EXAMPLES OF AREA-MINIMIZING SURFACES IN THE SUBRIEMANNIAN HEISENBERG GROUP $\mathbb{H}^1$ WITH LOW REGULARITY

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ABSTRACT. We give new examples of entire area-minimizing t-graphs in the subriemannian Heisenberg group $\mathbb{H}^1$. Most of the examples are locally lipschitz in Euclidean sense. Some regular examples have prescribed singular set consisting of either a horizontal line or a finite number of horizontal halflines extending from a given point. Amongst them, a large family of area-minimizing cones is obtained.

1. INTRODUCTION

Variational problems related to the subriemannian area in the Heisenberg group $\mathbb{H}^1$ have received great attention recently. A major question in this theory is the regularity of minimizers. A related one is the construction of examples with low regularity properties. The study of minimal surfaces in subriemannian geometry was initiated in the paper by Garofalo and Nhieu [22]. Later Pauls [27] constructed minimal surfaces in $\mathbb{H}^1$ as limits of minimal surfaces in Nil manifolds, the riemannian Heisenberg groups. Cheng, Hwang and Yang [9] have studied the weak solutions of the minimal surface equation for t-graphs and have proven existence and uniqueness results. Regularity of minimal surfaces, assuming that they are least C^1, has been treated in the papers by Pauls [28] and Cheng, Hwang and Yang [10]. We would like also to mention the recently distributed notes by Bigolin and Serra Cassano [5], where they obtain regularity properties of an $\mathbb{H}$-regular surface from regularity properties of its horizontal unit normal. Interesting examples of minimal surfaces which are not area-minimizing are obtained in [11]. See also [13]. Smoothness of lipschitz minimal intrinsic graphs in Heisenberg groups $\mathbb{H}^n$, for $n > 1$, has been recently obtained by Capogna, Citti and Manfredini [6].

Characterization in $\mathbb{H}^1$ of solutions of the Bernstein problem for C^2 surfaces has been obtained by Cheng, Hwang, Malchiodi and Yang [8], and Ritoré and Rosales [29] for t-graphs, and by Barone Adessi, Serra Cassano and Vittone [4] and Garofalo and Pauls [23] for vertical graphs.

Additional contributions concerning variational problems related to the subriemannian area in the Heisenberg groups include [26], [2], [8], [9], [10], [21], [21], [20], [19], [18], [17], [16], [25], [29]. The recent monograph by Capogna, Danielli, Pauls and Tyson [7] gives a recent overview of the subject with an exhaustive list of references. We would like

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to stress that, in $\mathbb{H}^1$, the condition $H \equiv 0$ is not enough to guarantee that a given surface of class C^2 is even a stationary point for the area functional, see Ritoré and Rosales [29], and Cheng, Hwang and Yang [9] for minimizing t-graphs.

The aim of this paper is to provide new examples in $\mathbb{H}^1$ of Euclidean locally lipschitz area-minimizing entire graphs over the xy-plane.

In section 3 we construct the basic examples. We start from a given horizontal line L, and a monotone angle function $\alpha : L \to (0, \pi)$ over this line. For each $p \in L$, we consider the two horizontal halflines extending from p making an angle $\pm\alpha(p)$ with L. We prove that in this way we always obtain an entire graph over the xy-plane which is Euclidean locally lipschitz and area-minimizing. The angle function α is only assumed to be continuous and monotone. Of course, further regularity on α yields more regularity on the graph. In case α is at least C^2 we get that the associated surface is $C^{1,1}$.

The surfaces in section 3 are the building blocks for our next construction in section 4. We fix a point $p \in \mathbb{H}^1$, and a family of counter-clockwise oriented horizontal halflines R_1, ..., R_n extending from p. We choose the bisector L_i of the wedge determined by R_{i-1} and R_i, and we consider angle functions $\alpha_i : L_i \to (0, \pi)$ which are continuous, nonincreasing as a function of the distance to p, and such that $\alpha(p)$ is equal to the angle between L_i and R_i. For every $q \in L_i$, we consider the halflines extending from q with angles $\pm\alpha_i(q)$. In this way we also a family of area-minimizing t-graphs which are Euclidean locally lipschitz. In case the obtained surface is regular enough we have that the singular set is precisely $\bigcup_{i=1}^n L_i$. If the angle functions α_i are constant, then we obtain area-minimizing cones (the original motivation of this paper), which are Euclidean locally $C^{1,1}$ minimizers, and C^∞ outside the singular set $\bigcup_{i=1}^n L_i$. For a single halfline L extending from the origin and an angle function $\alpha : L \to (0, \pi)$, continuous and nonincreasing as a function of the distance to 0, we patch the graph obtained over a wedge of the xy-plane with the plane $t = 0$ along the halflines extending from 0 making an angle $\alpha(0)$ with L. When α is constant we get again an area-minimizing cone which is Euclidean locally lipschitz. These cones are a generalization of the one obtained by Cheng, Hwang and Yang [9, Ex. 7.2].

An interesting consequence of this construction is that we get a large number of Euclidean locally $C^{1,1}$ area-minimizing cones with prescribed singular set consisting on either a horizontal line or a finite number of horizontal halflines extending from a given point. It is an open question to decide if these examples are the only area-minimizing cones, together with vertical halfspaces and the example by Cheng, Hwang and Yang [9, Ex. 7.2] with a singular halfline and its generalizations in the last section. The importance of tangent cones has been recently stressed in [1].

2. Preliminaries

The *Heisenberg group* $\mathbb{H}^1$ is the Lie group $(\mathbb{R}^3, *)$, where the product $*$ is defined, for any pair of points $[z, t]$, $[z', t'] \in \mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$, as

$$[z, t] * [z', t'] := [z + z', t + t' + \operatorname{Im}(z\overline{z}')], \qquad (z = x + iy).$$

For $p \in \mathbb{H}^1$, the *left translation* by p is the diffeomorphism $L_p(q) = p * q$. A basis of left invariant vector fields (i.e., invariant by any left translation) is given by

$$X := \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \qquad Y := \frac{\partial}{\partial y} - x\,\frac{\partial}{\partial t}, \qquad T := \frac{\partial}{\partial t}.$$

The *horizontal distribution* $\mathcal{H}$ in $\mathbb{H}^1$ is the smooth planar one generated by X and Y. The *horizontal projection* of a vector U onto $\mathcal{H}$ will be denoted by U_H. A vector field U is called *horizontal* if $U = U_H$. A *horizontal curve* is a C^1 curve whose tangent vector lies in the horizontal distribution.

We denote by $[U, V]$ the Lie bracket of two C^1 vector fields U, V on $\mathbb{H}^1$. Note that $[X, T] = [Y, T] = 0$, while $[X, Y] = -2T$. The last equality implies that $\mathcal{H}$ is a bracket generating distribution. Moreover, by Frobenius Theorem we have that $\mathcal{H}$ is nonintegrable. The vector fields X and Y generate the kernel of the (contact) 1-form $\omega := -y\,dx + x\,dy + dt$.

We shall consider on $\mathbb{H}^1$ the (left invariant) Riemannian metric $g = \langle \cdot, \cdot \rangle$ so that $\{X, Y, T\}$ is an orthonormal basis at every point, and the associated Levi-Civitá connection D. The modulus of a vector field U will be denoted by $|U|$.

Let $\gamma : I \to \mathbb{H}^1$ be a piecewise C^1 curve defined on a compact interval $I \subset \mathbb{R}$. The *length* of γ is the usual Riemannian length $L(\gamma) := \int_I |\dot\gamma|$, where $\dot\gamma$ is the tangent vector of γ. For two given points in $\mathbb{H}^1$ we can find, by Chow's connectivity Theorem [24, p. 95], a horizontal curve joining these points. The *Carnot-Carathédory distance* d_{cc} between two points in $\mathbb{H}^1$ is defined as the infimum of the length of horizontal curves joining the given points. A *geodesic* $\gamma : \mathbb{H}^1 \to \mathbb{R}$ is a horizontal curve which is a critical point of length under variations by horizontal curves. They satisfy the equation

$$D_{\dot\gamma}\dot\gamma + 2\lambda\, J(\dot\gamma) = 0, \tag{2.1}$$

where $\lambda \in \mathbb{R}$ is the *curvature* of the geodesic, and J is the $\pi/2$-degrees oriented rotation in the horizontal distribution. Geodesics in $\mathbb{H}^1$ with $\lambda = 0$ are horizontal straight lines. The reader is referred to the section on geodesics in [29] for further details.

The *volume* $|\Omega|$ of a Borel set $\Omega \subseteq \mathbb{H}^1$ is the Riemannian volume of the left invariant metric g, which coincides with the Lebesgue measure in $\mathbb{R}^3$. We shall denote this volume element by dv_g. The *perimeter* of $E \subset \mathbb{H}^1$ in an open subset $\Omega \subset \mathbb{H}^1$ is defined as

$$|\partial E|(\Omega) := \sup\left\{ \int_\Omega \operatorname{div} U\, dv_g : U \text{ horizontal and } C^1, |U| \leqslant 1, \operatorname{supp}(U) \subset \Omega \right\}, \tag{2.2}$$

where $\operatorname{supp}(U)$ is the support of U. A set $E \subset \mathbb{H}^1$ is of *locally finite perimeter* if $\mathcal{P}(E, \Omega) < +\infty$ for any bounded open set $\Omega \subset \mathbb{H}^1$. A set of locally finite perimeter has a measurable *horizontal unit normal* ν_E, that satisfies the following divergence theorem [17, Corollary 7.6]: if U is a horizontal vector field with compact support, then

$$\int_E \operatorname{div} U\, dv_g = \int_{\mathbb{H}^1} \langle U, \nu_E \rangle\, d|\partial E|.$$

If $E \subset \mathbb{H}^1$ has Euclidean lipschitz boundary, then [17, Corollary 7.7]

$$|\partial E|(\Omega) = \int_{\partial E \cap \Omega} |N_H|\, d\mathcal{H}^2, \tag{2.3}$$

where N is the outer unit normal to ∂E, defined $\mathcal{H}^2$-almost everywhere. Here $\mathcal{H}^2$ is the 2-dimensional riemannian Hausdorff measure.

Let $\Omega \subset \mathbb{H}^1$ be an open set. We say that $E \subset \mathbb{H}^1$ of locally finite perimeter is *area-minimizing* in Ω if, for any set F such that $E = F$ outside Ω we have

$$|\partial E|(\Omega) \leqslant |\partial F|(\Omega).$$

The following extension of the divergence theorem will be needed to prove the area-minimizing property of sets of locally finite perimeter

Theorem 2.1. *Let $E \subset \mathbb{H}^1$ be a set of locally finite perimeter, $B \subset \mathbb{H}^1$ a set with piecewise smooth boundary, and U a C^1 horizontal vector field in* $\operatorname{int}(B)$ *that extends continuously to the boundary of B. Then*

$$\int_{E\cap B} \operatorname{div} U \, dv_g = \int_B \langle U, \nu_E \rangle \, d|\partial E| + \int_E \langle U, \nu_B \rangle \, d|\partial B|. \tag{2.4}$$

Proof. The proof is modelled on [15, § 5.7]. Let s denote the riemannian distance function to $\mathbb{H}^1 - B$. For $\varepsilon > 0$, define

$$h_\varepsilon(p) := \begin{cases} 1, & \varepsilon \leqslant s(p), \\ s(p)/\varepsilon, & 0 \leqslant s(p) \leqslant \varepsilon, \end{cases}$$

Then h_ε is a lipschitz function (in riemannian sense). For any smooth h with compact support in B we have $\operatorname{div}(hU) = h \operatorname{div}(U) + \langle \nabla h, U \rangle$. By applying the divergence theorem for sets of locally finite perimeter [17] we get

$$\int_{\mathbb{H}^1} h \,\langle U, \nu_E \rangle \, d|\partial E| = \int_E h \, \operatorname{div}(U) + \int_E \langle \nabla h, U \rangle.$$

By approximation, this formula is also valid for h_ε. Taking limits when $\varepsilon \to 0$ we have $\mathcal{H}_\varepsilon \to \chi_B$. By the coarea formula for lipschitz functions

$$\frac{1}{\varepsilon} \int_{\{0 \leqslant s \leqslant \varepsilon\}} \chi_E \,\langle \nabla s, U \rangle = \frac{1}{\varepsilon} \int_0^\varepsilon \left\{ \int_{\{s=r\}} \chi_E \,\langle \nabla s, U \rangle \, d\mathcal{H}^2 \right\} dr,$$

and, taking again limits when $\varepsilon \to 0$ and calling N_B to the riemannian outer unit normal to ∂B (defined except on a small set), we have

$$\lim_{\varepsilon \to 0} \int_E \langle \nabla h_\varepsilon, U \rangle = \int_{\partial B} \chi_E \langle N_B, U \rangle \, d\mathcal{H}^2 = \int_E \langle \nu_B, U \rangle \, d|\partial B|.$$

Hence (2.4) is proved. □

For a C^1 surface $\Sigma \subset \mathbb{H}^1$ the *singular set* Σ_0 consists of those points $p \in \Sigma$ for which the tangent plane $T_p\Sigma$ coincides with the horizontal distribution. As Σ_0 is closed and has empty interior in Σ, the *regular set* $\Sigma - \Sigma_0$ of Σ is open and dense in Σ. It was proved in [14, Lemme 1], see also [3, Theorem 1.2], that, for a C^2 surface, the Hausdorff dimension with respect to the Riemannian distance on $\mathbb{H}^1$ of Σ_0 is less than two.

If Σ is a C^1 oriented surface with unit normal vector N, then we can describe the singular set $\Sigma_0 \subset \Sigma$, in terms of N_H, as $\Sigma_0 = \{p \in \Sigma : N_H(p) = 0\}$. In the regular part $\Sigma - \Sigma_0$, we can define the *horizontal unit normal vector* ν_H, as in [12], [30] and [23] by

$$\nu_H := \frac{N_H}{|N_H|}. \tag{2.5}$$

Consider the *characteristic vector field* Z on $\Sigma - \Sigma_0$ given by

$$Z := J(\nu_H). \tag{2.6}$$

As Z is horizontal and orthogonal to ν_H, we conclude that Z is tangent to Σ. Hence Z_p generates the intersection of $T_p\Sigma$ with the horizontal distribution. The integral curves of Z in $\Sigma - \Sigma_0$ will be called *characteristic curves* of Σ. They are both tangent to Σ and horizontal. Note that these curves depend on the unit normal N to Σ. If we define

$$S := \langle N, T\rangle\, \nu_H - |N_H|\, T, \tag{2.7}$$

then $\{Z_p, S_p\}$ is an orthonormal basis of $T_p\Sigma$ whenever $p \in \Sigma - \Sigma_0$.

In the Heisenberg group $\mathbb{H}^1$ there is a one-parameter group of *dilations* $\{\varphi_s\}_{s\in\mathbb{R}}$ generated by the vector field

$$W := xX + yY + 2tT. \tag{2.8}$$

We may compute φ_s in coordinates to obtain

$$\varphi_s(x_0, y_0, t_0) = (e^s x_0, e^s y_0, e^{2s} t_0). \tag{2.9}$$

Conjugating with left translations we get the one-parameter family of dilations $\varphi_{p,s} := L_p \circ \varphi_s \circ L_p^{-1}$ with center at any point $p \in \mathbb{H}^1$. A set $E \subset \mathbb{H}^1$ is a *cone of center* p if $\varphi_{p,s}(E) \subset E$ for all $s \in \mathbb{R}$.

Any isometry of $(\mathbb{H}^1, g)$ leaving invariant the horizontal distribution preserves the area of surfaces in $\mathbb{H}^1$. Examples of such isometries are left translations, which act transitively on $\mathbb{H}^1$. The Euclidean rotation of angle θ about the t-axis given by

$$(x, y, t) \mapsto r_\theta(x, y, t) = (\cos\theta\, x - \sin\theta\, y, \sin\theta\, x + \cos\theta\, y, t),$$

is also an area-preserving isometry in $(\mathbb{H}^1, g)$ since it transforms the orthonormal basis $\{X, Y, T\}$ at the point p into the orthonormal basis $\{\cos\theta\, X + \sin\theta\, Y, -\sin\theta\, X + \cos\theta\, Y, T\}$ at the point $r_\theta(p)$.

3. Examples with one singular line

Consider the x-axis in $\mathbb{H}^1 = \mathbb{R}^3$parametrized by $\Gamma(v) := (v, 0, 0)$. Take a non-increasing continuous function $\alpha : \mathbb{R} \to (0, \pi)$. For every $v \in \mathbb{R}$, consider two horizontal halflines L_v^+, L_v^- extending from $\Gamma(v)$ with angles $\alpha(v)$ and $-\alpha(v)$, respectively. The tangent vectors to these curves at $\Gamma(v)$ are given by $\cos\alpha(v)\, X_{\Gamma(v)} + \sin\alpha(v)\, Y_{\Gamma(v)}$ and $\cos\alpha(v)\, X_{\Gamma(v)} - \sin\alpha(v)\, Y_{\Gamma(v)}$, respectively.

The parametric equations of this surface are given by

$$(v, w) \mapsto \begin{cases} (v + w\cos\alpha(v), w\sin\alpha(v), -vw\sin\alpha(v)), & w \geqslant 0, \\ (v + |w|\cos\alpha(v), -|w|\sin\alpha(v), v|w|\sin\alpha(v)), & w \leqslant 0, \end{cases} \tag{3.1}$$

One can eliminate the parameters v, w to get the implicit equation

$$t + xy - y|y| \cot\alpha\left(-\frac{t}{y}\right) = 0.$$

Letting $\beta := \cot(\alpha)$, we get that β is a continuous non-decreasing function, and that the surface Σ_β defined by the parametric equations (3.1) is given by the implicit equation

$$0 = f_\beta(x,y,t) := t + xy - y|y|\,\beta\left(-\frac{t}{y}\right). \tag{3.2}$$

Observe that, because of the monotonicity condition on α, the projection of relative interiors of the open horizontal halflines to the xy-plane together with the planar x-axis L_x produce a partition of the plane. Since Σ_β is the union of the horizonal lifting of these planar halflines and the x-axis to $\mathbb{H}^1$, it is the graph of a continuous function $u_\beta : \mathbb{R}^2 \to \mathbb{R}$. For $(x,y) \in \mathbb{R}^2$, the only point in the intersection of Σ_β with the vertical line passing through (x,y) is precisely $(x,y,u_\beta(x,y))$. Obviously

$$f_\beta(x,y,u_\beta(x,y)) = 0. \tag{3.3}$$

For any $(x,y) \in \mathbb{R}^2$, denote by $\xi_\beta(x,y)$ the only value $v \in \mathbb{R}$ so that either $\Gamma(v) = (x,y,0)$, or $(x,y,u_\beta(x,y))$ is contained in one of the two above described halflines leaving $\Gamma(v)$. Trivially $\xi_\beta(x,0) = x$. Using (3.1) one checks that

$$\xi_\beta(x,y) = -\frac{u_\beta(x,y)}{y}, \qquad y \neq 0. \tag{3.4}$$

Recalling that $\alpha = \cot^{-1}(\beta)$, we see that the mapping

$$(v,w) \mapsto \begin{cases} (v + w\cos\alpha(v), w\sin\alpha(v)), & w \geqslant 0, \\ (v + |w|\cos\alpha(v), -|w|\sin\alpha(v)), & w \leqslant 0, \end{cases}$$

is an homeomorphism of $\mathbb{R}^2$ whose inverse is given by

$$(x,y) \mapsto (\xi_\beta(x,y), \operatorname{sgn}(y)\,|(x - \xi_\beta(x,y), y)|),$$

where $\operatorname{sgn}(y) := y/|y|$ for $y \neq 0$. Hence $\xi_\beta : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. By (3.4), the function $u_\beta(x,y)/y$ admits a continuous extension to $\mathbb{R}^2$.

Let us analyze first the properties of u_β for regular β

Lemma 3.1. *Let $\beta \in C^k(\mathbb{R})$, $k \geqslant 2$, be a non-decreasing function. Then*

(i) *u_β is a C^k function in $\mathbb{R}^2 - L_x$,*
(ii) *u_β is merely $C^{1,1}$ near the x-axis when $\beta \neq 0$,*
(iii) *u_β is C^∞ in $\xi^{-1}(I)$ when $\beta \equiv 0$ on any open set $I \subset \mathbb{R}$, and*
(iv) *Σ_β is area-minimizing.*
(v) *The projection of the singular set of Σ_β to the xy-plane is L_x.*

Proof. Along the proof we shall often drop the subscript β for f_β, u_β, ξ_β and Σ_β.

The proof of 1 is just an application of the Implicit Function Theorem since f_β is a C^k function for $y \neq 0$ when β is C^k.

To prove 2 we compute the partial derivatives of u_β for $y \neq 0$. They are given by

$$(u_\beta)_x(x,y) = \frac{-y}{1+|y|\,\beta'\big(\xi_\beta(x,y)\big)}, \tag{3.5}$$

$$(u_\beta)_y(x,y) = \frac{-x+|y|\,\big(2\beta\big(\xi_\beta(x,y)\big)-\beta'\big(\xi_\beta(x,y)\big)\,\xi_\beta(x,y)\big)}{1+|y|\,\beta'\big(\xi_\beta(x,y)\big)}. \tag{3.6}$$

Since $u_\beta(x,0)=0$ for all $x\in\mathbb{R}$ we get $(u_\beta)_x(x,0)=0$. On the other hand

$$(u_\beta)_y(x,0) = \lim_{y\to 0}\frac{u_\beta(x,y)}{y} = -\lim_{y\to 0}\xi_\beta(x,y) = -\xi_\beta(x,0) = -x.$$

The limits, when $y\to 0$, of (3.5) and (3.6) can be computed using (3.4). We conclude that the first derivatives of u_β are continuous functions and so u_β is a C^1 function on $\mathbb{R}^2$. To see that u_β is merely lipschitz, we get from (3.6) and (3.4)

$$\begin{aligned}(u_\beta)_{yy}(x,0) &= \lim_{y\to 0^\pm}\frac{(u_\beta)_y(x,y)+x}{y}\\ &= \lim_{y\to 0^\pm}\frac{|y|\,\big(2\beta\big(\xi_\beta(x,y)\big)-\beta'\big(\xi_\beta(x,y)\big)\,\xi_\beta(x,y)+x\beta'\big(\xi_\beta(x,y)\big)\big)}{y\,\big(1+|y|\,\beta'\big(\xi_\beta(x,y)\big)\big)}\\ &= \pm 2\beta(x).\end{aligned}$$

Hence side derivatives exist, but they do not coincide unless $\beta(x)=0$.

As $u_\beta\big|_{\xi^{-1}(I)} = -xy$, 3 follows easily .

To prove 4 we use a calibration argument. We shall drop the subscript β to simplify the notation. Let $F\subset\mathbb{H}^1$ such that $F=E$ outside a Euclidean ball B centered at the origin. Let $H^1 := \{(x,y,t) : y\geqslant 0\}$, $H^2 := \{(x,y,t) : y\leqslant 0\}$, $\Pi := \{(x,y,t) : y=0\}$. Vertical translations of the horizontal unit normal ν_E, defined outside Π, provide two vector fields U^1 on H^1, and U^2 on H^2. They are C^2 in the interior of the halfspaces and extend continuously to the boundary plane Π. As in the proof of Theorem 5.3 in [29], we see that

$$\operatorname{div} U^i = 0, \qquad i=1,2,$$

in the interior of the halfspaces. Here $\operatorname{div} U$ is the riemannian divergence of the vector field U. Observe that the vector field Y is the riemannian unit normal, and also the horizontal unit normal, to the plane Π. We may apply the divergence theorem to get

$$\begin{aligned}0 = \int_{E\cap\operatorname{int}(H^i)\cap B}\operatorname{div} U^i &= \int_E \big\langle U^i, \nu_{\operatorname{int}(H^i)\cap B}\big\rangle\, d|\partial(\operatorname{int}(H^i)\cap B)|\\ &\quad + \int_{\operatorname{int}(H^i)\cap B}\big\langle U^i,\nu_E\big\rangle\, d|\partial E|.\end{aligned}$$

Let $D := \Pi\cap\overline{B}$. Then, for every $p\in D$, we have $\nu_{\operatorname{int}(H^1)\cap B} = -Y$, $\nu_{\operatorname{int}(H^2)\cap B} = Y$, and $U^1 = J(v)$, $U^2 = J(w)$, where $v-w$ is proportional to Y, by the construction of Σ_β. Hence

$$\big\langle U^1,\nu_{\operatorname{int}(H^1)\cap B}\big\rangle + \big\langle U^2,\nu_{\operatorname{int}(H^2)\cap B}\big\rangle = \big\langle v-w, J(Y)\big\rangle = 0, \qquad p\in D.$$

Adding the above integrals we obtain

$$0=\sum_{i=1,2}\int_E\left\langle U^i,\nu_B\right\rangle d|\partial B|+\sum_{i=1,2}\int_{B\cap\operatorname{int}(H^i)}\left\langle U^i,\nu_E\right\rangle d|\partial E|.$$

We apply the same arguments to the set F and, since $E=F$ on ∂B we conclude

$$\sum_{i=1,2}\int_{B\cap\operatorname{int}(H^i)}\left\langle U^i,\nu_E\right\rangle d|\partial E|=\sum_{i=1,2}\int_{B\cap\operatorname{int}(H^i)}\left\langle U^i,\nu_F\right\rangle d|\partial F|. \tag{3.7}$$

As E is a subgraph, $|\partial E|(\Pi)=0$ and so

$$|\partial E|(B)=\sum_{i=1,2}\int_{B\cap\operatorname{int}(H^i)}\left\langle U^i,\nu_B\right\rangle d|\partial E|.$$

Cauchy-Schwarz inequality and the fact that $|\partial F|$ is a positive measure imply

$$\sum_{i=1,2}\int_{B\cap\operatorname{int}(H^i)}\left\langle U^i,\nu_F\right\rangle d|\partial F|\leqslant|\partial F|(B),$$

which implies 4.

To prove 5 simply take into account that the projection of the singular set of Σ_β to the xy-plane is composed of those points (x,y) such that $(u_\beta)-x-y=(u_\beta)_y+x=0$. From (3.5) we get that $(u_\beta)_x-y=0$ if and only if

$$y\left(2+|y|\,\beta'(\xi_\beta(x,y))\right)=0,$$

i.e, when $y=0$. In this case, from (3.6), we see that equation $(u_\beta)_y+x=0$ is trivially satisfied. □

We now prove the general properties of Σ_β from Lemma 3.1

Proposition 3.2. *Let $\beta:\mathbb{R}\to\mathbb{R}$ be a continuous non-decreasing function. Let u_β be the only solution of equation* (3.3)*, Σ_β the graph of u_β, and E_β the subgraph of u_β. Then*

(i) *u_β is locally lipschitz in Euclidean sense,*
(ii) *E_β is a set of locally finite perimeter in $\mathbb{H}^1$, and*
(iii) *Σ_β is area-minimizing in $\mathbb{H}^1$.*

Proof. Let

$$\beta_\varepsilon(x):=\int_{\mathbb{R}}\beta(y)\,\eta_\varepsilon(x-y)\,dy$$

the usual convolution, where η is a Dirac function and $\eta_\varepsilon(x):=\eta(x/\varepsilon)$, see [15]. Then β_ε is a C^∞ non-decreasing function, and β_ε converges uniformly, on compact subsets of $\mathbb{R}$, to β. Let $u=u_\beta$, $u_\varepsilon=u_{\beta_\varepsilon}$, $f=f_\beta$, $f_\varepsilon=f_{\beta_\varepsilon}$.

Let $D\subset\mathbb{R}^2$ be a bounded subset. To check that u is lipschitz on D it is enough to prove that the first derivatives of u_ε are uniformly bounded on D.

From (3.3) we get

$$\xi(x,y)+|y|\,\beta\big(\xi(x,y)\big)=x,\qquad y\neq 0.$$

For y fixed, define the continuous strictly increasing function

$$\rho_y(x):=x+|y|\,\beta(x).$$

Hence we get

$$\xi(x,y) = \rho_y^{-1}(x). \tag{3.8}$$

We can also define $(\rho_\varepsilon)_y(x) := x + |y|\beta_\varepsilon(x)$. Equation (3.8) holds replacing u, β by u_ε, β_ε.

Since $\rho_y^{-1}(x) = \xi(x,y)$, we conclude that ρ_y^{-1} is a continuous function that depends continuously on y.

Let us estimate

$$|(\rho_\varepsilon)_y^{-1}(x) - \rho_y^{-1}(x)|.$$

Let $z_\varepsilon := (\rho_\varepsilon)_y^{-1}(x)$, $z = \rho_y^{-1}(x)$. Then $x = (\rho_\varepsilon)_y(z_\varepsilon) = \rho_y(z)$ and we have, assuming $z_\varepsilon \geqslant z$.

$$\begin{aligned} 0 &= (\rho_\varepsilon)_y(z_\varepsilon) - \rho_y(z) = z_\varepsilon + |y|\beta_\varepsilon(z_\varepsilon) - \big(z + |y|\beta(z)\big) \\ &= (z_\varepsilon - z) + |y|\,\big(\beta_\varepsilon(z_\varepsilon) - \beta_\varepsilon(z)\big) + |y|\,\big(\beta_\varepsilon(z) - \beta(z)\big) \\ &\geqslant (z_\varepsilon - z) + |y|\,\big(\beta_\varepsilon(z) - \beta(z)\big). \end{aligned}$$

A similar computation can be performed for $z_\varepsilon \geqslant z$. The consequence is that

$$|z_\varepsilon - z| \leqslant |y|\,|\beta_\varepsilon(z) - \beta(z)|,$$

or, equivalently,

$$|(\rho_\varepsilon)_y^{-1}(x) - \rho_y^{-1}(x)| \leqslant |y|\,|\beta_\varepsilon(\rho_y^{-1}(x)) - \beta(\rho_y^{-1}(x))|.$$

As $\beta_\varepsilon \to \beta$ uniformly on compact subsets of $\mathbb{R}$, we have uniform convergence of $(\rho_\varepsilon)_y^{-1}(x)$ to $\rho_y^{-1}(x)$ on compact subsets of $\mathbb{R}^2$. This also implies the uniform convergence of $\xi_\varepsilon(x,y)$ to $\xi(x,y)$ on compact subsets. Hence also $u_\varepsilon(x,y)$ converges uniformly to $u(x,y)$ on compact subsets of $\mathbb{R}^2$.

From (3.5) and (3.6) we have

$$\begin{aligned} &|(u_\varepsilon)_x(x,y)| \leqslant |y|, \\ &|(u_\varepsilon)_y(x,y)| \leqslant |x| + 2\,|y|\,\big|\beta_\varepsilon\big(\xi_\varepsilon(x,y)\big)\big| + \big|\xi_\varepsilon(x,y)\big|. \end{aligned}$$

As $\beta_\varepsilon \to \beta$ and $\xi_\varepsilon(x,y) \to \xi(x,y)$ uniformly on compact subsets, we have that the first derivatives of u_ε are uniformly bounded on compact subsets. Hence u is locally lipschitz.

The subgraph of u_β is a set of locally finite perimeter in $\mathbb{H}^1$ since its boundary is locally lipschitz by 1. This follows from [17] and proves 2.

To prove 3 we use approximation and the calibration argument. Let $F \subset \mathbb{H}^1$ so that $F = E$ outside a Euclidean ball B centered at the origin. For the functions β_ε, consider the vector fields U_ε^i obtained by translating vertically the horizontal unit normal to the surface Σ_ε. We repeat the arguments on the proof of 4 in Lemma 3.1 to conclude as in (3.7) that

$$\sum_{i=1,2} \int_{B\cap \mathrm{int}(H^i)} \langle U_\varepsilon^i, \nu_E \rangle \, d|\partial E| = \sum_{i=1,2} \int_{B\cap \mathrm{int}(H^i)} \langle U_\varepsilon^i, \nu_F \rangle \, d|\partial F|.$$

Trivially we have

$$\sum_{i=1,2} \int_{B\cap \mathrm{int}(H^i)} \langle U_\varepsilon^i, \nu_F \rangle \, d|\partial F| \leqslant |\partial F|(B).$$

On the other hand, U^i_ε converges uniformly, on compact subsets, to U^i by Lemma 3.3. Passing to the limit when $\varepsilon \to 0$ and taking into account that $U^i = \nu_E$ we conclude

$$|\partial E|(B) \leqslant |\partial F|(B),$$

as desired. □

Lemma 3.3. *Let β be a continuous non-decreasing function. Then the horizontal unit normal of Σ_β is given, in $\{X, Y\}$-coordinates, by*

$$\nu_\beta(x,y) = \left(\frac{1}{(1+\beta^2)^{1/2}}, \frac{-\operatorname{sgn}(y)\,\beta}{(1+\beta^2)^{1/2}}\right)(\xi_\beta(x,y)), \qquad y \neq 0. \tag{3.9}$$

Moreover, ν_β admits continuous extensions to $y = 0$ from both sides of this line.

Proof. Since u_β is lipschitz, it is differentiable almost everywhere on $\mathbb{R}^2$. On these points,

$$\nu_\beta(x,y) = ((u_\beta)_x - y, (u_\beta)_y + x).$$

The function $-u_\beta(x,y)/y$ is constant along the lines $(x_0, 0) + \lambda\,(1+\beta^2)^{-1/2}(\beta, \pm 1)(x_0)$, for $\lambda \geqslant 0$. Let $y \geqslant 0$. From (3.2) we have

$$0 = -x_0 + x - y\,\beta(x_0).$$

Let $v := (1+\beta^2)^{-1/2}(\beta, 1)(x_0)$. Then $v(-u_\beta(x,y)/y) = 0$. Hence for almost every point on almost every line, we have

$$\beta(x_0)\,(u_\beta)_x + (u_\beta)_y = -x_0.$$

Hence we have

$$(u_\beta)_y + x = -x_0 - \beta(x_0)\,(u_\beta)_x + x_0 + y\,\beta(x_0) = \beta(x_0)\,(-(u_\beta)_x + y).$$

We conclude that the horizontal unit normal is proportional to $(1, -\beta)$, which implies (3.9). The case $y \leqslant 0$ is handled similarly. □

Example 3.4. Taking $\beta(x) := x$ we get

$$u_\beta(x,y) = -\frac{xy}{1+|y|},$$

which is a Euclidean $C^{1,1}$ graph.

Another family of interesting examples are the minimal cones obtained by taking the constant function $\beta(x) := \beta_0$. In this case we get

$$u_\beta(x,y) = -xy + \beta_0\,y|y|.$$

In this case Σ_β is a $C^{1,1}$ surface which is invariant by the dilations centered at any point of the singular line.

Take now

$$\beta(x) := \begin{cases} 0, & x \leqslant 0, \\ x, & x \geqslant 0. \end{cases}$$

In this case we obtain the graph

$$u_\beta(x,y) := \begin{cases} -xy, & x \leqslant 0, \\ -\frac{xy}{1+|y|}, & x \geqslant 0, \end{cases}$$

which is simply locally Lipschitz.

This example was mentioned to me by Scott Pauls. Consider now a continuous non-decreasing function $\beta : \mathbb{R} \to \mathbb{R}$, constant outside the Cantor set $C \subset [0,1]$ with $\beta(0) = 0$, $\beta(1) = 1$. Then the associated surface Σ_β is an area-minimizing surface in $\mathbb{H}^1$.

4. Examples with several singular halflines meeting at a point

Let $\alpha_1^0, \dots, \alpha_k^0$, be a family of positive angles so that

$$\sum_{i=1}^{k} \alpha_i^0 = \pi.$$

Let r_β be the rotation of angle β around the origin in $\mathbb{R}^2$. Consider a family of closed halflines $L_i \subset \mathbb{R}^2$, $i \in \mathbb{Z}_k$, extending from the origin, so that $r_{\alpha_i^0+\alpha_{i+1}^0}(L_i) = L_{i+1}$. Finally, define $R_i := r_{\alpha_i^0}(L_i)$. (An alternative way of defining this configuration is to start from a family of counter-clockwise oriented halflines $R_i \subset \mathbb{R}^2$, $i \in \mathbb{Z}_k$, choosing L_i, $i \in \mathbb{Z}_k$, as the bisector of the angle determined by R_{i-1} and R_i, and defining α_i^0 as the angle between L_i and R_i). Define W_i as the closed wedge, containing L_i, bordered by R_{i-1} and R_i.

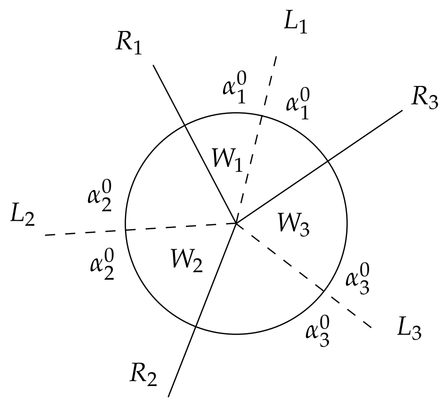

FIGURE 1. The initial configuration with three halflines L_1, L_2, L_3.

For every $i \in \mathbb{Z}_k$, let $\alpha_i : [0,\infty) \to (0,\pi)$ be a continuous nonincreasing function so that $\alpha_i(0) = \alpha_i^0$, and define, as in the previous section, $\beta_i := \cot(\alpha_i)$. Let $v_i \in \mathbb{S}^1$, $i \in \mathbb{Z}_k$, be such that $L_i = \{sv_i : s \geqslant 0\}$. For every $i \in \mathbb{Z}_k$ and $s \geqslant 0$, we take the two closed halflines $L_{s,i}^{\pm}$ in $\mathbb{R}^2$ extending from the point sv_i with tangent vectors $(\cos\alpha_i(s), \pm\sin\alpha_i(s))$. In this way we cover all of $\mathbb{R}^2$. We shall define $\alpha := (\alpha_1, \dots, \alpha_k)$.

Lift $L_1, \dots, L_k$ to horizontal halflines $L_1', \dots, L_k'$ in $\mathbb{H}^1$ from the origin, and $L_{s,i}^{\pm}$ to horizontal halflines in $\mathbb{H}^1$ extending from the unique point in L_i' projecting onto sv_i. In this way we obtain a continuous function $u_\alpha : \mathbb{R}^2 \to \mathbb{R}$. The graph Σ_α of u_α is a topological surface in $\mathbb{H}^1$.

Obviously the angle functions $\alpha_i(s)$ can be extended continuously and preserving the monotonicity, to an angle function $\widetilde{\alpha}_i : \widetilde{L}_i \to (0,\pi)$, where $\widetilde{L}_i$ is the straight line containing the halfline L_i. The graph of u_α restricted to W_i coincides with the Euclidean locally lipschitz area-minimizing surface $u_{\widetilde{\beta}_i}$, for $\widetilde{\beta}_i := \cot\widetilde{\alpha}_i$, constructed in the previous section. So the examples in this section can be seen as pieces of the examples of the previous one patched together.

Theorem 4.1. *Under the above conditions*

(i) *The function u_α is locally lipschitz in the Euclidean sense.*
(ii) *The surface Σ_α is area-minimizing.*

Proof. It is immediate that u_α is a graph which is locally lipschitz in Euclidean sense: choose a disk $D \subset \mathbb{R}^2$. Let p, $q \in D$. Assume first that (p,q) intersects the halflines $R_1,\ldots,R_k$ transversally at the points $x_1,\ldots,x_n$. Then $[p,x_1]$, $[x_1,x_2],\ldots,[x_n,p]$ are contained in wedges and hence

$$\begin{aligned}|u_\alpha(p)-u_\alpha(q)| &\leqslant |u_\alpha(p)-u_\alpha(x_1)|+\cdots+|u_\alpha(x_n)-u_\alpha(q)|\\ &C\left(|p-x_1|+\cdots|x_n-q|\right)=C\,|p-q|,\end{aligned}$$

where C is the supremum of the Lipschitz constants of $u_{\widetilde{\beta}_i}$ restricted to D. The general case is then obtained by approximating p and q by points in the condition of the assumption.

To prove that u_α is area minimizing we first approximate α_i by smooth angle functions $(\alpha_i)_\varepsilon$ with $(\alpha_i)_\varepsilon(0)=\alpha_i(0)$. In this way we obtain a calibrating vector field which is continuous along the vertical planes passing through R_i by Lemma 3.3. This allows us to apply the calibration argument to prove the area-minimizing property of Σ_α. □

Example 4.2 (Minimizing cones)**.** Let $\alpha_i(s)=\alpha_i^0$ be a constant for all i. Then the subgraph of Σ_α is a minimizing cone with center at 0. Restricted to the interior of the wedges W_i, the surface Σ_α is $C^{1,1}$. An easy computation shows that, taking $\beta(s):=\beta_0$ in the construction of the first section, the Riemannian normal to Σ_β along the halflines $\beta_0|y|=x$, $x\geqslant 0$ (that make angle $\pm\cot^{-1}(\beta_0)$ with the positive x-axis) is given by

$$N=\frac{-2y\,X+2\beta_0|y|\,Y-T}{\sqrt{1+4y^2+4\beta_0^2y^2}}=\frac{-2y\,X+2x\,Y-T}{\sqrt{1+4x^2+4y^2}}.$$

This vector field is invariant by rotations around the vertical axis. Hence in our construction, the normal vector field to Σ_α is continuous. It is straightforward to show that it is locally lipschitz in Euclidean sense.

Example 4.3 (Area-minimizing surfaces with a singular halfline)**.** These examples are inspired by [9, Example 7.2]. We consider a halfline L extending from the origin, and an angle function $\alpha : L \to (0,\pi)$ continuous and nonincreasing as a function of the distance to the origin. We consider the union of the halflines $L^+_{\alpha(q)}$, $L^-_{\alpha(q)}$ extending from $q\in L$ with angles $\alpha(q)$, $-\alpha(q)$, respectively. We patch the area-minimizing surface defined by α in the wedge delimited by the halflines L_0^+, L_0^-, with the plane $t=0$. In this way we get an entire area-minimizing t-graph, with lipschitz regularity. In case the angle function α is constant, we get an area-minimizing cone with center 0, which is defined by the equation

$$u(x,y) := \begin{cases} -xy + \beta_0\, y|y|, & -xy + \beta_0\, y|y| \geqslant 0, \\ 0, & -xy + \beta_0\, y|y| \leqslant 0. \end{cases}$$

This surface is composed of two smooth pieces patched together along the halflines $x = \beta_0|y|$.

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