A PROOF BY CALIBRATION OF AN ISOPERIMETRIC INEQUALITY IN THE HEISENBERG GROUP $\mathbb{H}^n$

MANUEL RITORÉ

Abstract. Let D be a closed disk centered at the origin in the horizontal hyperplane $\{t=0\}$ of the sub-Riemannian Heisenberg group $\mathbb{H}^n$, and C the vertical cylinder over D. We prove that any finite perimeter set E such that $D \subset E \subset C$ has perimeter larger than or equal to the one of the rotationally symmetric sphere with constant mean curvature of the same volume, and that equality holds only for the spheres using a recent result by Monti and Vittone [12].

1. Introduction

It was conjectured by P. Pansu in 1983 [14] that the isoperimetric regions, minimizing perimeter under a volume constraint, in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ are the topological balls enclosed by the one-parameter family $\{\mathbb{S}_\lambda\}_{\lambda>0}$ of rotationally symmetric spheres of constant mean curvature λ described in [14], see also [7], [17].

Existence of isoperimetric regions in Carnot groups was proven by G.-P. Leonardi and S. Rigot [6]. Every Carnot group $\mathbb{G}$ is equipped with an one-parameter family of dilations which has a well-known effect on the perimeter and the Haar measure, the volume of the group. Hence the isoperimetric profile of $\mathbb{G}$, the function assigning to each volume $v>0$ the infimum of the perimeter of the sets of volume v, is given by

$$I_{\mathbb{G}}(v) = C_{\mathbb{G}}\, v^{(Q-1)/Q},$$

where $Q>0$ is the homogeneous dimension of $\mathbb{G}$, and $C_{\mathbb{G}}>0$ is a constant. The isoperimetric profile must be seen as an optimal isoperimetric inequality in $\mathbb{G}$. For any finite perimeter set $E \subset \mathbb{G}$,

$$P_{\mathbb{G}}(E) \geqslant C_{\mathbb{G}}\, |E|^{(Q-1)/Q}, \tag{*}$$

where $P_{\mathbb{G}}$ is the sub-Riemannian perimeter and $|E|$ is the volume of E. Pansu's conjecture then states that, for $\mathbb{G}=\mathbb{H}^1$, which has homogeneous dimension $Q=4$, equality is attained in (*) precisely when E is the topological ball enclosed by some sphere $\mathbb{S}_\lambda$. This conjecture can be extended to the higher dimensional Heisenberg groups $\mathbb{H}^n$, $n \geqslant 2$.

Several attempts to solve this conjecture have been made. R. Monti [10] and G.-P. Leonardi and S. Masnou [7] have shown that there is no direct counterpart in $\mathbb{H}^1$ to the Brunn-Minkowski inequality in Euclidean space. In fact, such a Brunn-Minkowski type inequality would imply that the metric balls in the Carnot-Carathédory distance would be isoperimetric regions, which is known to be false [9]. The author and C. Rosales showed in [17] that the only compact rotationally symmetric C^2 hypersurfaces in

Date: July 14, 2010.
2000 *Mathematics Subject Classification.* 53C17, 53C42, 49Q20.
Key words and phrases. Sub-Riemannian geometry, Heisenberg group, area-stationary surface, constant mean curvature surface, isoperimetric problem, isoperimetric region, calibration.
Research supported by MCyT-Feder grant MTM2010-21206-C02-01.

$\mathbb{H}^n$ with constant mean curvature are the spheres $\mathbb{S}_\lambda$, see also [13]. The same authors proved in [18] that the spheres $\mathbb{S}_\lambda$ are the only compact C^2 surfaces in $\mathbb{H}^1$ which are area-stationary under a volume constraint, thus solving the isoperimetric problem assuming C^2 regularity of the solutions. R. Monti and M. Rickly [11] proved that the spheres $\mathbb{S}_\lambda$ are isoperimetric in $\mathbb{H}^1$ under the additional assumption of Euclidean convexity. In [3], D. Danielli et al provided a proof of the isoperimetric property of the spheres $\mathbb{S}_\lambda \subset \mathbb{H}^n$ in the class of sets that are the union of the graphs of a non-negative and a non-positive function and a negative graph of class C^2 over a Euclidean disk centered at the origin in the horizontal hyperplane $\{t = 0\}$ in $\mathbb{H}^n$, and enclosing the same volume above and below such hyperplane. For a description of these results, and some other approaches, the reader may consult Chapter 8 of the monograph by L. Capogna et al [2].

In this paper we extend the main result in [3]. We prove in Theorem 3.1 that, if C is the vertical cylinder in $\mathbb{H}^n$ over a closed disk D centered at the origin in the horizontal hyperplane $\{t = 0\}$, and $E \subset \mathbb{H}^n$ is a finite perimeter set so that $D \subset E \subset C$, then the perimeter of E is larger than or equal to the one of the ball $\mathbb{B}_\lambda$ enclosed by the sphere $\mathbb{S}_\lambda$ with $|\mathbb{B}_\lambda| = |E|$. Equality characterizes the spheres $\mathbb{S}_\lambda$ by a recent result of R. Monti and D. Vittone [12], who proved that a set in $\mathbb{H}^n$ of locally finite perimeter with continuous horizontal unit normal has $\mathbb{H}$-regular boundary. Theorem 3.1 can be applied to a set $E \subset \mathbb{H}^n$ rotationally symmetric with respect to a vertical axis passing through the origin and satisfying $D \subset E$. Assumption $D \subset E$ has been recently removed by R. Monti [8], who has proven, using Theorem 3.1 and a symmetrization argument, that the spheres $\mathbb{S}_\lambda \subset \mathbb{H}^n$ are isoperimetric in the class of rotationally symmetric sets of finite perimeter. Theorem 3.1 and the result of R. Monti and D. Vittone [12] together provide the only known characterization result for solutions of the isoperimetric problem in $\mathbb{H}^n$ in the class of finite perimeter sets.

One of the classical proofs of the isoperimetric inequality in $\mathbb{R}^3$ was given by H. A. Schwarz [23], and extended to higher dimensional Euclidean spaces, spheres and hyperbolic spaces in a series of papers by E. Schmidt [19], [20], [21], [22]. Schwarz's proof had two main ingredients: what it is known nowadays as Schwarz's symmetrization and a variational argument to prove that in the class of rotationally symmetric sets spheres have the smallest perimeter under a volume constraint. A unified argument, using calibrations, can be used can be used to give a proof of the second part of Schwarz's argument in a wide class of homogeneous Riemannian manifolds [16, § 1.3.1], and in sub-Riemannian manifolds. A symmetrization in the sub-Riemannian Heisenberg group $\mathbb{H}^n$ is difficult to produce due to the lack of reflections with respect to hyperplanes, on which all classical symmetrizations are based.

In the proof of Theorem 3.1 we consider the right cylinder $C \subset \mathbb{H}^n$ over a closed Euclidean disk D in the horizontal hyperplane $\{t = 0\}$. On C we construct two foliations by vertically translating the upper and the lower hemisphere of the only sphere $\mathbb{S}_\lambda$ intersecting $\{t = 0\}$ at ∂D. Using these foliations we prove that the sphere $\mathbb{S}_\lambda$ minimize the functional area $-\, n\lambda\,$volume in the class of finite perimeter sets $E \subset \mathbb{H}^n$ satisfying $D \subset E \subset C$. Then we minimize over the spheres $\mathbb{S}_\mu$ the functional area $-\, n\mu\,(\text{volume} - |E|)$ to get the desired result. The reader should compare our proof with the one given by E. Schmidt [20] of the isoperimetric property of balls in the n-dimensional sphere $\mathbb{S}^n$ in the class of rotationally symmetric sets.

We have organized this paper in two sections. In the following one we state some material needed in the proof of Theorem 3.1. Proofs of the results which are essential but cannot be found in the literature, in particular of Lemmae 2.2, 2.3, 2.4 are outlined. In section 3 we give the proof of our main result Theorem 3.1.

The author is extremely grateful to Roberto Monti and Davide Vittone for sending him a copy of their manuscript [12].

2. Preliminaries

2.1. **The Heisenberg group.** The Heisenberg group $\mathbb{H}^n$ is the Lie group $(\mathbb{R}^{2n+1}, *)$, where we consider in $\mathbb{R}^{2n+1} \equiv \mathbb{C}^n \times \mathbb{R}$ its usual differentiable structure and the product

$$(z,t) * (w,s) = (z + w, t + s + \sum_{i=1}^{n} \mathrm{Im}(z_i \bar{w}_i)).$$

A basis of left-invariant vector fields is given by $\{X_1, ..., X_n, Y_1, ..., Y_n, T\}$, where

$$X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n; \qquad T = \frac{\partial}{\partial t}.$$

The only non-trivial bracket relations are $[X_i, Y_i] = -2T$, $i = 1, \dots, n$. The *horizontal distribution* at a point $p \in \mathbb{H}^n$ is defined by $\mathcal{H}_p := \mathrm{span}\{(X_i)_p, (Y_i)_p : i = 1, \dots, n\}$.

We shall consider on $\mathbb{H}^n$ the left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ so that the basis $\{X_1, ..., X_n, Y_1, ..., Y_n, T\}$ is orthonormal. The *horizontal projection* of a vector field U in $\mathbb{H}^n$, denoted by U_H, is the orthogonal projection of U over $\mathcal{H}$.

The Levi-Civita connection on $(\mathbb{H}^n, g)$ is denoted by D. From Koszul formula and the Lie bracket relations we get

$$\begin{aligned} &D_{X_i} X_j = D_{Y_i} Y_j = D_T T = 0, \\ &D_{X_i} Y_j = -\delta_{ij}\, T, \qquad D_{X_i} T = Y_i, \qquad D_{Y_i} T = -X_i, \\ &D_{Y_i} X_j = \delta_{ij}\, T, \qquad D_T X_i = Y_i, \qquad D_T Y_i = -X_i. \end{aligned} \tag{2.1}$$

For any vector field U on $\mathbb{H}^n$ we define $J(U) := D_U T$. It follows from (2.1) that $J(X_i) = Y_i$, $J(Y_i) = -X_i$, and $J(T) = 0$, so that J defines a linear isometry when restricted to the horizontal distribution.

2.2. **Volume and sub-Riemannian perimeter.** The *volume* $|E|$ of a Borel set $E \subset \mathbb{H}^n$ is the Riemannian volume of E with respect to the metric g. The *area* $A(\Sigma)$ of a C^1 hypersurface $\Sigma \subset \mathbb{H}^n$ is defined as

$$A(\Sigma) := \int_\Sigma |N_H| \, d\Sigma,$$

where $d\Sigma$ is the area element induced on Σ by the Riemannian metric g, and N is a locally defined unit vector normal to Σ. For a C^2 hypersurface enclosing a bounded region E, the area coincides with the sub-Riemannian perimeter $|\partial E|$, defined as

$$|\partial E|(\Omega) := \sup \left\{ \int_\Omega \mathrm{div}\, U \, dv : U \text{ horizontal of class } C^1, |U| \leqslant 1, \mathrm{supp}(U) \subset \Omega \right\},$$

where $\Omega \subset \mathbb{H}^n$ is an open set, $\mathrm{div}\, U$ is the Riemannian divergence of the vector field U, and dv is the volume element associated to g.

A set $E \subset \mathbb{H}^n$ is of *locally finite perimeter* if $|\partial E|(\Omega) < +\infty$ for all bounded open sets $\Omega \subset \mathbb{H}^n$. It is of *finite perimeter* if $|\partial E| := |\partial E|(\mathbb{H}^1) < +\infty$. We normalize any finite perimeter set $E \subset \mathbb{H}^1$ to include its density one points and to exclude its density zero points [5, Prop. 3.1].

2.3. **Hypersurfaces in $\mathbb{H}^n$ and variational formulae.** For a C^1 hypersurface $\Sigma \subset \mathbb{H}^n$, the *singular set* $\Sigma_0 \subset \Sigma$ consists of the points where the tangent hyperplane coincides with the horizontal distribution. The set Σ_0 is closed and has empty interior in Σ, and so the *regular set* $\Sigma - \Sigma_0$ of Σ is open and dense in Σ. For any $p \in \Sigma - \Sigma_0$, the tangent hyperplane meets transversally the horizontal distribution, and so $T_p\Sigma \cap \mathcal{H}_p$ is $(2n-1)$-dimensional. We say that Σ is *two-sided* if there is a globally defined unit vector field normal to Σ. Every C^1 hypersurface is locally two-sided.

Let Σ be a C^2 hypersurface in $\mathbb{H}^n$, and N a unit vector normal to Σ. The singular set $\Sigma_0 \subset \Sigma$ can be described as $\Sigma_0 = \{p \in \Sigma : N_H(p) = 0\}$. In the regular part $\Sigma - \Sigma_0$, we can define the *horizontal unit normal vector* ν_H by $\nu_H := N_H/|N_H|$. Consider the unit vector field Z on $\Sigma - \Sigma_0$ given by $Z := J(\nu_H)$. As Z is horizontal and orthogonal to ν_H, it follows that Z is tangent to Σ. The integral curves of Z in $\Sigma - \Sigma_0$ will be called *characteristic curves*. Characteristic curves foliate the regular part of Σ.

Consider a C^1 vector field U with compact support on $\mathbb{H}^n$, and denote by $\{\varphi_t\}_{t\in\mathbb{R}}$ the associated group of diffeomorphisms. Let E be a bounded region enclosed by a hypersurface Σ. The families $\{E_t\}$, $\{\Sigma_t\}$, for t small, are the variations of E and Σ induced by U. Let $V(t) = |E_t|$ and $A(t) = A(\Sigma_t)$. We say that the variation is *volume-preserving* if $V'(0) = 0$. We say that Σ is *area-stationary* if $A'(0) = 0$ for any variation, and *volume-preserving area-stationary* if $A'(0) = 0$ for any volume preserving variation.

If Σ is a C^1 hypersurface enclosing a bounded region E, it is well-known that

$$V'(0) = \int_E \operatorname{div} U \, dv = -\int_\Sigma u \, d\Sigma, \tag{2.2}$$

where $u = \langle U, N\rangle$ and N is the unit vector normal to Σ pointing into E.

If Σ is C^2, and N is a unit vector field normal to Σ, the *mean curvature* of $\Sigma - \Sigma_0$ is given by $-nH := \operatorname{div}_\Sigma \nu_H$, where $\operatorname{div}_\Sigma U(p) := \sum_{i=1}^{2n} \langle D_{e_i} U, e_i\rangle$ for any orthonormal basis $\{e_i\}$ of $T_p\Sigma$. We say that Σ has constant mean curvature if H is constant on $\Sigma - \Sigma_0$. By combining [17, Lemma 2.3] and [18, Lemma 4.3 (4.7)] we have

Lemma 2.1. *Let $\Sigma \subset \mathbb{H}^n$ be a C^2 hypersurface enclosing a bounded region E, with inner unit normal vector N. Consider a variation induced by a vector field U, and let $u = \langle U, N\rangle$. Assume that Σ is volume-preserving area-stationary, and let H be the mean curvature of Σ. Then we have*

$$A'(0) = -\int_\Sigma nH \, u \, d\Sigma.$$

We remark that the variation associated to U in the statement of Lemma 2.1 is not assumed to be volume-preserving. From Lemma 2.1 and (2.2) we easily obtain

Lemma 2.2. *Let $\Sigma \subset \mathbb{H}^n$ be a C^2 hypersurface enclosing a bounded region E. Assume that Σ is volume-preserving area-stationary, and let H be the mean curvature of Σ. Then Σ is a critical point of the functional $A - nHV$ for any variation.*

Let Σ be a two-sided C^2 hypersurface without singular points. We can translate it vertically to get a foliation of the vertical cylinder C over Σ. Denote by N the unit normal to the foliation, and by ν_H the horizontal unit normal obtained from N. For any $p\in C$, let $\{e_i\}$ be an orthonormal basis of the leaf passing through p. Then

$$\operatorname{div}\nu_H(p)=\sum_{i=1}^{2n}\big\langle D_{e_i}\nu_H,e_i\big\rangle+\big\langle D_{N_p}\nu_H,N_p\big\rangle=-nH(p)+\big\langle D_{N_p}\nu_H,N_p\big\rangle.$$

Since $N=|N_H|\,\nu_H+\big\langle N,T\big\rangle\,T$, and $\big\langle D_U\nu_H,\nu_H\big\rangle$, $\big\langle D_T\nu_H,T\big\rangle$, and $\big\langle D_{\nu_H}T,\nu_H\big\rangle=\big\langle J(\nu_H),\nu_H\big\rangle$ vanish, we conclude that $\big\langle D_{N_p}\nu_H,N_p\big\rangle=0$. Hence

$$-nH=\operatorname{div}\nu_H. \tag{2.3}$$

In [17, Lemma 3.1 (3.3)] it is proven that

$$D_u\nu_H=|N_H|^{-1}\sum_{i=1}^{2n-1}\big(\big\langle D_uN,z_i\big\rangle-\big\langle N,T\big\rangle\big\langle J(u),z_i\big\rangle\big)\,z_i+\big\langle z,u\big\rangle\,T, \tag{2.4}$$

where $u\in T_p\Sigma$ for a given point $p\in\Sigma$, and $\{z_1,\ldots,z_{2n-1}\}$ is an orthonormal basis of $T_p\Sigma\cap\mathcal{H}_p$, with $z_1=z=J((\nu_H)_p)$. Completing $\{z_i\}$ to an orthonormal basis of $T_p\Sigma$ by adding a vector v, we obtain from (2.4) that $\big\langle D_v\nu_H,v\big\rangle=0$. Hence we conclude

$$-nH(p)=\sum_{i=1}^{2n-1}\big\langle D_{z_i}\nu_H,z_i\big\rangle,$$

where $\{z_i\}$ is an orthonormal basis of $T_p\Sigma\cap\mathcal{H}_p$. From (2.4), it follows that the endomorphism $v\mapsto -D_v\nu_H-|N_H|^{-1}\big\langle N,T\big\rangle\,J(v)^\top$, defined in the subspace $T_p\Sigma\cap\mathcal{H}_p$, $p\in\Sigma-\Sigma_0$, is selfadjoint. Thus there exists an orthonormal basis $\{v_1,\ldots,v_{2n-1}\}$ of $T_p\Sigma\cap\mathcal{H}_p$ composed of eigenvectors with eigenvalues $\kappa_1,\ldots,\kappa_{2n-1}$. By analogy with the Riemannian case they will be named *principal curvatures*, and we have $-nH=\kappa_1+\ldots+\kappa_{2n-1}$.

2.4. **Geodesics in $\mathbb{H}^n$.** We refer the reader to [17, § 3] for detailed arguments. Geodesics in $\mathbb{H}^n$ are horizontal curves $\gamma:I\to\mathbb{H}^n$ which are critical points of the Riemannian length $L(\gamma):=\int_I|\dot\gamma|$ for any variation by horizontal curves γ_ε. A vector field U along γ induces a variation by horizontal curves if and only if

$$\dot\gamma\big(\big\langle U,T\big\rangle\big)+2\,\big\langle\dot\gamma,J(U)\big\rangle=0. \tag{2.5}$$

The derivative of length for such a variation is given by

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}L(\gamma_\varepsilon)=-\int_I\big\langle D_{\dot\gamma}\dot\gamma,U\big\rangle. \tag{2.6}$$

Observe that $D_{\dot\gamma}\dot\gamma$ is orthogonal to both $\dot\gamma$ and T. Along γ consider the orthonormal basis of $T\mathbb{H}^n$ given by T, $\dot\gamma$, $J(\dot\gamma)$, Z_1, …, Z_{2n-2}. In the same way as for the case of $\mathbb{H}^1$, see [18, § 3], we take any smooth $f:I\to\mathbb{R}$ vanishing at the endpoints of I so that $\int_I f=0$. The vector field U along γ so that $U_H=f\,J(\dot\gamma)$, and $\big\langle U,T\big\rangle=2\int_I f$, satisfies (2.5). Hence (2.6) allows us to conclude that $\big\langle D_{\dot\gamma}\dot\gamma,J(\dot\gamma)\big\rangle$ is constant. Now let $f:I\to\mathbb{R}$ be any smooth function vanishing at the endpoints of I. Then the vector field $U=f\,Z_i$, for any $i=1,\ldots,2n-2$, satisfies (2.5), and hence $D_{\dot\gamma}\dot\gamma$ is orthogonal to Z_i for all $i=1,\ldots,2n-2$. So we obtain that the horizontal geodesic $\gamma:I\to\mathbb{H}^n$ satisfies the equation

$$D_{\dot\gamma}\dot\gamma+2\lambda\,J(\dot\gamma)=0, \tag{2.7}$$

for some constant $\lambda\in\mathbb{R}$. For $\lambda\in\mathbb{R}$, $p\in\mathbb{H}^n$, and $v\in T_p\mathbb{H}^n$, $|v|=1$, the geodesic $\gamma:I\to\mathbb{H}^n$ of curvature λ with initial conditions $\gamma(0)=p$, $\dot\gamma(0)=v$, will be denoted by $\gamma^\lambda_{p,v}$.

The equations of a geodesic can be computed in coordinates in the following way: let $\gamma(s) = (x_1(s), y_1(s), \ldots, x_n(s), y_n(s), t(s))$ be a horizontal geodesic. Then

$$\dot{\gamma}(s) = \sum_{i=1}^{n} \dot{x}_i(s)\,(X_i)_{\gamma(s)} + \dot{y}_i(s)\,(Y_i)_{\gamma(s)},$$

and

$$\dot{t}(s) = \sum_{i=1}^{n} (\dot{x}_i y_i - x_i \dot{y}_i)(s).$$

So equation (2.7) is transformed in

$$\begin{aligned} \ddot{x}_i &= 2\lambda\,\dot{y}_i, \\ \ddot{y}_i &= -2\lambda\,\dot{x}_i, \end{aligned}$$

with initial conditions $x_i(0) = (x_0)_i$, $y_i(0) = (y_0)_i$, and $\dot{x}_i(0) = A_i$, $\dot{y}_i(0) = B_i$, with $\sum_{i=1}^{n}(A_i^2 + B_i^2) = 1$.

Integrating these equations, for $\lambda = 0$, we obtain

$$\begin{aligned} x_i(s) &= (x_0)_i + A_i s, \\ y_i(s) &= (y_0)_i + B_i s, \\ t(s) &= t_0 + \sum_{i=1}^{n} (A_i (y_0)_i - B_i (x_0)_i)\, s, \end{aligned}$$

which are horizontal Euclidean straight lines in $\mathbb{H}^n$.

Integrating, for $\lambda \neq 0$, we obtain

$$\begin{aligned} x_i(s) &= (x_0)_i + A_i \left(\frac{\sin(2\lambda s)}{2\lambda}\right) + B_i \left(\frac{1-\cos(2\lambda s)}{2\lambda}\right), \\ y_i(s) &= (y_0)_i - A_i \left(\frac{1-\cos(2\lambda s)}{2\lambda}\right) + B_i \left(\frac{\sin(2\lambda s)}{2\lambda}\right), \\ t(s) &= t_0 + \frac{1}{2\lambda}\left(s - \frac{\sin(2\lambda s)}{2\lambda}\right) + \sum_{i=1}^{n} \left\{ (A_i (x_0)_i + B_i (y_0)_i)\left(\frac{1-\cos(2\lambda s)}{2\lambda}\right) \right. \\ &\qquad \left. - (B_i (x_0)_i - A_i (y_0)_i)\left(\frac{\sin(2\lambda s)}{2\lambda}\right) \right\}. \end{aligned}$$

In case $x_0 = y_0 = 0$, we obtain

$$\begin{aligned} x_i(s) &= A_i \left(\frac{\sin(2\lambda s)}{2\lambda}\right) + B_i \left(\frac{1-\cos(2\lambda s)}{2\lambda}\right), \\ y_i(s) &= -A_i \left(\frac{1-\cos(2\lambda s)}{2\lambda}\right) + B_i \left(\frac{\sin(2\lambda s)}{2\lambda}\right), \end{aligned}$$

and so $\dot{x}_i$, $\dot{y}_i$, $i = 1, \ldots, n$, can be expressed in terms of x_i, y_i in the following way

$$\dot{x}_i(s) = \frac{\lambda\,\sin(2\lambda s)}{1-\cos(2\lambda s)}\, x_i(s) + \lambda\, y_i(s), \tag{2.8}$$

$$\dot{y}_i(s) = -\lambda\, x_i(s) + \frac{\lambda\,\sin(2\lambda s)}{1-\cos(2\lambda s)}\, y_i(s). \tag{2.9}$$

2.5. **The spheres $\mathbb{S}_\lambda$.** For any $\lambda > 0$, $p \in \mathbb{H}^n$, consider the hypersurface $\mathbb{S}_{\lambda,p}$ defined by

$$\mathbb{S}_{\lambda,p} := \bigcup_{v\in\mathcal{H}_p, |v|=1} \gamma^v_{p,\lambda}([0,\pi/\lambda]).$$

If p is translated to the point $q = (0, -\pi/(4\lambda^2))$, then $\mathbb{S}_\lambda := \mathbb{S}_{\lambda,q}$ is the union of the graphs associated to the functions f and $-f$, where

$$f(z) = \frac{1}{2\lambda^2}\,\{\lambda|z|\,\sqrt{1-\lambda^2\,|z|^2} + \arccos(\lambda|z|)\}, \qquad |z| \leqslant \frac{1}{\lambda}.$$

The hypersurface $\mathbb{S}_\lambda$ is compact and homeomorphic to a $(2n)$-dimensional sphere. Its singular set consists of the two points $\pm(0, \pi/(4\lambda^2))$ on the t-axis, called the *poles*. It is known that the spheres $\mathbb{S}_\lambda$ are C^2 but not C^3 around the singular points. These hypersurfaces were conjectured to be the (smooth) solutions to the isoperimetric problem in $\mathbb{H}^1$ by P. Pansu [15]. It was proven in [17] that the hypersurfaces $\mathbb{S}_\lambda$ are the only compact hypersurfaces of revolution with constant mean curvature λ in $\mathbb{H}^n$. We shall denote by $\mathbb{B}_\lambda$ the topological closed ball enclosed by $\mathbb{S}_\lambda$. It is well known that the spheres $\mathbb{S}_\lambda$ consist of the union of segments of geodesics of curvature λ and length π/λ starting from a given point $p \in \mathbb{H}^n$.

Lemma 2.3. *The characteristic curves in $\mathbb{S}_\lambda$ are the geodesics of curvature λ joining the poles.*

Proof. Since $\mathbb{S}^\pm_\lambda$ is the graph of the function $\pm f$, the inner unit normal to $\mathbb{S}_\lambda$ is proportional to

$$\sum_{i=1}^n \bigg\{\bigg(\frac{\partial f}{\partial x_i} - y_i\bigg)X_i + \bigg(\frac{\partial f}{\partial y_i} + x_i\bigg)Y_i\bigg\} - T$$

on $\mathbb{S}^-_\lambda$, and proportional to

$$-\sum_{i=1}^n \bigg\{\bigg(\frac{\partial(-f)}{\partial x_i} - y_i\bigg)X_i + \bigg(\frac{\partial(-f)}{\partial y_i} + x_i\bigg)Y_i\bigg\} + T$$

on $\mathbb{S}^-_\lambda$. Let ν_λ be the horizontal unit normal to $\mathbb{S}_\lambda$. From

$$\frac{\partial f}{\partial x_i} = -\lambda|z|\,(1-\lambda^2|z|^2)^{-1/2}\,x_i, \qquad \frac{\partial f}{\partial y_i} = -\lambda|z|\,(1-\lambda^2|z|^2)^{-1/2}\,y_i,$$

we have that $J(\nu_\lambda)$ is given by

$$\sum_{i=1}^n \bigg(-\lambda x_i - \frac{(1-\lambda^2|z|^2)^{1/2}}{|z|}\,y_i\bigg)Y_i + \bigg(\lambda y_i - \frac{(1-\lambda^2|z|^2)^{1/2}}{|z|}\,x_i\bigg)X_i$$

on $\mathbb{S}^+_\lambda$, and by

$$\sum_{i=1}^n \bigg(-\lambda x_i + \frac{(1-\lambda^2|z|^2)^{1/2}}{|z|}\,y_i\bigg)Y_i + \bigg(\lambda y_i + \frac{(1-\lambda^2|z|^2)^{1/2}}{|z|}\,x_i\bigg)X_i$$

on $\mathbb{S}^-_\lambda$, where $|z|^2 = \sum_{i=1}^n (x_i^2 + y_i^2)$.

On the other hand, the tangent vector to a horizontal geodesic of curvature λ leaving from $(0, -\pi/(4\lambda^2))$, is given by $\dot\gamma(s) = \sum_{i=1}^n \dot x_i(s)\,X_i + \dot y_i(s)\,Y_i$, where $\dot x_i(s)$, $\dot y_i(s)$ satisfy (2.8) and (2.9). A direct computation shows

$$\frac{\lambda\,\sin(2\lambda s)}{1-\cos(2\lambda s)} = \pm\frac{(1-\lambda^2|z|^2)^{1/2}}{|z|}, \tag{2.10}$$

where the plus sign is chosen in case $s \in [0, \pi/(2\lambda)]$, that is, when $\gamma(s) \in \mathbb{S}^-_\lambda$, and the minus sign if $s \in [\pi/(2\lambda), \pi/\lambda]$, when $\gamma(s) \in \mathbb{S}^+_\lambda$. Replacing the value of

$\lambda\sin(2\lambda s)/(1-\cos(2\lambda s))$ in equations (2.8) and (2.9) by using (2.10), we conclude that $\dot\gamma$ is equal to $J(\nu_\lambda)$, i.e., γ is a characteristic curve of $\mathbb{S}_\lambda$. $\square$

Lemma 2.4. *Let $\Sigma\subset\mathbb{H}^n$ be a C^2 compact hypersurface with a finite number of isolated singular points. Assume that Σ has constant mean curvature. Then Σ is volume-preserving area-stationary. In particular, the spheres $\mathbb{S}_\lambda$ are volume-preserving area-stationary.*

Proof. Let U be a vector field inducing a volume-preserving variation of Σ. Let $u=\langle U,N\rangle$. By the first variation of volume (2.2) we have $\int_\Sigma u\,d\Sigma=0$. By the first variation of area [17, Lemma 3.2], we have

$$A'(0)=-\int_\Sigma \operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big)\,d\Sigma,$$

since u has mean zero and $\operatorname{div}_\Sigma\nu_H$ is constant.

To analyze the above integral, we consider open balls $B_\varepsilon(p_i)$ of radius $\varepsilon>0$ centered at the points $p_1,\ldots,p_k$ of the singular set Σ_0. By the divergence theorem in Σ, we have, for $\Sigma_\varepsilon=\Sigma-\bigcup_{i=1}^k B_\varepsilon(p_i)$,

$$-\int_{\Sigma_\varepsilon}\operatorname{div}_\Sigma\big(u\,(\nu_H)^\top\big)\,d\Sigma=\sum_{i=1}^k\int_{\partial B_\varepsilon(p_i)}u\,\langle\xi_i,(\nu_H)^\top\rangle\,d(\partial B_\varepsilon(p_i)),$$

where ξ_i is the inner unit normal vector to $\partial B_\varepsilon(p_i)$ in Σ, and $d(\partial B_\varepsilon(p_i))$ is the Riemannian volume element of $\partial B_\varepsilon(p_i)$. Note also that

$$\bigg|\sum_{i=1}^k\int_{\partial B_\varepsilon(p_i)}u\,\langle\xi_i,(\nu_H)^\top\rangle\,d(\partial B_\varepsilon(p_i))\bigg|\leqslant\big(\sup_\Sigma|u|\big)\sum_{i=1}^k V_{2n-1}(\partial B_\varepsilon(p_i)),$$

where $V_{2n-1}(\partial B_\varepsilon(p_i))$ is the Riemannian $(2n-1)$-volume of $\partial B_\varepsilon(p_i)$. Observe that the function $|\operatorname{div}_\Sigma(u\,(\nu_H^\top))|$ is bounded from above by $(\sup_\Sigma|u|)\,|\operatorname{div}_\Sigma\nu_H-|N_H|\,|\operatorname{div}_\Sigma N|+|\nabla_\Sigma u|$, which is also bounded. So we can apply the dominated convergence theorem and the fact that $V_{2n-1}(\partial B_\varepsilon(p_i))\to0$, when $\varepsilon\to0$, to prove that $A'(0)=0$. $\square$

2.6. **$\mathbb{H}$-regular surfaces ([1], [4]).** Let $\Omega\subset\mathbb{H}^n$ be an open set. Then $C^1_{\mathbb{H}}(\Omega)$ is the set of continuous real functions in Ω such that $\nabla_{\mathbb{H}}f$ is continuous, [4, def. 5.7], where $\nabla_{\mathbb{H}}f$ is defined by

$$\nabla_{\mathbb{H}}f:=\sum_{i=1}^n X_i(f)\,X_i+Y_i(f)\,Y_i.$$

Following [4, def. 6.1] we say that $\Sigma\subset\mathbb{H}^n$ is an $\mathbb{H}$-regular hypersurface if, for every $p\in\Sigma$, there is an open set Ω containing p, and a function $f\in C^1_{\mathbb{H}}(\Omega)$ such that

$$\Sigma\cap\Omega=\{q\in\Omega: f(q)=0\},\qquad \nabla_{\mathbb{H}}f(q)\neq0.$$

From [4, Thm. 6.5], we know that if Σ is an $\mathbb{H}$-regular hypersurface defined locally by a function $f\in C^1_{\mathbb{H}}(\Omega)$ with nonvanishing horizontal gradient $\nabla_{\mathbb{H}}f$, and we let $E:=\{q\in\Omega: f(q)<0\}$, then E is a finite perimeter set in Ω and $\nu_H=-\nabla_{\mathbb{H}}f/|\nabla_{\mathbb{H}}f|$.

We have the following

Lemma 2.5. *Let Σ be an $\mathbb{H}$-regular hypersurface, and let ν_H be its horizontal unit normal vector. If $\gamma: I\to\mathbb{H}^n$ is an integral curve of $J(\nu_H)$ with $\gamma(0)\in\Sigma$, then $\gamma(I)\subset\Sigma$.*

Proof. Locally $\Sigma = \{p \in \Omega : f(p) = 0\}$, for some open set $\Omega \subset \mathbb{H}^n$, and some $f \in C^1_{\mathbb{H}}(\Omega)$ with non vanishing horizontal gradient. If γ is an integral curve of $J(\nu_H)$, then

$$\frac{d}{dt} f(\gamma(t)) = \big\langle \nabla_H f(\gamma(t)), \dot\gamma(t)\big\rangle = 0.$$

This implies that $f \circ \gamma$ is a constant function and, since $f(\gamma(0)) = 0$, we have $f \circ \gamma \equiv 0$ in Ω. By the connectedness of I, we conclude that $\gamma(I) \subset \Sigma$. □

3. Proof of the isoperimetric inequality

We shall denote by $D_r := \{(z, 0) : |z| \leqslant r\}$ the closed Euclidean disk of radius $r > 0$ contained in the Euclidean hyperplane $\Pi_0 := \{t = 0\}$, and by $C_r := \{(z, t) : |z| \leqslant r\}$ the vertical cylinder over D_r. The vertical t-axis $\{(0, t) : t \in \mathbb{R}\}$ will be denoted by L. For any set $B \subset \mathbb{H}^n$, define $B^+ := B \cap \{(z, t) \in \mathbb{H}^n : t \geqslant 0\}$, $B^- := B \cap \{(z, t) \in \mathbb{H}^n : t \leqslant 0\}$.

We recall that to any finite perimeter set $E \subset \mathbb{H}^1$ we may add its density one points and remove its density zero points, without changing the perimeter and the volume of E. We shall always normalize a finite perimeter set in this way.

Theorem 3.1. *Let $E \subset \mathbb{H}^n$ be a finite perimeter set such that $D \subset E \subset C$, where $D = D_r$, $C = C_r$, for some $r > 0$. Then*

$$|\partial E| \geqslant |\partial \mathbb{B}_\mu|, \tag{3.1}$$

where $\mathbb{B}_\mu$ is the ball with $|\mathbb{B}_\mu| = |E|$. Equality holds in (3.1) *if and only if $E = \mathbb{B}_\mu$.*

Proof. It can be easily proven that $E^\pm := E \cap (\mathbb{H}^n)^\pm$ are finite perimeter sets. The reduced boundary $\partial^* E^+$ of E^+ is contained in $(\partial^* E \cap \{t > 0\}) \cup \operatorname{int}(D)$, where $\operatorname{int}(D)$ is the interior of D inside Π_0.

We choose two families of functions. For $0 < \varepsilon < 1$ we consider a function φ_ε, depending on the distance to the vertical axis L, so that

$$\begin{aligned} \varphi_\varepsilon(p) &= 0, & d(p, L) &\leqslant \varepsilon^2, \\ \varphi_\varepsilon(p) &= 1, & d(p, L) &\geqslant \varepsilon, \\ |\nabla \varphi_\varepsilon(p)| &\leqslant 2/\varepsilon, & \varepsilon \leqslant d(p, L) &\leqslant \varepsilon^2. \end{aligned}$$

Again for $0 < \varepsilon < 10$ we consider a function ψ_ε, depending on the distance to the Euclidean hyperplane Π_0, so that

$$\begin{aligned} \psi_\varepsilon(p) &= 1, & d(p, \Pi_0) &\leqslant \varepsilon^{-1}, \\ \psi_\varepsilon(p) &= 0, & d(p, \Pi_0) &\geqslant \varepsilon^{-1} + 1, \\ |\nabla \psi_\varepsilon(p)| &\leqslant 2, & \varepsilon^{-1} \leqslant d(p, \Pi_0) &\leqslant \varepsilon^{-1} + 1. \end{aligned}$$

Let $\lambda := 1/r$. Then the ball $\mathbb{B}_\lambda$ satisfies $\mathbb{B}_\lambda \cap \Pi_0 = D$. Translate vertically the closed halfspheres $\mathbb{S}^+_\lambda$ to get a foliation of C. Let X be the vector field on $C \setminus L$ given by the horizontal unit normal to the leaves of the foliation. By (2.3), on $C \setminus L$ we have

$$\operatorname{div} X = -n\lambda.$$

We consider the horizontal vector field $\psi_\varepsilon \varphi_\varepsilon X$, which has compact support on $\mathbb{H}^n$.

$$\int_{E^+} \operatorname{div}(\psi_\varepsilon \varphi_\varepsilon X)\, dv = \int_{E^+} \psi_\varepsilon \varphi_\varepsilon \operatorname{div} X\, dv + \int_{E^+} \big\langle \nabla(\psi_\varepsilon \varphi_\varepsilon), X\big\rangle\, dv.$$

Observe that

$$\lim_{\varepsilon\to 0}\int_{E^+}\psi_\varepsilon\varphi_\varepsilon\,\operatorname{div}X\,dv=-n\lambda\,|E^+|,$$

by Lebesgue's Dominated Convergence Theorem since $\psi_\varepsilon\varphi_\varepsilon\operatorname{div}X$ is uniformly bounded, E^+ has finite volume, and $\lim_{\varepsilon\to 0}\psi_\varepsilon\varphi_\varepsilon=1$. On the other hand

$$\lim_{\varepsilon\to 0}\int_{E^+}\big\langle\nabla(\psi_\varepsilon\varphi_\varepsilon),X\big\rangle\,dv=0,$$

since $\big\langle\varphi_\varepsilon\nabla\psi_\varepsilon,X\big\rangle$ is bounded and converges pointwise to 0, and

$$\lim_{\varepsilon\to 0}\int_{E^+}|\big\langle\psi_\varepsilon\nabla\varphi_\varepsilon,X\big\rangle|\,dv\leqslant\lim_{\varepsilon\to 0}\int_{E^+}|\nabla\varphi_\varepsilon|\,dv=0.$$

So we conclude

$$\lim_{\varepsilon\to 0}\int_{E^+}\operatorname{div}(\psi_\varepsilon\varphi_\varepsilon X)\,dv=-n\lambda\,|E^+|. \tag{3.2}$$

By applying the Divergence Theorem for finite perimeter sets [4] to E^+ and to the vector field $\psi_\varepsilon\varphi_\varepsilon X$ we have

$$\int_{E^+}\operatorname{div}(\psi_\varepsilon\varphi_\varepsilon X)\,dv=-\int_D\big\langle\psi_\varepsilon\varphi_\varepsilon X,N_D\big\rangle\,dD-\int_{\partial^*E\cap\{t>0\}}\big\langle\psi_\varepsilon\varphi_\varepsilon X,\nu_H\big\rangle\,d\,|\partial E|,$$

where N_D is the Riemannian unit normal to D pointing into $(\mathbb{H}^n)^+$, dD is the Riemannian area element on D, and ν_H is the inner horizontal unit normal to ∂^*E. Taking limits when $\varepsilon\to 0$, we get from (3.2) and inequality $\big\langle X,\nu_H\big\rangle\leqslant 1$,

$$-\,n\lambda\,|E^+|\geqslant-\int_D\big\langle X,N_D\big\rangle\,dD-\int_{\partial^*E\cap\{t>0\}}d\,|\partial E|, \tag{3.3}$$

with equality if and only if, $|\partial E|$-a.e., $X=\nu_H$ on $\partial E\cap(\mathbb{H}^n)^+$.

We may replace E^+ by $\mathbb{B}_\lambda^+$ in the previous reasoning to obtain

$$-\,n\lambda\,|\partial\mathbb{B}_\lambda^+|=-\int_D\big\langle X,N_D\big\rangle\,dD-\int_{\partial^*\mathbb{B}_\lambda\cap\{t>0\}}d\,|\partial\mathbb{B}_\lambda|. \tag{3.4}$$

Hence from (3.3) and (3.4) we obtain

$$\int_{\partial^*E\cap\{t>0\}}d\,|\partial E|\geqslant\int_{\partial\mathbb{B}_\lambda\cap\{t>0\}}d\,|\partial\mathbb{B}_\lambda|+n\lambda\,\big(|E^+|-|\mathbb{B}_\lambda^+|\big), \tag{3.5}$$

with equality if and only if, $|\partial E|$-a.e., $X=\nu_H$ on $\partial E\cap(\mathbb{H}^n)^+$.

We consider now the foliation of C by vertical translations of the closed halfspheres $\mathbb{S}_\lambda^-$. Let Y the vector field on $C\setminus L$ given by the horizontal unit normal to the leaves of the foliation. By applying the previous argument we get a similar estimate

$$\int_{\partial^*E\cap\{t<0\}}d\,|\partial E|\geqslant\int_{\partial\mathbb{B}_\lambda\cap\{t<0\}}d\,|\partial\mathbb{B}_\lambda|+n\lambda\,\big(|E^-|-|\mathbb{B}_\lambda^-|\big), \tag{3.6}$$

with equality if and only if, $|\partial E|$-a.e., $Y=\nu_H$ on $\partial E\cap(\mathbb{H}^n)^-$.

Hence, adding (3.5) and (3.6), and taking into account that $\partial E\cap\Pi_0$ and $\partial\mathbb{B}_\lambda\cap\Pi_0$ do not contribute to the perimeter of E and $\mathbb{B}_\lambda$, and that $E\cap\Pi_0$ and $\mathbb{B}_\lambda\cap\Pi_0$ do not contribute to the volume of E and $\mathbb{B}_\lambda$, we get

$$|\partial E|\geqslant|\partial\mathbb{B}_\lambda|+n\lambda\,\big(|E|-|\mathbb{B}_\lambda|\big), \tag{3.7}$$

with equality if and only if, $|\partial E|$-a.e, $X=\nu_H$ on $\partial E\cap(\mathbb{H}^n)^+$ and $Y=\nu_H$ on $\partial E\cap(\mathbb{H}^n)^-$.

Let $f(\rho) := n\rho\,|E| + |\partial\mathbb{B}_\rho| - n\rho\,|\mathbb{B}_\rho|$. By Lemmae 2.2 and 2.4, the sphere $\mathbb{S}_\rho$ is a critical point of $A - n\rho V$, with ρ fixed, for any variation. So we have $A(\mathbb{S}_\rho)' - n\rho\, V(\mathbb{B}_\rho)' = 0$, where primes indicate the derivative with respect to ρ. Hence we have

$$f'(\rho) = n\,(|E| - |\mathbb{B}_\rho|).$$

Since the function $\rho \mapsto |\mathbb{B}_\rho|$ is decreasing and takes its values in the interval $(0, +\infty)$, we obtain that $f(\rho)$ is a convex function with a unique minimum μ for which $|E| = |\mathbb{B}_\mu|$. Hence we obtain from (3.7)

$$|\partial E| \geqslant f(\lambda) \geqslant f(\mu) = |\partial\mathbb{B}_\mu|, \tag{3.8}$$

which implies (3.1).

Assume now that equality holds in (3.8). Then we have $\nu_H = X$ on $\partial E \cap (\mathbb{H}^n)^+$ and $\nu_H = Y$ on $\partial E \cap (\mathbb{H}^n)^-$ $|\partial E|$-almost everywhere. By [12, Thm 1.2], ∂E is an $\mathbb{H}$-regular hypersurface. By Lemma 2.5, the integral curves of $J(\nu_H)$ are contained in $\partial E \setminus L$.

For every $p \in \partial D$, consider the integral curve $\gamma_p : I_p \to \mathbb{H}^n$ of $J(\nu_H)$, where I_p is the maximal interval for which γ_p is defined. The trace $\gamma_p(I_p)$ is contained in $\partial E \setminus L$. Such a curve is also an integral curve of $J(X)$ in $(\mathbb{H}^n)^+$ and an integral curve of $J(Y)$ in $(\mathbb{H}^n)^-$, and so it is contained in the sphere $\mathbb{S}_\mu$. In fact, it is part of a characteristic curve of $\mathbb{S}_\mu$.

It is easy to check that $\bigcup_{p\in\partial D} \gamma_p(I_p) = \mathbb{S}_\mu \setminus (\mathbb{S}_\mu)_0$. This implies that $\mathbb{S}_\mu \setminus (\mathbb{S}_\mu)_0 \subset \partial E \setminus L$. Hence a connected component of ∂E coincides with $\mathbb{S}_\mu$. Since $|E| = |\mathbb{B}_\mu|$, we obtain $\partial E = \mathbb{S}_\mu$.

□

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Departamento de Geometría y Topología, Universidad de Granada, E–18071 Granada, España

E-mail address: ritore@ugr.es