LARGE ISOPERIMETRIC REGIONS IN THE PRODUCT OF A COMPACT MANIFOLD WITH EUCLIDEAN SPACE

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ABSTRACT. Given a compact Riemannian manifold M without boundary, we show that large isoperimetric regions in $M\times\mathbb{R}^k$ are tubular neighborhoods of $M\times\{x\}$, with $x\in\mathbb{R}^k$.

1. INTRODUCTION

We consider the *isoperimetric problem* of minimizing perimeter under a given volume constraint inside $M\times\mathbb{R}^k$, where $\mathbb{R}^k$ is k-dimensional Euclidean space and M is an m-dimensional compact Riemmanian manifold without boundary. The dimension of the product manifold $N=M\times\mathbb{R}^k$ will be $n=m+k$. Our main result is the following

Theorem 1.1. *Let M be a compact Riemannian manifold. There exists a constant $v_0>0$ such that any isoperimetric region in $M\times\mathbb{R}^k$ of volume $v\geqslant v_0$ is a tubular neighborhood of $M\times\{x\}$, with $x\in\mathbb{R}^k$.*

This result, in case $k=1$, was first proven by Duzaar and Steffen [3, Prop. 2.11]. As observed by Frank Morgan, an alternative proof for $k=1$ can be given using the monotonicity formula and properties of the isoperimetric profile of $M\times\mathbb{R}$. Gonzalo [8] considered the general problem in his Ph.D. Thesis. In $\mathbb{S}^1\times\mathbb{R}^k$, the result follows from the classification of isoperimetric regions by Pedrosa and Ritoré [15]. Large isoperimetric regions in asymptotically flat manifolds have been recently characterized by Eichmair and Metzger [4].

In our proof we use symmetrization and prove in Corollary 2.2 that an anisotropic scaling of symmetrized isoperimetric regions of large volume L^1-converge to a tubular neighborhood of $M\times\{0\}$. This convergence can be improved in Lemma 2.4 to Hausdorff convergence of the boundaries from density estimates on tubes, obtained in Lemma 2.3, similar to the ones obtained by Ritoré and Vernadakis [16]. Results of White [17] and Grosse-Brauckmann [9] on stable submanifolds then imply that the scaled boundaries are cylinders, Theorem 3.2. For small dimensions, it is also possible to use a result by Morgan and Ros [14] to get the same conclusion only using L^1-convergence. Once it is shown that the symmetrized set is a tube, it is not difficult to show that the original isoperimetric region is also a tube.

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Given a set $E \subset N$, their perimeter and volume will be denoted by $|E|$ and $P(E)$, respectively. We refer the reader to Maggi's book [11] for background on finite perimeter sets. The r-dimensional Hausdorff measure of a set E will be denoted by $H^r(E)$.

On $M \times \mathbb{R}^k$ we shall consider the anisotropic dilation of ratio $t > 0$ defined by

$$\varphi_t(p,x) = (p,tx), \qquad (p,x) \in M \times \mathbb{R}^k.$$

Since the Jacobian of the map φ_t is t^k we have

$$|\varphi_t(E)| = t^k|E|, \qquad E \subset M \times \mathbb{R}^k. \tag{1.1}$$

Let $\Sigma \subset M \times \mathbb{R}^k$ be an $(n-1)$-rectifiable set. At a regular point $p \in \Sigma$, the unit normal ξ can be decomposed as $\xi = av + bw$, with $a^2 + b^2 = 1$, v tangent to M and w tangent to $\mathbb{R}^k$. Then the Jacobian of $\varphi_t|\Sigma$ is equal to $t^{k-1}(t^2a^2+b^2)^{1/2}$. For $t \geqslant 1$ we get

$$t^k H^{n-1}(\varphi_t(\Sigma)) \geqslant H^{n-1}(\varphi_t(\Sigma)) \geqslant t^{k-1}H^{n-1}(\varphi_t(\Sigma)), \tag{1.2}$$

and the reversed inequalities when $t \leqslant 1$. A similar property holds for the perimeter. Equality holds in the right hand side of (1.2) if and only if $a = 0$, what implies that ξ is tangent to $\mathbb{R}^k$.

An open ball of radius $r > 0$ and center $x \in \mathbb{R}^k$ will be denoted $D(x,r)$. If it is centered at the origin, then $D(r) = D(0,r)$. We shall also denote by $T(x,r)$ the set $M \times D(x,r)$, and by $T(r)$ the set $M \times D(r)$. Observe that $\varphi_t(T(x,r)) = T(tx,tr)$ and that $T(x,r)$ is the tubular neighborhood of radius $r > 0$ of $M \times \{x\}$. If $E \subset N$ and $r > 0$, we shall denote by E_r the set $E \cap \big(N \setminus T(r)\big)$.

Given any set $E \subset N$ of finite perimeter, we can replace it by a *normalized* set $\operatorname{sym} E$ by requiring $\operatorname{sym} E \cap (\{p\} \times \mathbb{R}^k) = \{p\} \times D(r(p))$, where $H^k(D(r(p))$ is equal to the H^k-measure of $\operatorname{sym} E \cap (\{p\} \times \mathbb{R}^k)$. For such a set we get

Theorem 1.2. *In the above conditions, we have*

(1) $|\operatorname{sym} E| = |E|$,
(2) $P(\operatorname{sym} E) \leqslant P(E)$.

The proof of Theorem 1.2 is similar to the one of symmetrization in $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$ with respect to one of the factors, see Burago and Zalgaller [1] (or Maggi [11] for the case $m = 1$). The main ingredients are a corresponding inequality for the Minkowski content and approximation of finite perimeter sets by sets with smooth boundary.

Given $E \subset N$, we shall denote by E^* its orthogonal projection over M. If E is normalized, and $u : E^* \to \mathbb{R}^+$ measures the radius of the disk obtained projecting $E \cap (\{p\} \times \mathbb{R}^k)$ to $\mathbb{R}^k$, we get, assuming enough regularity on u,

$$|E| = \omega_k \int_{E^*} u^k dH^m,$$
$$H^{n-1}(\partial E) = k\omega_k \int_{E^*} u^{k-1}\sqrt{1+|\nabla u|^2}\, dH^m.$$

where $\omega_k = H^k(D(1))$, and $k\omega_k = H^{k-1}(\mathbb{S}^{k-1})$. Observe that

$$|T(r)| = \omega_k r^k H^m(M),$$
$$P(T(r)) = k\omega_k r^{k-1} H^m(M),$$

so that

$$P(T(r)) = k\left(\omega_k H^m(M)\right)^{1/k} |T(r)|^{(k-1)/k}. \tag{1.3}$$

Observe also that, in case E is normalized and $0 < r < s$, we have $(E_s)^* \subset (E_r)^*$.

The isoperimetric profile of $M \times \mathbb{R}^k$ is the function

$$I(v) = \inf\{P(E); |E| = v\}.$$

An isoperimetric region $E \subset M \times \mathbb{R}^k$ is one that satisfies $I(|E|) = P(E)$. Existence of isoperimetric regions in $M \times \mathbb{R}^k$ is guaranteed by a result of Frank Morgan [13, pp. 129], since the quotient of $M \times \mathbb{R}^k$ by its isometry group is compact. From his arguments, it also follows that isoperimetric regions are bounded in M. See also [6]. Observe that, from (1.3), we get

$$I(v) \leqslant k\left(\omega_k H^m(M)\right)^{1/k} v^{(k-1)/k}, \tag{1.4}$$

for any $v > 0$. The regularity of isoperimetric regions in Riemannian manifolds is well-known, see Morgan [12] and Gonzales-Massari-Tamanini [7]. The boundary is regular except for a singular set of vanishing H^{n-7} measure.

Proposition 1.3. *The isoperimetric profile I of N is non-decreasing and continuous.*

Proof. Let $v_1 < v_2$, and $E \subset N$ an isoperimetric region of volume v_2. Let $0 < t < 1$ so that $|\varphi_t(E)| = v_1$. By (1.2) we have

$$I(v_1) \leqslant P(\varphi_t(E)) \leqslant P(E) = I(v_2).$$

This shows that I is non-decreasing.

Since I is a monotone function, it can only have jump discontinuities. If E is an isoperimetric region of volume v, using a smooth vector field supported in the regular part of the boundary of E, one can find a continuous function f, defined in a neighborhood of v, so that $I \leqslant f$. This implies that I cannot have jump discontinuities at v. □

We shall also use the following well-known isoperimetric inequalities in M and $M \times \mathbb{R}^k$

Lemma 1.4 ([3])**.** *Given $0 < v_0 < H^m(M)$, there exist a constant $a(v_0) > 0$ such that*

$$H^{m-1}(\partial E) \geqslant a(v_0) H^m(E),$$

for any set $E \subset M$ satisfying $0 < H^m(E) < v_0$.

Lemma 1.5. *Given $v_0 > 0$, there exists a constant $c(v_0) > 0$ so that*

$$I(v) \geqslant c(v_0)\, v^{(n-1)/n}, \tag{1.5}$$

for any $v \in (0, v_0)$.

Lemma 1.5 follows from the facts that $I(v)$ is strictly positive for $v > 0$ and is asymptotic to the Euclidean isoperimetric profile when v approaches 0.

2. LARGE ISOPERIMETRIC REGIONES IN C

In this Section we shall prove that normalized isoperimetric regions of large volume, when scaled down to have constant volume v_0, have their boundaries uniformly close to the boundary of the normalized tube of volume v_0.

If $E \subset N$ is any finite perimeter set and $T(E)$ is the tube with the same volume as E, we define

$$E^- = E \cap T(E), \quad E^+ = E \setminus T(E)$$

Let $t > 0$, and $\Omega = \varphi_t(E)$. Since $\varphi_t(E^+) = \Omega^+$, (1.1) implies

$$\frac{|E^+|}{|E|} = \frac{|\Omega^+|}{|\Omega|}. \tag{2.1}$$

A similar equality holds replacing E^+ by E^-.

Proposition 2.1. *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of normalized sets with volumes $|E_i| \to \infty$. Let $v_0 > 0$ and $0 < t_i < 1$ so that $|\varphi_{t_i}(E_i)| = v_0$ for all $i \in \mathbb{N}$. Let T be the tube of volume v_0 around M_0.*

If $\varphi_{t_i}(E_i)$ does not converges to T in the L^1-topology, then there is a constant $c > 0$, only depending on $\{E_i\}_{i\in\mathbb{N}}$, so that, passing to a subsequence we get,

$$H^{n-1}(\partial E_i) \geqslant c|E_i|. \tag{2.2}$$

Proof. Assume $T = M \times D(r)$, and set $\Omega_i = \varphi_{t_i}(E_i)$. As $|\Omega_i| = |T|$, we get $2\,|\Omega_i^+| = |\Omega_i \triangle T|$ and, since $|\Omega_i \triangle T|$ does not converge to 0, the sequence $|\Omega_i^+|$ does not converge to 0 either. Let $c_1 > 0$ be a constant so that $\limsup_{i\to\infty}(|\Omega_i^+|/|\Omega_i|) > c_1$. From (2.1) we obtain

$$\limsup_{i\to\infty} \frac{|E_i^+|}{|E_i|} > c_1. \tag{2.3}$$

Now we claim that

$$\liminf_{i\to\infty} H^m((\Omega_i \cap \partial T)^*) < H^m(M). \tag{2.4}$$

To prove (2.4) we argue by contradiction. Assume that $\liminf_{i\to\infty} H^m((\Omega_i \cap \partial T)^*) = H^m(M)$. As Ω_i is normalized, we have $(\Omega_i \cap \partial T)^* \subset (\Omega_i \cap T)^*$ and so $(T \setminus \Omega_i) \subset (M \setminus (\Omega_i \cap \partial T)^*) \times D(r)$. This implies $\limsup_{i\to\infty} |T \setminus \Omega_i| = 0$. Since $|\Omega_i| = |T|$, we get $\lim_{i\to\infty} |\Omega_i \triangle T| = 2 \lim_{i\to\infty} |T \setminus \Omega_i| = 0$, a contradiction that proves the claim.

Hence there exists $w \in (0, H^m(M))$ so that

$$\liminf_{i\to\infty} H^m((\Omega_i \cap \partial T)^*) < w. \tag{2.5}$$

Let $r_i > 0$ be the radius of the tube with the same volume as E_i. As $(E_i^+)^* = (\Omega_i^+)^*$ and E_i is normalized, we have

$$\liminf_{i\to\infty} H^m((E_i \cap \partial T(s))^*) < w, \qquad s \geqslant r_i. \tag{2.6}$$

The above arguments imply, replacing the original sequence by a subsequence, that

$$|E_i^+| > c_1\,|E_i|, \qquad H^m((E_i \cap \partial T(s))^*) < w, \qquad i \in \mathbb{N}, s \geqslant r_i. \tag{2.7}$$

Let $a=a(w)$ be the constant in Lemma 1.4. For the elements of the subsequence satisfying (2.7) we have

$$\begin{aligned}H^{n-1}(\partial E_i) &\geqslant H^{n-1}(\partial E_i\cap(N\setminus T(r_i)))\\ &\geqslant \int_{r_i}^{\infty} H^{n-2}(\partial E_i\cap\partial T(s))\,ds\\ &\geqslant \int_{r_i}^{\infty} H^{n-2}(\partial(E_i\cap\partial T(s)))\,ds\\ &= \int_{r_i}^{\infty} H^{m-1}(\partial(E_i\cap\partial T(s))^*)\,H^{k-1}(\partial D(s))\,ds\\ &\geqslant \int_{r_i}^{\infty} a\,H^{m}((E_i\cap\partial T(s))^*)\,H^{k-1}(\partial D(s))\,ds\\ &= a\int_{r_i}^{\infty} H^{n-1}(E_i\cap\partial T(s))\,ds = a\,|E_i^+| > a\,c_1|E_i|,\end{aligned}$$

what proves the result. In the previous inequalities we have used the coarea formula for the distance function to $M\times\{0\}$; that $\partial(E_i\cap\partial T(s))\subset\partial E_i\cap\partial T(s)$, where the first ∂ denotes the boundary operator in $\partial T(s)$; the fact that for an $O(k)$-invariant set F we have $F\cap\partial T(s)=(F\cap\partial T(s))^*\times\partial D(s)$, and so $H^{r+k-1}(F\cap\partial T(s))=H^r((F\cap\partial T(s))^*)H^{k-1}(\partial D(s))$; that $(\partial(E_i\cap\partial T(s)))^*=\partial(E_i\cap\partial T(s))^*$; and the isoperimetric inequality on M given in Lemma 1.4. □

Corollary 2.2. *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of normalized isoperimetric sets with volumes $\lim_{i\to\infty}|E_i|=\infty$. Let $v_0>0$ and $0<t_i<1$ such that $\Omega_i=\varphi_{t_i}(E_i)$ has volume v_0 for all $i\in\mathbb{N}$. Then $\Omega_i\to T$ in the L^1-topology, where T is the tube of volume v_0.*

Proof. Regularity results for isoperimetric regions imply that $P(E_i)=H^{n-1}(\partial E_i)$. If Ω_i does not converge to T in the L^1-topology then, using (2.2) in Lemma 2.1 and (1.4), we get,

$$c\,|E_i|\leqslant P(E_i)\leqslant k\left(\omega_k H^m(M)\right)^{1/k}|E_i|^{(k-1)/k},$$

for a subsequence, thus yielding a contradiction by letting $i\to\infty$ since $|E_i|\to\infty$. □

Using density estimates, we shall show now that the L^1 convergence of the scaled isoperimetric regions can be improved to Hausdorff convergence.

In a similar way to Leonardi and Rigot [10, p. 18] (see also [16] and David and Semmes [2]), given $E\subset N$, we define a function $h:\mathbb{R}^k\times(0,+\infty)\to\mathbb{R}^+$ by

$$h(x,R)=\frac{\min\{|E\cap T(x,R)|,|T(x,R)\setminus E|\}}{R^n},$$

for $x\in\mathbb{R}^k$ and $R>0$. We remark that the quantity $h(x,R)$ is not homogeneous in the sense of being invariant by scaling since $h(x,R)\leqslant\frac{1}{2}(k\omega_k H^m(M))R^{k-n}$, which goes to infinity when R goes to 0. When the set E should be explicitly mentioned, we shall write

$$h(E,x,R)=h(x,R).$$

Lemma 2.3. *Let $E \subset N$ be an isoperimetric region of volume $v > v_0$. Let $\tau > 1$ such that $\Omega = \varphi_\tau^{-1}(E)$ has volume v_0. Choose ε so that*

$$0 < \varepsilon < \left\{ v_0, \left(\frac{c(v_0)\, v_0^{1/k}}{2H^m(M)} \right)^n, \left(\frac{c(v_0)}{8n} \right)^n \right\}, \tag{2.8}$$

where $c(v_0)$ the one in (1.5).

Then, for any $x \in \mathbb{R}^k$ and $R \leqslant 1$ so that $h(\Omega, x, R) \leqslant \varepsilon$, we get

$$h(\Omega, x, R/2) = 0.$$

Moreover, in case $h(\Omega, x, R) = |\Omega \cap T(x,R))| R^{-n}$, we get $|\Omega \cap T(x, R/2)| = 0$ and, in case $h(\Omega, x, R) = |T(x,R) \setminus \Omega| R^{-n}$, we have $|T(x, R/2) \setminus \Omega| = 0$.

Proof. Using Lemma 1.5 we get a positive constant $c(v_0)$ so that (1.5) is satisfied, i.e., $I(w) \geqslant c(v_0)\, w^{(n-1)/n}$, for all $0 \leqslant w \leqslant v_0$.

Assume first that

$$h(x,R) = h(\Omega, x, r) = \frac{|\Omega \cap T(x,R)|}{R^n}.$$

Define

$$m(r) = |\Omega \cap T(x,r)|, \quad 0 < r \leqslant R.$$

The function $m(r)$ is non-decreasing and, for $r \leqslant R \leqslant 1$, we get

$$m(r) \leqslant m(R) \leqslant |\Omega \cap T(x,R)| \leqslant \varepsilon R^n \leqslant \varepsilon < v_0, \tag{2.9}$$

by (2.8). Hence $v_0 - m(r) > 0$ for $0 < r \leqslant R$.

By the coarea formula, when $m'(r)$ exists, we get

$$m'(r) = \frac{d}{dr} \int_0^r H^{n-1}(\Omega \cap \partial T(x,s))\, ds = H^{n-1}(\Omega \cap \partial T(x,r)).$$

Now define

$$\lambda(r) = \frac{v_0^{1/k}}{(v_0 - m(t))^{1/k}} = \frac{v^{1/k}}{(v - |T(\tau x, \tau r)|)^{1/k}} \geqslant 1,$$

and

$$\Omega(r) = \varphi_{\lambda(r)}(\Omega \setminus T(x,r)),$$

so that $|\Omega(r)| = |\Omega|$. Then

$$E(r) = \varphi_\tau(\Omega(r)) = \varphi_{\lambda(r)}(E \setminus T(\tau x, \tau r)),$$

and $|E(r)| = |E|$. Then, using (1.2) for $\lambda(r) \geqslant 1$ and standard properties of finite perimeter sets, we have

$$\begin{aligned} I(v) &\leqslant P(E(r)) \leqslant \lambda(r)^k \left(P(E \setminus T(\tau x, \tau r)) \right) \\ &\leqslant \frac{v_0}{v_0 - m(r)} \left(P(E) - P(E \cap T(\tau x, \tau r)) + 2H^{n-1}(E \cap \partial T(\tau x, \tau r)) \right). \end{aligned} \tag{2.10}$$

Since $\tau \geqslant 1$ and $E \cap \partial T(\tau x, \tau r)$ is part of a cylinder, using (1.2) again we get

$$P(E \cap T(\tau x, \tau r) \geqslant \tau^{k-1} P(\Omega \cap T(x,r)) \geqslant \tau^{k-1} c(v_0)\, m(r)^{(n-1)/n},$$

$$H^{n-1}(E \cap \partial T(\tau x, \tau r)) = \tau^{k-1} H^{n-1}(\Omega \cap \partial T(x,r)) = \tau^{k-1} m'(r),$$

Replacing them in (2.10), taking into account that $P(E) = I(v)$ and $\tau^k v_0 = v$, we have

$$
\begin{aligned}
2m'(r) &\geqslant m(r)^{(n-1)/n}\left(c(v_0) - \frac{m(r)^{1/n}}{\tau^{k-1}v_0}\, I(v)\right)\\
&\geqslant m(r)^{(n-1)/n}\left(c(v_0) - \frac{m(r)^{1/n}}{v_0^{1/k}}\,\frac{I(v)}{v^{(k-1)/k}}\right)\\
&\geqslant m(r)^{(n-1)/n}\left(c(v_0) - \frac{\varepsilon^{1/n}}{v_0^{1/k}}\,(k\omega_k H^m(M))\right)\\
&\geqslant \frac{c(v_0)}{2}\, m(r)^{(n-1)/n},
\end{aligned}
\tag{2.11}
$$

where we have used $m(r) \leqslant \varepsilon$, (1.4), and (2.8)

If there were $r \in [R/2, R]$ such that $m(r) = 0$ then, by the monotonicity of the function $m(r)$, we would conclude $m(R/2) = 0$ as well. So we assume $m(r) > 0$ in $[R/2, R]$. Then by (2.11), we get

$$
\frac{c(v_0)}{4} \leqslant \frac{m'(t)}{m(t)^{(n-1)/n}}, \qquad H^1\text{-a.e.}
$$

By (2.9) we get $m(R) \leqslant \varepsilon R^n$. Integrating between $R/2$ and R

$$
c(v_0)R/8 \leqslant n\,(m(R)^{1/n} - m(R/2)^{1/n}) \leqslant n\, m(R)^{1/n} \leqslant n\,\varepsilon^{1/n} R.
$$

This is a contradiction, since $\varepsilon < (c(v_0)/8n)^n$ by (2.8). So the proof in case $h(x,R) = |\Omega \cap T(x,R)|\, R^{-n}$ is completed.

Now we deal with the case $h(x,R) = |T(x,R) \setminus \Omega|\, R^{-n}$. Define

$$
m(r) = |T(x,r) \setminus \Omega|.
$$

Then $m(r)$ is a non-decreasing function and

$$
m'(r) = H^{n-1}(\Omega^c \cap \partial T(x,r)) = \frac{1}{\tau^{k-1}}\, H^{n-1}(E^c \cap \partial T(\tau x, \tau r)),
\tag{2.12}
$$

since $E^c \cap \partial T(\tau x, \tau r)$ is part of a tube. We also have $m(r) \leqslant m(R) \leqslant \varepsilon R^n \leqslant \varepsilon < v_0$ by (2.8). Observe that

$$
P(E \cup T(\tau x, \tau r) \leqslant P(E) - P(T(\tau x, \tau r) \setminus E) + 2H^{n-1}(E^c \cap \partial E(\tau x, \tau r)).
\tag{2.13}
$$

Since $\varphi_\tau(T(x,r) \setminus \Omega) = T(\tau x, \tau r) \setminus E$ and $\tau \geqslant 1$, we get

$$
\begin{aligned}
P(T(\tau x, \tau r) \setminus E) &= P(\varphi_\tau(T(x,r) \setminus \Omega))\\
&\geqslant \tau^{k-1} P(T(x,r) \setminus \Omega) \geqslant \tau^{k-1}\, c(v_0)\, m(r)^{(n-1)/n}.
\end{aligned}
\tag{2.14}
$$

Now, using that I is a non-decreasing function we easily obtain $P(E) = I(v) \leqslant I(|E \cup T(\tau x, \tau r)|) \leqslant P(E \cup T(\tau x, \tau r))$. We estimate $P(E \cup T(\tau x, \tau r))$ from (2.13). Using (2.14) and (2.12), we get

$$
I(v) = P(E) \leqslant P(E \cup T(\tau x, \tau r)) \leqslant I(v) - \tau^{k-1} c(v_0)\, m(r)^{(k-1)/k} + 2\tau^{k-1} m'(r)
\tag{2.15}
$$

and so

$$
\frac{c(v_0)}{2} \leqslant \frac{m'(r)}{m(r)^{(n-1)/n}}, \qquad H^1\text{-a.e.}
$$

By (2.9) we get $m(R) \leqslant \varepsilon R^n$. Integrating between $R/2$ and R

$$c(v_0)R/4 \leqslant n\,(m(R)^{1/n} - m(R/2)^{1/n}) \leqslant n\,m(R)^{1/n} \leqslant n\,\varepsilon^{1/n}R,$$

we get a contradiction since by (2.8) we have $\varepsilon < (c(v_0)/(8n))^n < (c(v_0)/(4n))^n$. This concludes the proof. □

Let $F \subset N$, then $F_r = \{x \in N : d(x,F) \leqslant r\}$. We improve now the L^1-convergence of normalized isoperimetric regions obtained in Corollary 2.2 to Hausdorff convergence of their boundaries

Lemma 2.4. *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of isoperimetric sets in N with $\lim_{i\to\infty}|E_i| = \infty$. Let $v_0 > 0$ and $\{t_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} t_i = 0$ and $|\Omega_i| = v_0$ for all $i \in \mathbb{N}$, where $\Omega_i = \varphi_{t_i}(E_i)$. Then for every $r > 0$, $\partial\Omega_i \subset (\partial T)_r$, for large enough $i \in \mathbb{N}$, where T is the tube of volume v_0.*

Proof. Since $|\Omega_i| = v_0$, using (2.8) we can choose a uniform $\varepsilon > 0$ so that Lemma 2.3 holds with this ε for all Ω_i, $i \in \mathbb{N}$. This means that, for any $x \in N$ and $0 < r \leqslant 1$, whenever $h(\Omega_i, x, r) \leqslant \varepsilon$ we get $h(\Omega_i, x, r/2) = 0$.

As $\Omega_i \to T$ in $L^1(N)$ by Corollary 2.2, we can choose a sequence $r_i \to 0$ so that

$$|\Omega_i \triangle T| < r_i^{n+1}. \tag{2.16}$$

Now fix some $0 < r < 1$. We reason by contradiction assuming that, for some subsequence, there exist

$$x_i \in \partial\Omega_i \setminus (\partial T)_r. \tag{2.17}$$

We distinguish two cases.

First case: $x_i \in N \setminus T$, for a subsequence. Choosing i large enough, (2.17) implies $T(x_i, r_i) \cap T = \emptyset$ and (2.16) yields

$$|\Omega_i \cap T(x_i, r_i)| \leqslant |\Omega_i \setminus T| \leqslant |\Omega_i \triangle T| < r_i^{n+1}.$$

So, for i large enough, we get

$$h(\Omega_i, x_i, r_i) = \frac{|\Omega_i \cap T(x_i, r_i)|}{r_i^n} < r_i \leqslant \varepsilon.$$

By Lemma 2.3, we conclude that $|\Omega_i \cap T(x_i, r_i/2)| = 0$, a contradiction.

Second case: $x_i \in T$. Choosing i large enough, (2.17) implies $T(x_i, r_i) \subset T$ and so

$$|T(x_i, r_i) \setminus \Omega_i| \leqslant |T \setminus \Omega_i|, \quad \text{for every } r_i < r.$$

Then, by (2.16), we get

$$|T(x_i, r_i) \setminus \Omega_i| \leqslant |T \setminus \Omega_i| \leqslant |\Omega_i \triangle T| < r_i^{n+1}.$$

So, for i large enough, we get

$$h(\Omega_i, x_i, r_i) = \frac{|T(x_i, r_i) \setminus \Omega_i|}{r_i^n} < r_i \leqslant \varepsilon.$$

By Lemma 2.3, we conclude that $|T(x_i, r_i/2) \setminus \Omega_i| = 0$, and we get again contradiction that proves the Lemma. □

3. Strict $O(k)$-stability of tubes with large radius

In his Section we consider the orthogonal group $O(k)$ acting on the product $M \times \mathbb{R}^k$ through the second factor.

Let $\Sigma \subset M \times \mathbb{R}^k$ be a compact hypersurface with constant mean curvature. It is well-known that Σ is a critical point of the area functional under volume-preserving deformations, and that Σ is a second order minima of the area under volume-preserving variations if and only if

$$\int_\Sigma \big(|\nabla u|^2 - q\,u^2\big)\,d\Sigma \geqslant 0, \tag{3.1}$$

for any smooth function $u : \Sigma \to \mathbb{R}$ with mean zero on Σ. In the above formula ∇ is the gradient on Σ and q is the function

$$q = \operatorname{Ric}(N,N) + |\sigma|^2,$$

where $|\sigma|^2$ is the sum of the squared principal curvatures in Σ, N is a unit vector field normal to Σ, and Ric is Ricci curvature on N.

A hypersurface satisfying (3.1) is usually called *stable* and condition (3.1) is referred to as *stability condition*. In case Σ is $O(k)$-invariant we can consider an equivariant stability condition: we shall say that Σ is *strictly $O(k)$-stable* if there exists a positive constant $\lambda > 0$ such that

$$\int_\Sigma \big(|\nabla u|^2 - q\,u^2\big)\,d\Sigma \geqslant \lambda \int_\Sigma u^2\,d\Sigma$$

for any function $u : \Sigma \to \mathbb{R}$ with mean zero which is $O(k)$-invariant.

We consider now the tube $T(r) = M \times D(r)$. The boundary of $T(r)$ is the cylinder $\Sigma(r) = M \times \partial D(r)$, which is $O(k)$-invariant, and has k principal curvatures equal to $1/r$. Hence its mean curvature is equal to k/r and the squared norm of the second fundamental form satisfies $|\sigma|^2 = k/r^2$. The inner unit normal to $\Sigma(r)$ is the normal to $\partial D(r)$ in $\mathbb{R}^k$ (it is tangent to the factor $\mathbb{R}^k$). This implies that $\operatorname{Ric}(N,N) = 0$.

We have the following result

Lemma 3.1. *The cylinder $\Sigma(r)$ is strictly $O(k)$-stable if and only if*

$$r^2 > \frac{k}{\lambda_1(M)},$$

where $\lambda_1(M)$ is the first positive eigenvalue of the Laplacian in M.

Proof. Let $\Sigma=\Sigma(r)=M\times D(r)$. Observe that an $O(k)$-invariant function on Σ is just a function $u:M\to\mathbb{R}$, that has mean zero on Σ if and only if $\int_M u\,dM=0$. Hence

$$\begin{aligned}\int_\Sigma (|\nabla u|^2-q\,u^2)\,d\Sigma &= k\omega_k r^{k-1}\int_M (|\nabla_M u|^2-\frac{k}{r^2}u^2)\,dM\\ &\geqslant k\omega_k r^{k-1}\left(\lambda_1(M)-\frac{k}{r^2}\right)\int_M u^2\,dM\\ &=\left(\lambda_1(M)-\frac{k}{r^2}\right)\int_\Sigma u^2\,d\Sigma.\end{aligned}$$

This proves the Lemma. □

Using the results of White [17] and Grosse-Brauckmann [9], we deduce the following result

Theorem 3.2. *Let T be a normalized tube so that $\Sigma=\partial T$ is a strictly $O(k)$-stable cylinder. Then there exists $r>0$ so that any $O(k)$-invariant finite perimeter set E with $|E|=|T|$ and $\partial E\subset T_r$ has larger perimeter than T unless $E=T$.*

Proof. Since Σ is strictly $O(k)$-stable, Grosse-Brauckmann [9, Lemma 5] implies that, for some $C>0$, Σ has strictly positive second variation for the functional

$$F_C=\text{area}+H\,\text{vol}+\frac{C}{2}(\text{vol}-\text{vol}(T))^2,$$

in the sense that the second variation of F_C in the normal direction of a function u satisfies

$$\delta_u^2F_C=\int_\Sigma (|\nabla u|^2-q\,u^2)\,d\Sigma+C\left(\int_\Sigma u\,d\Sigma\right)^2\geqslant\lambda\int_\Sigma u^2\,d\Sigma,$$

for any smooth $O(k)$-invariant function u (see the discussion in the proof of Theorem 2 in Morgan and Ros [14]). White's proof of Theorem 3 in [17] observes that a sequence of minimizers of F_C in tubular neighborhoods of radius $1/n$ of Σ are *almost minimizing* and hence $C^{1,\alpha}$ submanifolds that converge Hölder differentiably to Σ, contradicting the positivity of the second variation of Σ. Theorem 1.2 implies that the symmetrization of these minimizers are again minimizers. Thus we get a family of $O(k)$-minimizers of F_C converging Hölder differentiably to Σ, thus contradicting the strict $O(k)$-stability of Σ. □

4. Proof of Theorem 1.1

First we claim that there exists $v_0>0$ such that, for any isoperimetric region E of volume $|E|\geqslant v_0$, the set $\operatorname{sym}E$ is a tube.

To prove this, consider a sequence of isoperimetric regions $\{E_i\}_{i\in\mathbb{N}}$ with $\lim_{i\to\infty}|E_i|=\infty$. We know that $\{\operatorname{sym}E_i\}_{i\in\mathbb{N}}$ are also isoperimetric regions. Let $T=M\times D$ be a strictly $O(k)$-stable tube, that exists by Lemma 3.1. For large i, we scale down the sets $\operatorname{sym}E_i$ so that $\Omega_i=\varphi_{t_i}^{-1}(\operatorname{sym}E_i)$ has the same volume as T. As $\operatorname{sym}E_i$ is isoperimetric and $t_i>1$, we get from (1.4) and (1.2) that $P(\Omega_i)\leqslant P(T)$. By Corollary 2.2, the sets $\{\partial\Omega_i\}_{i\in\mathbb{N}}$ converge to ∂T in Hausdorff distance. By Theorem 3.2, $\Omega_i=T$ for large i and so $\operatorname{sym}E_i$ is a tube. This proves the claim. In particular, $H^m(E\cap(\{p\}\times\mathbb{R}^k))=H^m(D)$ for any $p\in M$.

Hence the isoperimetric profile satisfies $I(v)=C\,v^{(k-1)/k}$ for the constant C in (1.3) and any $v\geqslant v_0$. We conclude

$$I(t^kv)=t^{k-1}I(v),\qquad t^kv\geqslant v_0. \tag{4.1}$$

Let E be an isoperimetric region with volume $|E|>v_0$, and $t<1$ so that $t^k|E|=v_0$. Then

$$I(t^k|E|)\leqslant P(\varphi_t(E))\leqslant t^{k-1}P(E)=t^{k-1}I(|E|)$$

by the inequality corresponding to (1.2) when $t\leqslant 1$. By (4.1), equality holds and the unit normal ξ to $\operatorname{reg}(\partial E)$, the regular part of ∂E, is tangent to the $\mathbb{R}^k$ factor. This implies that the m-Jacobian of the restriction f of the projection $\pi_1:M\times\mathbb{R}^k\to M$ to the regular part of ∂E is equal to 1. By Federer's coarea formula for rectifiable sets [5, 3.2.22] we get

$$H^{n-1}(\partial E)=\int_M H^{k-1}(f^{-1}(p))\,dH^m.$$

Assume that $\operatorname{sym}E$ is the tube $T(E)=M\times D$. The Euclidean isoperimetric inequality implies $H^{k-1}(f^{-1}(p))\geqslant H^{k-1}(\{p\}\times\partial D)$ and so $H^{n-1}(\partial E)\geqslant H^{n-1}(\partial T(E))$, again by the coarea formula. As $P(E)=P(\operatorname{sym}E)=P(T(E))$, we get $H^{k-1}(f^{-1}(p))=H^{k-1}(\partial D)$ for H^m-a.e. $p\in M$ and so $\pi_1^{-1}(p)$ is equal to a disc $\{p\}\times D_p$ for H^m- a.e. $p\in M$.

The fact that ξ is tangent to $\mathbb{R}^k$ in $\operatorname{reg}(\partial E)$ implies that $\operatorname{reg}(\partial E)$ is locally a cylinder of the form $U\times S$, where $U\subset M$ is an open set and $S\subset\mathbb{R}^k$ is a smooth hypersurface. Hence the discs D_p are centered at the same point, i.e., E is the translation of a normalized tube, what proves the theorem.

Remark 4.1. The equivariant version of Theorem 2 in Morgan and Ros [14], together with Corollary 2.2, can be used to prove Theorem 1.1 for small dimensions.

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