mathbb{R}^n$ be an arbitrary subset with $\mathcal{H}^m(A) < \infty$. Then A can be decomposed as the union of two disjoint sets $A = B \cup C$, where B is $(\mathcal{H}^m, m)$-rectifiable and $\mathcal{J}^m(C) = 0$.*

In this theorem $\mathcal{J}^m$ is the m-dimensional integralgeometric measure. Equality $\mathcal{J}^m(C) = 0$ implies that the m-dimensional measure of the orthogonal projection of C onto a m-dimensional linear subspace is zero for almost all subspaces.

One can define an orientation on a rectifiable set, just by choosing a measurable orientation in the set of tangent planes. Recall that there is a tangent plane for almost all the points of the set. Of course one can define an uncountably number of such orientations.

3. Currents

3.1. **Definition and properties** [27, Ch. 4]**.** The concept of current was introduced by Federer and Fleming [11] in 1960. Consider the space $\mathcal{D}^m$ of smooth differential m-forms with compact support in $\mathbb{R}^n$, endowed with the topology of C^∞ convergence on compact subsets. The space of continuous linear maps $T : \mathcal{D}^m \to \mathbb{R}$ will be denoted by $\mathcal{D}_m$. A *current* is an element of $\mathcal{D}_m$.

An oriented m-rectifiable set $A \subset \mathbb{R}^n$ defines a current $T_A \in \mathcal{D}_m$ in the following way. For any $a \in A$ for which there is a tangent plane, let $\vec{S}(a)$ denote the m-vector associated to the oriented tangent plane, and let μ be an integer-valued function in $L^1(A)$. Then

$$T_A(\varphi) = \int_A \mu \left\langle \vec{S}, \varphi \right\rangle d\mathcal{H}^m.$$

The function $\mu(a)$ measures the multiplicity of the rectifiable set in the point a. A current induced by a *compact* oriented m-rectifible set will be called a *rectifiable current*.

The *support* of a current T is the smallest closed set C such that, for any $\varphi \in \mathcal{D}^m$ such that $\operatorname{spt}(\varphi) \cap C = \emptyset$ one has $T(\varphi) = 0$.

One can define the *boundary* of a current in the following way: if $T \in \mathcal{D}_m$ and $\varphi \in \mathcal{D}^{m-1}$, then

$$\partial T(\varphi) = T(d\varphi).$$

Property $d^2 = 0$ of the exterior derivative of forms implies that ∂T has no boundary. The boundary of a rectifiable current T need not be rectifiable. If this is the case, then we say that T is an *integral current.*

The *mass* of a current T is

$$\mathbf{M}(T) = \sup\{T(\varphi) : |\varphi| \leqslant 1\},$$

where $|\varphi|$ is the supremum norm of the differential form φ. If $T \in \mathcal{D}_m$ is a rectifiable current associated to a rectifiable set A with multiplicity $\mu \equiv 1$ then $\mathbf{M}(T) = \mathcal{H}^m(A)$.

A natural topology in the space of currents is the weak topology. We say that $T_i \to T$ weakly in $\mathcal{D}_m$ if and only if $T_i(\varphi) \to T(\varphi)$ for any differential form $\varphi \in \mathcal{D}^m$. If T_i converges weakly to T then one obtains easily

$$\mathbf{M}(T) \leqslant \liminf_i \mathbf{M}(T_i).$$

Another seminorm in the space of curremts is the *flat norm*. It is defined as

$$\mathcal{F}(T) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) : T = A + \partial B, A \ m\text{-rect.}, B \ (m+1)\text{-rect.}\},$$

with $T \in \mathcal{D}_m$. If one has two m-currents T_1 and T_2, then saying that they are close in the flat norm is equivalent to say that there is an $(m+1)$-current of small mass such that its boundary is T_1 plus T_2 plus another m-current of small mass.

The flat norm is weaker in general than the mass norm, and stronger than the weak topology.

Another classical definition is that of a *normal current*, which is a current T with compact support and $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$.

3.2. **The deformation theorem and consequences** [27, Ch. 5]**.** One of the most interesting results concerning integral currents is the deformation theorem, which deforms a given integral current $T \in \mathcal{D}_m$ to a current contained in a given m-dimensional grid of prescribed mesh.

Theorem 3.1 (Deformation Theorem [27, 5.1])**.** *Let $T \in \mathcal{D}_m$ be an integral current in $\mathbb{R}^n$. Then there are integral currents $Q \in \mathcal{D}_m$, $S \in \mathcal{D}_{m+1}$, and a current $P \in \mathcal{D}_m$ such that*

$$T = P + Q + \partial S,$$

so that $\mathbf{M}(P) \leqslant \gamma\, \mathbf{M}(T)$, $\mathbf{M}(Q) \leqslant \varepsilon\, \mathbf{M}(\partial T)$ and $\mathbf{M}(S) \leqslant \varepsilon\gamma\, \mathbf{M}(T)$, where $\gamma = 2n^{2m+2}$, and hence $\mathcal{F}(T - P) \leqslant \varepsilon\gamma\, (\mathbf{M}(T) + \mathbf{M}(\partial T))$.

Moreover $\operatorname{spt}(T)$ *is contained in a given m-dimensional 2ε grid, and* $\operatorname{spt}(\partial T)$ *is contained in the $(m-1)$-dimensional 2ε grid.*

An important first consequence of the deformation theorem is

Theorem 3.2 (Isoperimetric Inequality [27, 5.3])**.** *An m-dimensional cycle T $(\partial T = 0)$ in $\mathbb{R}^n$ bounds an $(m+1)$-dimensional integral current S with*

$$\mathbf{M}(S)^{m/(m+1)} \leqslant \gamma\, \mathbf{M}(T),$$

with $\gamma = 2n^{2m+2}$.

Almgren [5] improved this theorem by showing that the best constant γ one can place in the above inequality corresponds to the classical isoperimetric inequality in $\mathbb{R}^{m+1}$, assuming that S is area-minimizing.

Another consequence of the deformation theorem is

Theorem 3.3. *The set of integral currents T in $\mathbf{B}(0, R)$ such that $\mathbf{M}(T) \leqslant C$, $\mathbf{M}(\partial T) \leqslant C$ is totally bounded in the flat norm topology.*

A consequence of this result is the coincidence of the mass norm and the flat norm in the subset of integral currents T in $\mathbf{B}(0, R)$ such that $\mathbf{M}(T) \leqslant C$, $\mathbf{M}(\partial T) \leqslant C$, for any constant $C > 0$.

Tha previous theorem is the first ingredient in the proof of the

Theorem 3.4 (Compactness Theorem)**.** *Given $R > 0$, $C > 0$, the set of integral currents with support in $\mathbf{B}(0, R)$ such that $\mathbf{M}(T) \leqslant C$, $\mathbf{M}(\partial T) \leqslant C$ is compact in the flat norm topology.*

Now we are in position of applying these results to prove existence of area-minimizing surfaces for the Plateau problem.

Corollary 3.5. *Let B an $(m-1)$-rectifiable current in $\mathbb{R}^n$ with $\partial B = 0$. Then there is an m-dimensional area-minimizing rectifiable current S with $\partial S = B$.*

Proof. The proof is given in several steps

(i) First show that B bounds some surface by taking a cone over B.
(ii) Take a sequence of rectifiable currents S_i with $\partial S_i = B$ approaching the infimum of the area.
(iii) Project the S_i into a ball containing B. Since the projection to the boundary of the ball is distance nonincreasing, the mass is reduced.
(iv) By the Compactness Theorem, get a limit S.
(v) By lower semicontinuity $\mathbf{M}(S) \leqslant \liminf_i \mathbf{M}(S_i)$, which shows that S is minimizing.

The limit we have obtained in this process is an integral current. □

The existence of this limit is not obvious since it is not clear that there is an area-minimizing surface among the infinitely many candidates.

An example of an area-minimizing surface is a minimal graph over a convex region. This follows from projection over the right cylinder over the convex region and a calibration argument. To get the proof we consider slices of an integral current by hypersurfaces, and the Gauss-Green Theorem; see [27, Ch. 6].

3.3. **Regularity Theorems** [27, Ch. 8]**.** We are interested now in seeing how regular is the solution to the Plateau problem we have obtained in the previous subsection.

In this direction, the first regularity result was proved by Fleming in 1962, who showed [12] that an area-minimizing rectifiable current in $\mathbb{R}^3$ is an embedded surface in the interior. This regularity theorem was generalized to three-dimensional surfaces in $\mathbb{R}^4$ by Almgren [2] in 1966, and to m-dimensional surfaces in $\mathbb{R}^{m+1}$, $m \leqslant 6$, by Simons [39] in 1968. In 1969, Bombieri, De Giorgi and Giusti [6] gave an example of a seven-dimensional, area-minimizing rectifiable current in $\mathbb{R}^8$ which is not smooth at the origin (the cone over $\mathbb{S}^3(1/\sqrt{2}) \times \mathbb{S}^3(1/\sqrt{2})$ in $\mathbb{R}^8$). The complete regularity result was given by Federer.

Theorem 3.6. [10] *An $(n-1)$-dimensional, area-minimizing rectifiable current in $\mathbb{R}^n$ is a smooth, embedded manifold on the interior except for a singular set of Hausdorff dimension at most $n - 8$.*

Similarly in a smooth Riemannian manifold M^n, for $n \leqslant 7$, an area-minimizing rectifiable current is a smooth embedded manifold and, for $n \geqslant 8$, the dimension of the singular set is less than or equal to $n - 8$. The same result is true with a volume constraint (almost minimizing is enough). This was proved by Gonzalez, Massari and Tamanini [16] for finite perimeter sets. We state the equivalent result for currents.

Theorem 3.7. [16] *Let T be an $(n-1)$-dimensional rectifiable current of least area in the unit ball $B \subset \mathbb{R}^n$, with prescribed boundary in ∂B, bounding an oriented region of prescribed volume.*

Then T is a smooth hypersurface of constant mean curvature in its interior, except for a singular set of dimension at most $n-8$.

The result also holds in Riemannian manifolds (see [28]).

Regularity in higher codimension is much weaker. It was already proved by Federer [9, 5.3.16] that the set of regular points is dense in the interior. On the other hand, it was known that complex analytic varieties are area-minimizing, with singular sets of codimension 2. The most conclusive result was proved by Almgren.

Theorem 3.8. [4] *An m-dimensional, area-minimizing rectifiable current in $\mathbb{R}^n$ is a smooth, embedded manifold on the interior except for a singular set of Hausdorff dimension at most $m-2$.*

In 1988, Chang [7] showed that there are only isolated singularities for 2-dimensional currents in $\mathbb{R}^n$.

The only known examples in $\mathbb{R}^4$ are complex curves. It is known that the tangent cone must consist on complex planes (with multiplicities). The only known examples of 3-dimensional surfaces in Euclidean 5-space are the 2-dimensional examples in 4-space cross a line.

Why are regularity results stronger in codimension 1? First of all one can reduce to multiplicity 1, since a nesting lemma yields decomposition into nested, multiplicity 1 area-minimizing surfaces. One then proves that each one is regular. If they touch, they must coincide by the maximum principle. All the ingredients also hold in Riemannian manifolds.

There are several open questions concerning singularities. A first one is about the structure of singularities. A second one is if area-minimizing surfaces are stratified manifolds, or could they have fractional dimensional singular sets? Also, what are the possible tangent cones?

Concerning boundary regularity, the definitive result is the one by Hardt and Simon.

Theorem 3.9. [17] *Let T be an $(n-1)$-dimensional, area-minimizing rectifiable current in $\mathbb{R}^n$, bounded by a C^2 oriented submanifold with multiplicity 1. Then at every boundary point, $\operatorname{spt} T$ is a C^1 embedded manifold with boundary.*

In a smooth Riemannian manifold M^3 with nonempty boundary ∂M, area-minimizing surfaces S in homology classes of M (mod. ∂M) are also regular along the boundary.

The main tool of regularity theory is the monotonicity in r of (mass inside r-ball)/r^m [27, Ch. 9]. More precisely, for an m-dimensional rectifiable current T, we define the mass ratio $M(r)$ as $\mathbf{M}(T \llcorner \mathbf{B}(a,r))/\alpha_m r^m$, which is an increasing function. It is inmediate that the density $\Theta(T,a)$ equals $\lim_{r\to 0} M(r)$. Then we have

Theorem 3.10. *Let T be an area-minimizing locally rectifiable current, and let $a \in \operatorname{spt}(T)$. Then, for $0 < r < \operatorname{dist}(a, \operatorname{spt}(T))$, the mass ratio $M(r)$ is an incresing function of r.*

Proof. As $f(r) = \mathbf{M}(T \llcorner \mathbf{B}(a,r))$ is increasing, it is differentiable almost everywhere. By slicing we have $\mathbf{M}(\partial(T \llcorner \mathbf{B}(a,r))) \leqslant f'(r)$. If C_r is the cone over $\partial(T \llcorner \mathbf{B}(a,r))$ with vertex a, we get $\mathbf{M}(C_r) = (r/m)\,\mathbf{M}(\partial(T \llcorner \mathbf{B}(a,r)))$ by the coarea formula. Hence

$$f(r) \leqslant \mathbf{M}(C_r) \leqslant \frac{r}{m}\, f'(r),$$

and

$$\alpha_m\, M'(r) = \left(\alpha_m\, \frac{f(r)}{r^m}\right)' = \frac{m\alpha_m}{r^{m+1}} \left[\frac{r}{m}\, f'(r) - f(r)\right] \geqslant 0,$$

and one easily concludes that $M(r)$ is increasing. □

It is not difficult to show that equality holds if and only if T is a cone. A first corollary of the above result is that

$$\mathbf{M}(T \llcorner \mathbf{B}(a,r)) \geqslant \Theta(T,a)\, \alpha_m r^m.$$

As $\Theta(T,a) = 1$ for almost all points of a rectifiable current T, we easily conclude that $\Theta(T,a) \geqslant 1$ everywhere on $\operatorname{spt} T - \operatorname{spt} \partial T$. In particular, if T is a 2-dimensional, area-minimizing surface in $\mathbb{R}^3$, then $\mathbf{M}(T,a) \geqslant \pi r^2$.

Another interesting consequence of monotonicity is the existence of a tangent cone (limit of homothetic expansions) at any point of an area-minimimizing rectifiable current [27, 9.8].

When proving the regularity of an area-minimizing surface there are two steps: the first one is to show that the tangent cone is a plane. The second one is to conclude that the existence of the tangent plane implies that the surface is regular.

Allard [1] generalized monotonicity to stationary surfaces (mean curvature 0) and proved that for bounded mean curvature surfaces in a Riemannian manifold

$$M(r)\, e^{Cr}\, r^{-m}$$

is monotone, where $M(r)$ is the mass inside a ball of radius r. This result was proved in the very general context of varifolds. It is not valid for surfaces stationary for general integrands Φ, although for Φ-minimizers we still have

$$M(r) \geqslant C\, r^m.$$

4. Other types of surfaces

The Belgian physicist Plateau [30] (see also [41]) observed around 1870 two kind of singularities in soap films: either three sheets meet along a seam at 120^o, or four such seams meet at a point at about 109^o. The latter phenomenon corresponds to the triangles obtained from any given edge of a regular tetrahedron and from the segments which join the barycenter with the vertexes of the edge. This behaviour of soap films cannot be modelled by currents since the singular set is very large and contradicts the regularity results of the previous section. Hence other definitions of generalized surfaces are necessary to treat this problem.

For instance, to treat nonorientable surfaces one can study rectifiable currents modulo 2. Two rectifiable currents T_1 and T_2 are congruent modulo 2 if there is some rectifiable current Q such that $T_1 - T_2 = 2Q$. In general one can consider rectifiable currents modulo any integer number ν. However these new objects are not appropiate to model the tetrahedral soap film (see the discussion in [27, pp. 107 ff.]).

We discuss now several different types of generalized surfaces.

4.1. **Varifolds** [27, 11.2]**.** A varifold is a measure concentrated on a set in space and certain "tangent planes". The notion of varifold allows multiplicity, but neither cancellation nor obvious definition of orientation. Also tangent planes need not be associated with the set. More precisely, an m-dimensional *varifold* is a Radon measure on $\mathbb{R}^n \times G_m\mathbb{R}^n$, where $G_m\mathbb{R}^n$ is the Grassmannian of unoriented unit m-planes through the origin in $\mathbb{R}^n$.

An *integral varifold* is one associated to a rectifiable set. Let $S \subset \mathbb{R}^n$ be a rectifiable set, say with multiplicity 1, and m-tangent plane $\vec{S}$. Define

$$\mathbf{v}(S) = \mathcal{H}^m \llcorner \{(x, \vec{S}(x)) : x \in S\}.$$

Instead of the notion of boundary we get the first variation of a varifold. $\delta V(\vec{v})$ is the initial rate of change of area in $\mathbb{R}^n$ under a smooth variation with initial velocity $\vec{v}$. A *stationary varifold* is one for which $\delta V = 0$ for all $\vec{v}$. Stationary integral varifolds include area-minimizing rectifiable currents (modulo ν also), and soap films. For integral varifolds there are a compactness theorem and isoperimetric inequalities. Also the following regularity result holds

Theorem 4.1 (Allard's regularity [1])**.** *An integral k-dimensional varifold with mean curvature bounded or in L^p, $p > k$, is a $C^{1,\alpha}$ submanifold at points of density* 1*.*

An important open question for varifolds is whether a two-dimensional stationary integral varifold in an open subset of $\mathbb{R}^3$ is a smooth embedded manifold almost everywhere.

4.2. **Finite perimeter sets and functions of bounded variation.** The notion of a set of finite perimeter was introduced by Cacciopoli and De Giorgi [8] and the theory was developed independently for some time. Later it became clear that a finite perimeter set in $\mathbb{R}^n$ is $(\mathcal{H}^n, n)$-rectifiable [15, Ch. 4], [38, §14].

We shall say that $A \subset \mathbb{R}^n$ is a *finite perimeter set* if, for any vector field X with compact support in $\mathbb{R}^n$, we have

$$\int_A \operatorname{div} X \, d\mathcal{H}^n \leqslant c(A)|X|,$$

where $|X|$ is the sup norm of the vector field X, and $c(A) > 0$ is a constant depending on the set A. It can be shown that this property is equivalent to saying that the characteristic function χ_A of the set A is of *bounded variation.* The perimeter $\mathcal{P}(A)$ of the set A is defined as

$$\mathcal{P}(A) = \sup_{|X| \leqslant 1} \int_A \operatorname{div} X \, d\mathcal{H}^n.$$

This definition coincides with $\mathcal{H}^{n-1}(\partial A)$ if ∂A is a smooth hypersurface.

From the definition and Riesz representation theorem, one can get the existence of an $\mathcal{H}^{n-1}$-measurable vector valued function $\nu : A \to \mathbb{R}^n$ and a Borel measure μ

on $\mathbb{R}^n$ such that

$$\int_A \operatorname{div} X \, d\mathcal{H}^n = \int_{\mathbb{R}^n} \langle \nu, X \rangle \, d\mu.$$

Both the measure μ and the function ν has support contained in the topological boundary of A.

For finite perimeter sets the natural convergence is in measure. We shall say that $A_i \to A$ if $\chi_{A_i} \to \chi_A$ in L^1. With this notion of convergence the perimeter is lower semicontinuous, i.e., if $A_i \to A$ then

$$\mathcal{P}(A) \leqslant \liminf_i \mathcal{P}(A_i).$$

Finite perimeter sets also admit slicing and isoperimetric inequalities. The interested reader should consult Giusti [15], Massari-Miranda [23], Simon [38], and Ziemer [43].

4.3. $(\mathbf{M}, \varepsilon, \delta)$**-minimal sets** [27, 11.3]**.** Almgren modeled soap films as $(\mathbf{M}, 0, \delta)$-minimal sets, which satisfy two conditions: they touch the boundary, and tiny portions (of diameter $\leqslant \delta$) cannot be pinched or deformed to make the area go down. To be more precise, $S \subset \mathbb{R}^n - B$ bounded, with $\mathcal{H}^m(S) < \infty$, is $(\mathbf{M}, 0, \delta)$-minimal with respect to a closed set B (the boundary) if, for every Lipschitz deformation φ of $\mathbb{R}^n$ which differs from the identity map only in a δ-ball disjoint from B,

$$\mathcal{H}^m(S) \leqslant \mathcal{H}^m(\varphi(S)).$$

The Lipschitz function φ need not be a diffeomorphism. For more general functions $\varepsilon(r) = C\, r^\alpha$, $\alpha > 0$, the weaker inequality,

$$\mathcal{H}^m(S) \leqslant (1 + \varepsilon(r))\, \mathcal{H}^m(\varphi(S)),$$

defines $(\mathbf{M}, \varepsilon, \delta)$-minimal sets, which include soap bubbles with volume constraints.

These sets were treated by Almgren in his monograph [3]. Three basic properties of $(\mathbf{M}, \varepsilon, \delta)$-minimal sets are $(\mathcal{H}^m, m)$-rectifiability [3, II.3(9)], monotonicity formula [40, II.1], and existence of $(\mathbf{M}, 0, \delta)$-minimal cone at every point [40, II.2]. The regularity of $(\mathbf{M}, \varepsilon, \delta)$-minimal sets in $\mathbb{R}^3$ was studied by Taylor [40], who proved

Theorem 4.2. *$(\mathbf{M}, \varepsilon, \delta)$-minimal sets in $\mathbb{R}^3$ can have two kind of singularities: the ones predicted by Plateau.*

Proof. The proof consists in showing that there are only two possible singularities. To prove this, take any singularity and consider its linear approximation, which is a cone determined over a geodesic net in the sphere. This idea goes back to Lamarle (1864). There are precisely 10 such geodesic nets [27, pp. 134–135] (Lamarle missed one). One then shows that all but three of them are unstable. One corresponds to a great circle in the sphere, which yields regular portions, and the other two correspond to the singular sets.

Hence, for approximating cones, there are only two possible singularities. The really hard part is to show that the cone is a good enough approximation of the original soap film. Reifenberg's epiperimetric inequality is used to show that the minimizers converge rapidly to an asymptotic cone. □

For singularities of $(\mathbf{M}, 0, \delta)$-minimal sets in $\mathbb{R}^4$ one has the following conjecture.

Conjecture 4.3. *A cluster in $\mathbb{R}^4$ has only the following singularities:*

(i) 3 *hypersurfaces meeting along surfaces at* 120^o.
(ii) 4 *such surfaces meeting along a curve at almost* 109^o.
(iii) 5 *or* 16 *of such curves meeting at a point as in the cone over a regular simplex or the cone over a hypercube.*

Some progress on this conjecture has been made by Brakke, Sullivan and White.

Many questions remain open.

- Boundary regularity.
- The existence of a least-area $(\mathbf{M}, 0, \delta)$-minimal set in $\mathbb{R}^3$ with a given smooth boundary curve.
- The cone over the regular tetrahedron is the least-area separator of the four faces. Is there a smaller equilibrium surface not separating the four regions?
- It is known that a smooth minimal surface is locally area-minimizing in arbitrary codimension. Is this property still true whenever the tangent cone is area-minimizing?

5. Double bubbles

5.1. **Existence and regularity of area-minimizing clusters.** Bubble clusters seek the least-area way to enclose and separate several regions of prescribed volume. But they do *not* always find the absolute least area shape (they could be stable, but not minimizers).

A *cluster* is a collection of disjoint regions $R_1, \ldots, R_m$ (n-dimensional locally integral currents of multiplicity 1 in $\mathbb{R}^{n+1}$), with surface area

$$\frac{1}{2}\bigg(\sum_i \mathbf{M}(\partial R_i) + \mathbf{M}(\partial \sum_i R_i)\bigg).$$

A region is not assumed to be connected. The existence of an area-minimizing cluster enclosing prescribed volumes is guaranteed by Almgren's work [3] in a very general context. A simplified proof was given by Morgan [26].

Theorem 5.1. *In $\mathbb{R}^n$, given volumes $V_1, \ldots, V_m > 0$, there exists an area-minimizing cluster for those volumes.*

The proof of this result needs some lemmas, the first of which is the following observation.

Lemma 5.2. *Given any cluster, there exists $C > 0$, such that arbitrary small changes in volume may be accomplished inside small balls at a cost*

$$|\Delta A| \leqslant C\,|\Delta V|.$$

Another interesting lemma which follows from the monotonicity formula and the isoperimetric inequality is

Lemma 5.3. *An area-minimizing cluster is bounded in $\mathbb{R}^n$.*

With these ingredients we are ready to prove Theorem 5.1. The proof of this result is complicated because when taking a minimizing sequence the volume could disappear at infinity. Let us take a minimizing sequence $\mathcal{C}_\alpha$ with the prescribed volumes.

Step 1. Get a convergent subsequence to a non zero limit $\mathcal{C}$. One can choose p_α such that

$$\mathrm{vol}(R_{1,\alpha} \cap \mathbf{B}(p_\alpha, c_1)) \geqslant c_2,$$

where $R_{1,\alpha}$ is the first region of $\mathcal{C}_\alpha$. This is a technical result known as the Concentration Lemma. The proof follows easily since we partition $\mathbb{R}^n$ into cubes K_j. Choosing appropiately the length of the edges we have inside K_j,

$$\mathrm{area}(\partial R_{1,\alpha} \cap K_j) \geqslant \gamma \, (\mathrm{vol}(R_{1,\alpha} \cap K_j))^{(n-1)/n},$$

for some isoperimetric constant γ, and so

$$\mathrm{area}(\partial R_{1,\alpha} \cap K_j) \geqslant \gamma \, \frac{\mathrm{vol}(R_{1,\alpha} \cap K_j)}{\max_i \mathrm{vol}(R_{1,\alpha} \cap K_i)^{1/n}}.$$

Summing over j we get

$$\max_i \mathrm{vol}(R_{1,\alpha} \cap K_i)^{1/n} \geqslant \frac{\gamma}{A} \, V_1 = c_2,$$

which proves the desired result. (This is a minor correction of the statement in [27, 13.7(1)].)

Translate to assume $p_\alpha = 0$.

Step 2. Show that the limit $\mathcal{C}$ is area-minimizing for its volumes. If not then one can get another (minimizing) sequence with less area than the original one. If no volume is lost then we are done.

Step 3. If some volume disappears at infinity then repeat the process with some p'_α.

Then countably many repetitions capture the total volume and yield a solution. This solution is bounded by Lemma 5.3, and so only a finite number of repetitions is needed.

Concerning regularity the first observation is that an area-minimizing cluster is $(\mathbf{M}, \varepsilon, \delta)$-minimal [27, 13.8]. The proof of this result relies on Lemma 5.2. Then one can apply Taylor's regularity result in $\mathbb{R}^3$, which implies that an $(\mathbf{M}, \varepsilon, \delta)$-minimal set consists on smooth, constant mean curvature surfaces meeting in threes at 120^o along curves, in turn meeting in fours about 109^o.

In higher dimensions Almgren [3] showed that $(\mathbf{M}, \varepsilon, \delta)$-minimal sets in $\mathbb{R}^n$, $n \geqslant 3$, are $C^{1,\alpha}$ almost everywhere. A simple treatment of $(\mathbf{M}, \varepsilon, \delta)$-minimal sets in $\mathbb{R}^2$ can be found in Morgan [25]. White [42] proved that they consists on constant mean curvature hypersurfaces meeting in threes at 120^o along $(n-2)$-dimensional submanifolds, which meet in fours along smooth $(n-3)$-dimensional surfaces, which meet in an $(n-4)$-dimensional set.

5.2. **Characterization of area-minimizing clusters.** For an account of results on single bubbles the interested reader can consult [27, 13.2].

For double bubbles the only known optimal shapes are those of the double bubble in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$. For the remanining volumes and dimensions the problem is still open.

The *standard double bubble* in $\mathbb{R}^n$ is composed of three $(n-1)$-spherical caps meeting along an $(n-2)$-sphere at 120^o. In case we consider two equal volumes one of the spherical caps degenerates to a flat disc. The whole configuration is obtained by rotating three circles (or two circles and a segment) on a plane meeting at 120^o around an $(n-2)$-dimensional subspace.

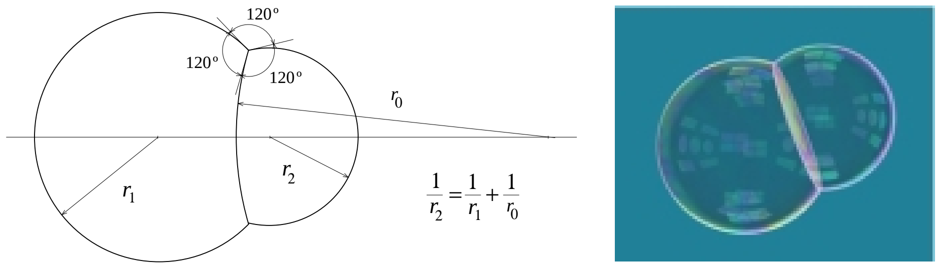

FIGURE 1. The standard double bubble

The double bubble conjecture states that the standard double bubble provides the least perimeter way to enclose and separate two regions of given volumes in $\mathbb{R}^n$.

This conjecture appeared in Foisy's undergraduate thesis [13]. The *planar* double bubble conjecture was proved in 1990 by Foisy, Alfaro, Brock, Hodges and Zimba [14]. The double bubble conjecture in $\mathbb{S}^2$ was solved by Masters [24] in 1994 .

The study of double bubbles is complicated since we do not know anything about their structure. The regions can be disconnected with any number of components, and also the exterior of the bubble can be disconnected, with precisely one non-compact component, and possibly more compact ones which will be called *empty spaces* or *empty chambers*.

The first results towards proving the double bubble conjecture in Euclidean 3-space were

(i) The bubble is a surface of revolution around some line in $\mathbb{R}^3$, White, Foisy [13], Hutchings [20].
(ii) The isoperimetric profile $A(v_1, v_2)$, which assigns to every pair of volumes (v_1, v_2) the perimeter of the area-minimizing double bubble of these volumes is a *concave* function. This implies that A is increasing in each variable, which shows that there are no empty spaces inside, Hutchings [20].

It was also proved in Foisy [13] and Hutchings [20] that the bubble must touch the axis of revolution. In order to prove this one consider the transformations $r \mapsto (r^2 - \varepsilon)^{1/2}$, where r is the distance to the axis of revolution. This deformation decreases area and keeps constant the volume enclosed. For two equal volumes Hutchings also proved that each region is connected. By using this result Hass and Schlaffly proved, by making use of computer techniques, that

Theorem 5.4. [19] *The standard double bubble is the least perimeter way to enclose and separate two equal volumes in* $\mathbb{R}^3$.

In the general case it was proved by Hutchings, Morgan, Ritoré and Ros [21], [22] that there are at most three components (one of the regions can be disconnected and has at most two connected components). Finally they proved

Theorem 5.5. [22] *The standard double bubble is the least perimeter way to enclose and separate two arbitrary given volumes in* $\mathbb{R}^3$.

We already know that an area-minimizing bubble must be a surface of revolution, that there are no empty chambers, and that the number of components is less than or equal to three. This is enough to reduce the number of candidates to two possible configurations which are depicted in Figure 2. Both can be obtained by rotating a graph Γ in a half-plane about the boundary line

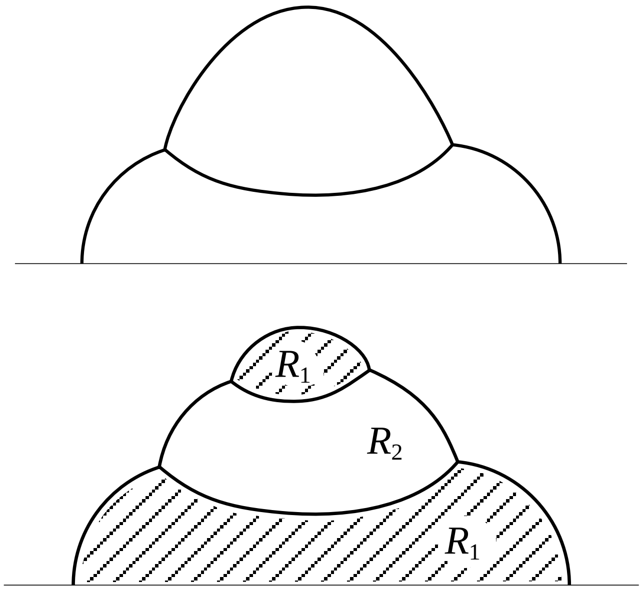

FIGURE 2. Candidate graphs to produce area-minimizing bubbles when rotated

The main ingredient in the proof is an unstability argument, based on Courant's Nodal Domain Theorem. It essentially says that a Killing field cannot induce too many nodal regions in the bubble. If X is a Killing field, the one induced by a one-parameter group of isometries, a nodal region as a connected component of the function $\langle X, N\rangle$, where N is a unit vector field normal to the sheets of the bubble. This argument has been previously used to study stability and isoperimetric questions in [32], [34], [33] and [29]. From this principle one can prove the following Proposition

Proposition 5.6. *Consider a double bubble of revolution $\Sigma \subset \mathbb{R}^n$, $n \geqslant 3$, with axis L, as in Figure 2. Assume there is a finite number of nonsingular points $p_1, \ldots, p_k$, with $x = f(p_1) = \ldots = f(p_k)$, which separate the generating graph Γ.*

Then Σ is unstable.

In the above Proposition, $f(p)$ is the intersection of the normal line to p, with the axis of revolution L. Of course $f(p)$ can be ∞. This Proposition reduces the proof of the double bubble conjecture to elementary planar geometry modulo some properties of Delaunay hypersurfaces (hypersurfaces of revolution with constant mean curvature). For instance, we can easily discard the two components case. Consider Figure 3. Let L' be the line equidistant from a and b. Assume that this line meets the axis of revolution L at some point x. In each one of the curves joining a and b there is at least a point at maximum or minimum distance from x. Call them p_1 and p_2. Then p_1 and p_2 separate the generating graph Γ, which implies that the bubble is unstable.

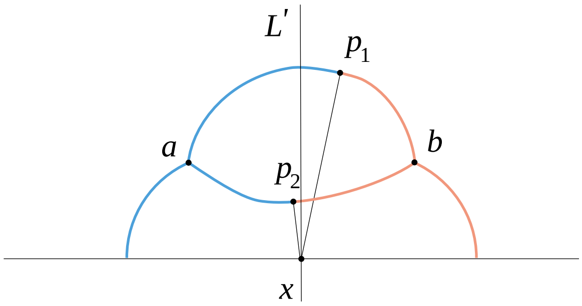

FIGURE 3. The partition method

The three component case is more involved, and some special properties of hypersurfaces of revolution with constant mean curvature are used. We refer to the interested reader to [22] for the discussion of this case.

By using the arguments in [22], Reichardt, Heilmann, Lai and Spielman [31] were able to prove the double bubble conjecture in $\mathbb{R}^4$ and in some particular cases in $\mathbb{R}^n$.

6. Integrands and isoperimetric inequalities

6.1. **Integrands.** An *integrand* is a map $\Phi : \mathbb{R}^n \times \Lambda_m \mathbb{R}^n \to \mathbb{R}$ which is homogeneous and continuous. Sometimes it is required that Φ be smooth except at the origin, positive, even ($\Phi(x,\xi) = \Phi(x,-\xi)$) or convex ($\Phi(x,\xi_1+\xi_2) \leqslant \Phi(x,\xi_1)+\Phi(x,\xi_2)$). An integrand defines a functional on the space of m-dimensional rectifiable sets by

$$\Phi(S) = \int_S \Phi(x,\vec{S}(x))\, d\mathcal{H}^m(x)$$

For instance the integrand $\Phi(x,\xi) = |\xi|$ defines the area functional.

Existence of Φ-minimizers follows from the compactness theorem. Interior regularity of a hypersurface minimizing a smooth elliptic integrand Φ in $\mathbb{R}^n$ or in a Riemannian manifold M^n except for a singular set of $\mathcal{H}^{n-3}$-measure 0 was proved by Schoen, Simon, and Almgren [36]. In higher codimension, regularity on an open dense set was proved by Almgren and Federer [9, 5.3.16–17]. Boundary regularity is known just if the surface has density 1/2, Hardt [18].

The Φ-isoperimetric problem for curves in $\mathbb{R}^n$ seeks for curves of least Φ-energy bounding a fixed least-area. In $\mathbb{R}^2$ the optimal shape is the Wulff shape (the unit ball in the dual norm).

6.2. **Isoperimetric inequalities.** It is known since ancient times that the circle in $\mathbb{R}^2$ encloses the most area for fixed length. The first rigorous proof of this fact was given by Weierstrass in the late 1800s. That a round sphere in $\mathbb{R}^3$ encloses the most volume for fixed area was proved by Schwarz [37]. Later Schmidt [35] proved that a geodesic ball in Euclidean, hyperbolic or spherical space encloses a fixed volume with the least possible perimeter.

In higher codimension there are many submanifolds with a given boundary, so that one cannot bound the perimeter boundary of a submanifold by its volume. However isoperimetric inequalities can be obtained in $\mathbb{R}^n$ if we require that the spanning surface be area-minimizing

Theorem 6.1 (Almgren [5])**.** *An m-dimensional area-minimizing integral current in $\mathbb{R}^n$ $(2 \leqslant m \leqslant n)$ has no more area than a round disc of the same boundary area, with equality only for the round disc itself.*

Let us give the main ideas of the proof for surfaces in $\mathbb{R}^3$ (although this case was known earlier). Amongst area-minimizing surfaces in $\mathbb{R}^3$ with area π there is one Q with boundary C of least length L. We assume regularity to simplify the arguments. Of course, by direct comparison with the planar disc of area π, we get

$$L \leqslant 2\pi.$$

Let us see first that the curvature κ of the curve satisfies inequality $\kappa \leqslant 1$. Otherwise we could find a local variation of C (around some point where $\kappa > 1$) so that

$$\frac{dL}{dA} > 1.$$

We use this variation to reduce the length of C reducing at the same time the area of Q. On the other hand, scaling up to restore the area we get

$$\frac{dL}{dA} = \frac{L}{2A} \leqslant 1,$$

so that the combination of both transformations keeps the area constant but reduce boundary length, a contradiction to the minimality of the length of C.

Finally we look at the Gauss map of the boundary B of the convex hull of C (Q lies inside by the convex hull property for minimal surfaces). The total curvature of B is 4π and the only contribution to this total curvature comes from $B \cap C$ since $B - C$ is a flat surface. Geometrical considerations [27, p. 183] imply that this contribution to the total curvature equals $(2/\pi)\,\kappa\alpha$, where $\alpha \in (0, \pi)$ is an angle. Hence

$$4\pi = \frac{2}{\pi} \int_C \kappa\,\alpha\,ds \leqslant \frac{2}{\pi}\,1\,\pi\,L = 2L.$$

This shows that $L = 2\pi$. If equality holds then $\kappa \equiv 1$ and $\alpha \equiv \pi$ and this is enough to show that Q is a flat disc bounded by the circle C.

To make this argument completely rigorous, Almgren [5] considered the equidistants (sets at fixed distance) from the boundary B of the convex hull of B.

Finally we state some open questions concerning isoperimetric inequalities

(i) We know that amongst area-minimizing surfaces the disc of area π is the largest minimal surface bounded by a system of curves of total length 2π. Is this still true for all minimal surfaces? Is it true for stable minimal surfaces?

(ii) Does any isoperimetric inequality hold for Φ-minimal surfaces?

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