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Proof of the Double Bubble Conjecture

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Abstract

We prove that the standard double bubble provides the least-area way to enclose and separate two regions of prescribed volume in $\mathbb{R}^3$.

1. Introduction

Archimedes and Zenodorus (see [K, p. 273]) claimed and Schwarz [S] proved that the round sphere is the least-perimeter way to enclose a given volume in $\mathbb{R}^3$. The Double Bubble Conjecture, long believed (see [P, pp. 300–301], [B, p. 120]) but only recently stated as a conjecture [F1, §3], says that the familiar double soap bubble of Figure 1, consisting of two spherical caps separated by a spherical cap or a flat disc, meeting at 120 degree angles, provides the least-perimeter way to enclose and separate two given volumes.

Theorem (see 7.1). *In $\mathbb{R}^3$, the unique perimeter-minimizing double bubble enclosing and separating regions R_1 and R_2 of prescribed volumes v_1 and v_2 is a standard double bubble as in Figure* 1, *consisting of three spherical caps meeting along a common circle at* 120-*degree angles*. (*For equal volumes, the middle cap is a flat disc.*)

The analogous result in $\mathbb{R}^2$ was proved by the 1990 Williams College "SMALL" undergraduate research Geometry Group [F2]. The case of equal volumes in $\mathbb{R}^3$ was proved with the help of a computer in 1995 by Hass, Hutchings, and Schlafly [HHS], [Hu], [HS2] (see [M1], [HS1], [M2, Chapt. 13]). In this paper we give a complete, computer-free proof of the Double Bubble Conjecture for arbitrary volumes in $\mathbb{R}^3$, using stability arguments, as announced in [HMRR].

Reichardt, Heilmann, Lai and Spielman [RHLS] have generalized our results to $\mathbb{R}^4$ and certain higher dimensional cases (when at least one region is known to be connected). The 2000 edition of [M2] treats bubble clusters through these current results.

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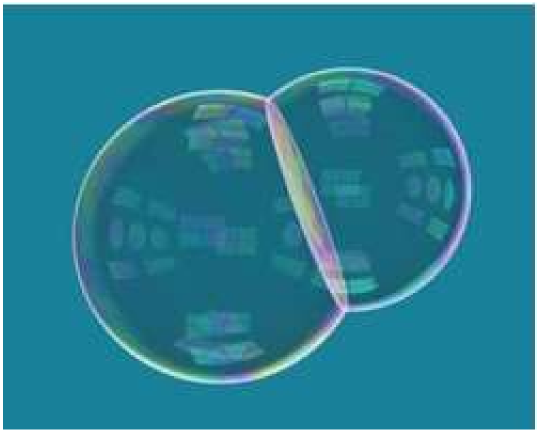

Figure 1. The standard double bubble provides the least-perimeter way to enclose and separate two prescribed volumes. Computer graphics copyright John M. Sullivan, University of Illinois (http://www.math.uiuc.edu/~jms/Images/)

Previous results (see [M2, Chapts. 13 and 14]). Our strategy for proving Theorem 7.1 is to assume that a given double bubble minimizes perimeter and to use this assumption to deduce that the double bubble is standard. This strategy is valid only if we know that a perimeter-minimizing double bubble exists. F. Almgren [A, Thm. VI.2] (see [M2, Chapt. 13]) proved the existence and almost-everywhere regularity of perimeter-minimizing bubble clusters enclosing k prescribed volumes in $\mathbb{R}^{n+1}$, using geometric measure theory. J. Taylor [T] proved that minimizers in $\mathbb{R}^3$ consist of smooth constant-mean-curvature surfaces meeting in threes at 120-degree angles along curves, which in turn could meet only in fours at isolated points. An argument suggested by White, which was written up by Foisy [F1, Thm. 3.4] and Hutchings [Hu, Thm. 2.6], shows further that any perimeter-minimizing double bubble in $\mathbb{R}^{n+1}$ (for $n \geqslant 2$) has rotational symmetry about some line.

Unfortunately, the existence proofs depend on allowing the enclosed regions R_1 and R_2 to be disconnected. The complementary "exterior" region could also *a priori* be disconnected. (If one tries to require the regions to be connected, they might in principle disconnect in the minimizing limit, as thin connecting tubes shrink away.) Hutchings [Hu] partially dealt with this complication, using concavity and decomposition arguments to show for a perimeter-minimizing double bubble that both regions have positive pressure (see 4.1) and hence that the exterior is connected. Moreover there is a Basic Estimate (see §6) which puts upper bounds on the numbers of components of R_1 and R_2, depending on the dimension n and the volumes v_1, v_2.

For equal volumes in $\mathbb{R}^3$, the Basic Estimate implies that both enclosed regions are connected. It can then be shown that a nonstandard perimeter-minimizing double bubble would have to consist of two spherical caps with a toroidal band between them (Fig. 8). Any such bubble can be described by two parameters, and Hass and Schlafly [HS2] used a rigorous computer search of the parameter space to rule out all such possibilities in the equal volume case, thus proving the Double Bubble Conjecture for equal volumes in $\mathbb{R}^3$. Earlier computer experiments of Hutchings and Sullivan had suggested that in fact no such nonstandard double bubbles were stable, and we confirm that in this paper, without using a computer.

Our proof. In the present paper we consider arbitrary volumes v_1, v_2 in $\mathbb{R}^3$. We give a short proof using the Hutchings Basic Estimate that the larger region is connected (Proposition 6.2), and we use a stability argument (Proposition 6.5) to show that the smaller region has at most two components, as in Figure 2. (That the smaller region has at most two components can also be deduced from the Hutchings Basic Estimate using careful computation; see [HLRS, Prop. 4.6], [M2, 14.11–14.13].)

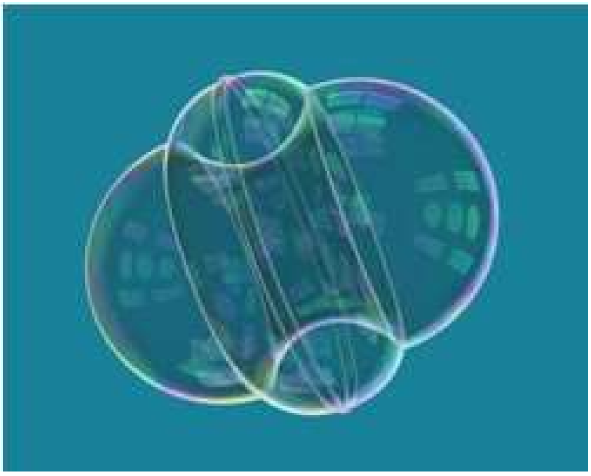

Figure 2. A nonstandard double bubble. One region has two components (a central bubble and a thin toroidal bubble); the second region is another toroidal bubble in between. Computer graphics copyright John M. Sullivan, University of Illinois (http://www.math.uiuc.edu/˜jms/Images/)

To prove that an area-minimizing double bubble Σ is standard, consider rotations about an axis orthogonal to the axis of symmetry. At certain places on Σ, the rotation vector field may be tangent to Σ; i.e., the corresponding normal variation vector field v on Σ may vanish. The axis can be chosen so that these places separate Σ into (at least) four pieces (Proposition 5.8). Some nontrivial combinations w of the restrictions of v to the four pieces vanish on one piece and respect the two volume constraints. By stability, w satisfies a nice differential equation, and hence vanishes on more parts of Σ, which must therefore be pieces of spheres (Proposition 5.2). It follows that Σ must be the standard double bubble.

The foregoing argument in the proof of Proposition 5.2 was inspired by Courant's Nodal Domain Theorem [CH, p. 452], which says for example that the first eigenfunction is nonvanishing. Other applications of this principle to isoperimetric problems and to the study of volume-preserving stability have been given by Ritoré and Ros [RR], by Ros and Vergasta [RV], by Ros and Souam [RS] and by Pedrosa and Ritoré [PR].

Open questions. We conjecture that the standard double bubble in $\mathbb{R}^{n+1}$ is the unique stable double bubble. Sullivan [SM, Prob. 2] has conjectured that the standard k-bubble in $\mathbb{R}^{n+1}$ ($k \leqslant n+2$) is the unique minimizer enclosing k regions of prescribed volume. This remains open even for the triple bubble in $\mathbb{R}^2$, although Cox, Harrison, Hutchings, Kim, Light, Mauer and Tilton [CHK] have proved it minimizing in a category of bubbles with connected regions (which *a priori* in principle might bump up against each other).

One can consider the Double Bubble Conjecture in hyperbolic space $\mathbb{H}^{n+1}$ or in the round sphere $\mathbb{S}^{n+1}$. The symmetry and concavity results still hold [Hu, 3.8–3.10]. The case of $\mathbb{S}^2$ was proved by Masters [Ma]. The cases of $\mathbb{H}^2$ and equal volumes in $\mathbb{H}^3$ and in $\mathbb{S}^3$ when the exterior is at least ten percent of $\mathbb{S}^3$ were proved by Cotton and Freeman [CF].

There is also the very physical question in $\mathbb{R}^3$ of whether the standard double bubble is the unique stable double bubble with connected regions. By our Corollary 5.3, it would suffice to prove rotational symmetry. In $\mathbb{R}^2$, Morgan and Wichiramala [MW] have proved that the standard double bubble is the unique stable double bubble, except of course for two single bubbles.

Contents. Section 2 gives the precise definition of double bubble and a proof that there is a unique standard double bubble enclosing two given volumes. Section 3 provides variational formulas for our stability arguments. Section 4 gives some preliminary results on the geometry of hypersurfaces of revolution with constant mean curvature ("surfaces of Delaunay"). Section 5 uses stability arguments to show that a perimeter-minimizing double bubble in $\mathbb{R}^{n+1}$ must be standard if one enclosed region is connected and the other region has at most two components. Section 6 proves the requisite component bounds for perimeter-minimizing double bubbles in $\mathbb{R}^3$. This completes the proof of The Double Bubble Conjecture, as summarized in Section 7.

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2. Double bubbles

A *double bubble* in $\mathbb{R}^{n+1}$ is the union of the topological boundaries of two disjoint regions of prescribed volumes. A *smooth double bubble* $\Sigma \subset \mathbb{R}^{n+1}$ is a piecewise smooth oriented hypersurface consisting of three compact pieces Σ_1, Σ_2 and Σ_0 (smooth up to the boundary), with a common $(n-1)$-dimensional smooth boundary C such that $\Sigma_1 + \Sigma_0$ (resp. $\Sigma_2 - \Sigma_0$) encloses a region R_1 (resp. R_2) of prescribed volume v_1 (resp. v_2). None of these objects is assumed to be connected. The unit normal vector field N along Σ will be always chosen according to the following criterion: N points into R_1 along ∂R_1 and points into R_2 along Σ_2. We denote by σ and H the second fundamental form and the mean curvature of Σ. Note that these objects are not univalued along the singular set C but they depend on the sheet Σ_i we use to compute them. We will also use the notation N_i, σ_i and H_i to indicate the restriction of N, σ and H to Σ_i, $i = 0, 1, 2$.

Since by Theorem 4.1 perimeter-minimizing double bubbles are smooth double bubbles (geometric measure theory automatically ignores negligible hair and dirt), throughout the rest of this paper by "double bubble" we will mean "smooth double bubble."

A *standard double bubble* in $\mathbb{R}^{n+1}$ consists of two exterior spherical pieces and a separating surface (which is either spherical or planar) meeting in an equiangular way along a given $(n-1)$-dimensional sphere C.

Proposition 2.1. *There is a unique standard double bubble (up to rigid motions) for given volumes in* $\mathbb{R}^{n+1}$. *The mean curvatures satisfy* $H_0 = H_1 - H_2$.

Proof. Consider a unit sphere through the origin and a congruent or smaller sphere intersecting it at the origin (and elsewhere) at 120 degrees as in Figure 3. There is a unique completion to a standard double bubble. Varying the size of the smaller sphere yields all volume ratios precisely once. Scaling yields all pairs of volumes precisely once.

The condition on the curvatures follows by plane geometry for $\mathbb{R}^2$ and hence for $\mathbb{R}^{n+1}$ (see [M2, Prop. 14.1]). ☐

Remark 2.2. Montesinos [Mon] (see [SM, Prob. 2]) has proved that there is a unique standard k-bubble in $\mathbb{R}^{n+1}$ for $k \leqslant n + 2$.

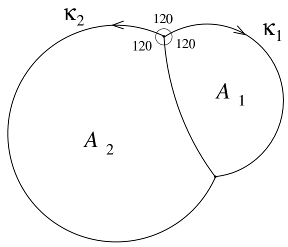

Figure 3. Varying the size of the smaller cap yields standard double bubbles of all volume ratios. Then scaling yields all pairs of volumes.

3. Variation formulae

In this section we will consider one-parameter variations $\{\varphi_t\}_{|t|<\varepsilon} : \Sigma \to \mathbb{R}^{n+1}$ of a double bubble $\Sigma \subset \mathbb{R}^{n+1}$ which are univalued along the singular set C and when restricted to each one of the pieces Σ_i are smooth (up to the boundary). Denote by $X = d\varphi_t/dt$ the associated infinitesimal vector field at $t = 0$. Taking into account our choice of normal vectors to Σ it is a standard fact that the derivative of the volume of the regions R_1 and R_2 are given by

$$(3.1) \qquad -\int_{\Sigma_1} \langle X, N_1\rangle - \int_{\Sigma_0} \langle X, N_0\rangle, \quad \text{and} \quad -\int_{\Sigma_2} \langle X, N_2\rangle + \int_{\Sigma_0} \langle X, N_0\rangle,$$

respectively. On the other hand the first derivative of area of the bubble is given by

$$\sum_{i=0,1,2} \int_{\Sigma_i} \operatorname{div}_{\Sigma_i} X,$$

where $\operatorname{div}_{\Sigma_i}$ is the divergence in Σ_i of a vector field in $\mathbb{R}^{n+1}$. If $\{e_j\}$ is an orthonormal basis of $T\Sigma_i$ and X is a vector field in $\mathbb{R}^{n+1}$ then $\operatorname{div}_{\Sigma_i} X = \sum_j \left\langle D_{e_j} X, e_j \right\rangle$, where D is the Levi-Civitá connection in $\mathbb{R}^{n+1}$. As $\operatorname{div}_{\Sigma_i} X = \operatorname{div}_{\Sigma_i} X^T - nH_i \langle X, N_i\rangle$, where X^T is the projection of X to $T\Sigma_i$, the Divergence Theorem then implies the following well-known result.

LEMMA 3.1 (First variation of area for double bubbles). *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a double bubble consisting of smooth hypersurfaces Σ_0, Σ_1, Σ_2, meeting smoothly along an $(n-1)$-dimensional submanifold C. Then the first derivative of the area along a deformation $\varphi_t(\Sigma)$ at $t = 0$, where φ_t is a variation with associated vector field X, is given by*

$$(3.2) \qquad -\sum_{i=0,1,2} \int_{\Sigma_i} nH_i \langle X, N_i\rangle - \int_C \langle X, \nu_0 + \nu_1 + \nu_2\rangle,$$

where N_i are the normal vectors to the smooth parts Σ_i of Σ and ν_i are the inner conormals to C inside Σ_i.

Suppose that a double bubble Σ is *stationary* for any variation preserving the volume of the regions R_1 and R_2. By Lemma 3.1 this is equivalent to

(i) the mean curvatures H_i are constant, with $-H_1 + H_2 + H_0 = 0$, and

(ii) $\nu_0 + \nu_1 + \nu_2 = 0$ along C.

The mean curvature H_1 (resp. H_2) is called the *pressure* of the region R_1 (resp. R_2). From (i) above, we get that if $H_0 > 0$, then R_1 has larger pressure than R_2.

The functions $u_i = \langle X, N_i \rangle$ are the normal components of the variational field X. If the variation preserves volumes, from (3.1) they satisfy

$$\int_{\Sigma_1} u_1 + \int_{\Sigma_0} u_0 = 0, \qquad \int_{\Sigma_2} u_2 - \int_{\Sigma_0} u_0 = 0, \tag{3.3}$$

and, since at the points of the singular set we have $-N_1 + N_2 + N_0 = 0$, we get that

$$-u_1 + u_2 + u_0 = 0 \quad \text{along } C. \tag{3.4}$$

Now we follow the arguments in [BCE, Lemma 2.2] to show that any volume preserving infinitesimal variation is integrable.

LEMMA 3.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a stationary double bubble. Given smooth functions $u_i : \Sigma_i \to \mathbb{R}$ such that* (3.3) *and* (3.4) *are satisfied, there is a variation $\{\varphi_t\}$ of Σ which leaves constant the volume of the regions enclosed by $\varphi_t(\Sigma)$ and such that the normal components of the associated infinitesimal vector field X are the functions u_i, $i = 0, 1, 2$.*

Proof. The boundary condition (3.4) allows us to construct a smooth vector field Z on C such that $\langle Z, N_i \rangle = u_i$, which can be extended smoothly along each Σ_i so that $\langle Z, N_i \rangle = u_i$. Let $\{\psi_t\}$ be a one-parameter variation of Σ associated to Z (we can take $\psi_t = \psi + t\,Z$). We choose nonnegative smooth functions $f_i : \Sigma_i \to \mathbb{R}$, $f_i \neq 0$, with compact support inside $\operatorname{int} \Sigma_i$, extended by 0 to Σ. For t, s_1, $s_2 \in \mathbb{R}$ close to 0, we consider the three-parameter deformation

$$\psi_t + s_1\, f_1\, N_1^t + s_2\, f_2\, N_2^t,$$

where N_i^t is the normal vector to $\psi_t(\Sigma_i)$. Let $v_i(t, s_1, s_2)$, $i = 1\ 2$, be the volume of the deformed region R_i. Then

$$\frac{\partial v_i}{\partial s_j}(0,0,0) = \int_{\Sigma_i} f_j,$$

which equals 0 if $i \neq j$ and is positive if $i = j$. From conditions (3.3)

$$\frac{\partial v_i}{\partial t}(0,0,0) = 0.$$

Applying the Implicit Function Theorem we find smooth functions $s_1(t)$, $s_2(t)$ with $s_i(0) = 0$ such that the volume of the regions R_i is preserved along the deformation. Let X be the vector field associated to this deformation. Using that the variation is volume preserving we get that $s_i'(0) = 0$. Hence the normal components of X are the functions u_i. □

Now we wish to compute the second derivative of area for a variation of a double bubble keeping constant the volume of the two enclosed regions.

PROPOSITION 3.3 (Second variation of area for stationary double bubbles). *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a stationary double bubble, and let φ_t be a one-parameter variation with associated vector field X which preserves the volumes of R_1 and R_2. Then the second derivative of the area of $\varphi_t(\Sigma)$ at $t = 0$ is given by*

$$-\int_\Sigma u\,(\Delta u + |\sigma|^2 u) - \sum_{i=0,1,2} \int_C u_i \left\{ \frac{\partial u_i}{\partial \nu_i} + q_i u_i \right\}, \tag{3.5}$$

where $u = \langle X, N\rangle$, $u_i = \langle X, N_i\rangle$, Δ is the Laplacian of Σ, $|\sigma|^2$ is the squared norm of the second fundamental form, ν_i is the unit inner normal to C inside Σ_i, and the functions q_i are given by $q_1 = (\kappa_0 - \kappa_2)/\sqrt{3}$, $q_2 = (-\kappa_1 - \kappa_0)/\sqrt{3}$ and $q_0 = (\kappa_1 + \kappa_2)/\sqrt{3}$ with $\kappa_i = \sigma_i(\nu_i, \nu_i)$, $i = 0, 1, 2$.

Proof. First we recall that the derivative of the mean curvature H is given by

$$n\,\frac{dH}{dt}(0) = \Delta u + |\sigma|^2 u. \tag{3.6}$$

To obtain the second derivative of area we differentiate (3.2) with respect to t. The derivative of the integrals over Σ_i in (3.2) equals

$$-\int_\Sigma u\,(\Delta u + |\sigma|^2 u) - \sum_{i=0,1,2} nH_i\, \frac{d}{dt}\bigg|_{t=0} \left(\int_{\Sigma_i} \langle X, N_i\rangle \right). \tag{3.7}$$

Let us see that the last sum vanishes. Let $a_i = \frac{d}{dt}\big|_{t=0} \left(\int_{\Sigma_i} \langle X, N_i\rangle \right)$. Since the variation $\varphi_t(\Sigma)$ preserves volume, we obtain from (3.1) that $a_1 + a_0 = 0$ and $a_2 - a_0 = 0$. As $-H_1 + H_2 + H_0 = 0$ we conclude

$$H_1 a_1 + H_2 a_2 + H_0 a_0 = a_0\,(-H_1 + H_2 + H_0) = 0,$$

which shows that the latter sum in (3.7) vanishes as we claimed.

It remains to treat the boundary term in (3.2). Since $\nu_0 + \nu_1 + \nu_2 = 0$ on C, differentiating with respect to t we have

$$\frac{d}{dt}\bigg|_{t=0} \left(\int_C \langle X, \nu_0 + \nu_1 + \nu_2\rangle \right) = \int_C \left\langle X, D_X (\nu_0 + \nu_1 + \nu_2) \right\rangle.$$

Equation (3.5) is then obtained from Lemma 3.6 below. To compute $D_X\nu_i$ the vector ν_i has been extended as ν_i^t along the integral curves of X, so that ν_i^t is the unit inner conormal to $\varphi_t(C)$ in $\varphi_t(\Sigma_i)$. □

Remark 3.4. For bubbles in $\mathbb{R}^3$, the second variation formula and proof admit isolated singularities, such as tetrahedral soap film singularities. For bubbles in $\mathbb{R}^{n+1}$, C need only be piecewise smooth, including pieces meeting along an $(n-2)$-dimensional submanifold. In addition, the second variation is insensitive to any sets of $\mathcal{H}^{n-2}$ measure 0 (see [MR, Lemmas 3.1, 3.3]).

In a smooth Riemannian ambient manifold M^{n+1}, the second variation has an additional term

$$-\int_\Sigma \mathrm{Ric}(N,N)\,u^2$$

involving the Ricci curvature in the normal direction N (see [BP, §7]).

Remark 3.5. By approximation, the second variation formula (3.5) holds in a distributional sense (see (3.12)) for X piecewise C^1 or in H^1.

LEMMA 3.6. *Under the hypotheses of Proposition* 3.3 *we have*

$$D_X(\nu_0+\nu_1+\nu_2) = \sum_i \left\{\frac{\partial u_i}{\partial \nu_i} + q_i u_i\right\} N_i = -\left\{\frac{d\theta_2}{dt}(0)N_2 + \frac{d\theta_0}{dt}(0)N_0\right\}, \tag{3.8}$$

where $\theta_2 = \angle(\nu_1,\nu_2)$, $\theta_0 = \angle(\nu_1,\nu_0)$ *are the angles determined by the sheets of* $\varphi_t(\Sigma)$ *along the singular set.*

Proof. Let $Y = X^C$ be the orthogonal projection of X to the tangent bundle TC. For each i we have $X = Y + \langle X,\nu_i\rangle\,\nu_i + \langle X,N_i\rangle\,N_i$. As Y is tangent to C we have $\langle Y, D_X(\sum_i \nu_i)\rangle = 0$. We also have $\langle \nu_i, D_X\nu_i\rangle = 0$ for each i since $|\nu_i| = 1$. Moreover

$$\langle D_X\nu_i, N_i\rangle = -\langle \nu_i, D_X N_i\rangle = \sigma_i(\nu_i, X^i) + \frac{\partial u_i}{\partial \nu_i},$$

where X^i is the orthogonal projection of X to $T\Sigma_i$. Hence

$$D_X(\nu_0+\nu_1+\nu_2) = \sum_i \left\{\frac{\partial u_i}{\partial \nu_i} + \sigma_i(\nu_i,\nu_i)\,\langle X,\nu_i\rangle + \sigma_i(Y,\nu_i)\right\} N_i.$$

Observe that $\langle D_Y\nu_i,\nu_i\rangle = 0$, and that $\sum_i D_Y\nu_i = D_Y(\sum_i \nu_i)$ vanishes since Y is tangent to the singular set, where $\sum_i \nu_i$ is identically zero. Of course this implies $\sum_i (D_Y\nu_i)^C = (D_Y(\sum_i \nu_i))^C = 0$. Hence we see that

$$\begin{aligned} 0 = \sum_i D_Y\nu_i &= \sum_i \left\{(D_Y\nu_i)^C + \langle D_Y\nu_i,\nu_i\rangle\,\nu_i + \langle D_Y\nu_i, N_i\rangle\,N_i\right\} \\ &= \sum_i \langle D_Y\nu_i, N_i\rangle\,N_i = \sum_i \sigma_i(Y,\nu_i)\,N_i. \end{aligned}$$

Therefore we get that $D_X(\nu_0+\nu_1+\nu_2) = \sum_i \left\{\frac{\partial u_i}{\partial \nu_i} + \kappa_i\,\langle X,\nu_i\rangle\right\} N_i$.

Taking into account that

$$(3.9)\quad \nu_1=\frac{1}{\sqrt{3}}\,(N_0-N_2),\qquad \nu_2=\frac{1}{\sqrt{3}}\,(-N_1-N_0),\qquad \nu_0=\frac{1}{\sqrt{3}}\,(N_1+N_2),$$

we obtain that

$$\langle X,\nu_1\rangle=\frac{1}{\sqrt{3}}\,(u_0-u_2),\quad \langle X,\nu_2\rangle=\frac{1}{\sqrt{3}}\,(-u_1-u_0),\quad \langle X,\nu_0\rangle=\frac{1}{\sqrt{3}}\,(u_1+u_2).$$

As $-N_1+N_2+N_0=0$ we have

$$\sqrt{3}\sum_i \kappa_i\,\langle X,\nu_i\rangle\;N_i=\sqrt{3}\sum_i q_iu_iN_i\;+\;\{(\kappa_0u_2-\kappa_2u_0)\,N_1+(\kappa_1u_0-\kappa_0u_1)\,N_2 +(\kappa_2u_1-\kappa_1u_2)\,N_0\}\,.$$

The summand between brackets is a vector whose coordinates coincide, up to sign, with the determinant of the matrix

$$\begin{pmatrix} N_1^i & \kappa_1 & u_1\\ N_2^i & \kappa_2 & u_2\\ N_0^i & \kappa_0 & u_0\end{pmatrix},$$

where $N_j=(N_j^1,\ldots,N_j^n)$. But this determinant vanishes since $-N_1^i+N_2^i+N_0^i=0$, $-u_1+u_2+u_0=0$, and $-\kappa_1+\kappa_2+\kappa_0=0$. This last equality holds because $-H_1+H_2+H_0=0$, and $-\sigma_1(Z,T)+\sigma_2(Z,T)+\sigma_0(Z,T)=0$ for any vector Z and T in TC. Hence the first part of (3.8) follows.

To prove the remaining part we write $\nu_2=R(\theta_2)\nu_1$, $\nu_0=R(\theta_0)\nu_1$, where $R(\theta)$ is the rotation in the plane spanned by ν_1 and N_1 given, in this basis, by

$$R(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

We have

$$D_X(\nu_0+\nu_1+\nu_2)=(\,\mathrm{Id}+R(\theta_2)+R(\theta_0))\frac{d\nu_1}{dt}+\frac{d\theta_2}{dt}(-N_2)+\frac{d\theta_0}{dt}(-N_0),$$

and (3.8) follows since the first summand vanishes at $t=0$. □

Remark 3.7. For a variation such that the angles of the sheets are preserved, we have $D_X(\nu_0+\nu_1+\nu_2)=0$ (since $\nu_0+\nu_1+\nu_2=0$ for all t), so by (3.8), the boundary term in the second variation formula (3.5) vanishes.

Consider a stationary bubble Σ. We say that a function $u:\bigcup\Sigma_i\to\mathbb{R}$ defined on the disjoint union of the Σ_i is *admissible* if the restrictions u_i to the smooth pieces Σ_i of Σ, lie in the Sobolev space H^1 (of functions in L^2 whose gradient is squared integrable) and satisfy the boundary condition

$$(3.10)\qquad u_1=u_2+u_0\quad\text{along the singular set } C.$$

The space of admissible functions satisfying the zero mean value conditions

$$\int_{\Sigma_1} u_1 + \int_{\Sigma_0} u_0 = \int_{\Sigma_2} u_2 - \int_{\Sigma_0} u_0 = 0 \tag{3.11}$$

will be denoted by $\mathcal{F}(\Sigma)$. From the results at the beginning of this section, we see that admissible functions correspond to deformations of Σ and that $\mathcal{F}(\Sigma)$ are the infinitesimal variations of those deformations which preserve the volume of the regions R_1 and R_2. The bilinear form on the space of admissible functions for the second variation of the area (3.5) will be denoted by Q, and it is given by

$$\begin{aligned} Q(u,v) &= \int_\Sigma \Big\{ \langle \nabla u, \nabla v\rangle - |\sigma|^2 uv \Big\} - \sum_{i=0,1,2} \int_C q_i u_i v_i \\ &= -\int_\Sigma (\Delta u + |\sigma|^2 u)\, v - \sum_{i=0,1,2} \int_C \left\{ \left(\frac{\partial u_i}{\partial \nu_i} + q_i u_i \right) v_i \right\}, \end{aligned} \tag{3.12}$$

where ν_i is the inner normal to C inside Σ_i and q_i are the functions defined in the statement of Proposition 3.3. We will say that a (smooth) double bubble Σ is *stable* if it is stationary and $Q(u,u) \geqslant 0$ for any $u \in \mathcal{F}(\Sigma)$. We shall say that it is *unstable* if it is not stable. By Lemma 3.2 a perimeter-minimizing double bubble is stable.

LEMMA 3.8. *Let Σ be a stable double bubble and $u \in \mathcal{F}(\Sigma)$ such that $Q(u,u) = 0$. Then u is smooth on the interior of Σ_i, $i = 0, 1, 2$, and there exist real numbers $\lambda_0, \lambda_1, \lambda_2$, with $\lambda_1 = \lambda_0 + \lambda_2$, such that*

$$\Delta u_i + |\sigma|^2 u_i = \lambda_i, \quad \text{on } \Sigma_i.$$

Proof. The stability of Σ implies that $Q(u+tv, u+tv) \geqslant 0$ for any $v \in \mathcal{F}$ and $t \in \mathbb{R}$. Therefore $Q(u,v) = 0$ and so, taking arbitrary functions with mean zero and support inside the interior of Σ_i we conclude that the displayed equation holds in a distributional sense. From elliptic regularity, u is smooth on the interior of Σ_i. ☐

A smooth admissible function u is said to be a *Jacobi function* if it corresponds to an infinitesimal deformation of Σ which preserves the mean curvature of the pieces Σ_k, and the fact that these pieces meet in an equiangular way along its singular set. By formulae (3.6) and (3.8), we have that u is a Jacobi function if and only if

$$\begin{cases} \Delta u + |\sigma|^2 u = 0, & \text{on } \Sigma, \\ -\left(\dfrac{\partial u_1}{\partial \nu_1} + q_1 u_1 \right) = \dfrac{\partial u_2}{\partial \nu_2} + q_2 u_2 = \dfrac{\partial u_0}{\partial \nu_0} + q_0 u_0, & \text{along } C. \end{cases} \tag{3.13}$$

Any Killing vector field Y of $\mathbb{R}^{n+1}$ gives a Jacobi function on Σ, $u = \langle Y, N \rangle$.

LEMMA 3.9. *Let $S \subset \Sigma$ be a subdomain with piecewise-smooth boundary and u a Jacobi function on Σ which vanishes on ∂S (in particular we assume that all the u_i vanish at $\partial S \cap C$). If w is defined by*

$$w = \begin{cases} u, & \text{on } S, \\ 0, & \text{on } \Sigma - S, \end{cases}$$

then w is an admissible function and $Q(w, w) = 0$.

Let $S' \subset \Sigma$ be a second subdomain, with the same properties of S, and w' its associated admissible function. If the interiors of S and S' are disjoint, then $Q(w, w') = 0$.

Proof. A Jacobi function u satisfies $Q(u, v) = 0$ for any admissible function v by (3.12) and (3.13). Therefore the equalities $Q(w, w) = Q(u, w) = 0$ prove the first assertion. The second one is trivial. □

4. Area-minimizing double bubbles and Delaunay hypersurfaces

As described in the Previous Results section of the introduction, F. Almgren [A, Thm. VI.2] (see [M2, Chapt. 13]) proved the existence and almost-everywhere regularity of perimeter-minimizing bubble clusters enclosing k prescribed volumes in $\mathbb{R}^{n+1}$, using geometric measure theory. Using symmetry, concavity, and decomposition arguments, Hutchings analyzed the structure of minimizing double bubbles.

THEOREM 4.1.

(a) (after White, [F1, Thm. 3.4], [Hu, Thm. 2.6]). *An area-minimizing double bubble in $\mathbb{R}^{n+1}$ (for $n \geqslant 2$) is a hypersurface of revolution about some line L.*

(b) ([Hu, Cor. 3.3]). *In an area-minimizing double bubble, both enclosed regions have positive pressure.*

(c) ([Hu, Thm. 5.1]). *An area-minimizing double bubble is either the standard double bubble or consists of a topological sphere with a finite tree of annular bands attached as in Figure 4. The two caps are pieces of spheres, and the root of the tree has just one branch. All pieces are smooth (Delaunay) hypersurfaces meeting in threes at 120-degree angles along round $(n-1)$-spheres.*

Let Ω a connected component of the regions R_1 or R_2 in a nonstandard minimizing double bubble. Then either the smooth pieces in the boundary of Ω are all annuli or $\partial\Omega$ is the union of two spherical caps D_1 and D_2 and one annulus M_0. In the first case we shall refer to Ω as a *torus component*, and in the latter one as the *spherical component*.

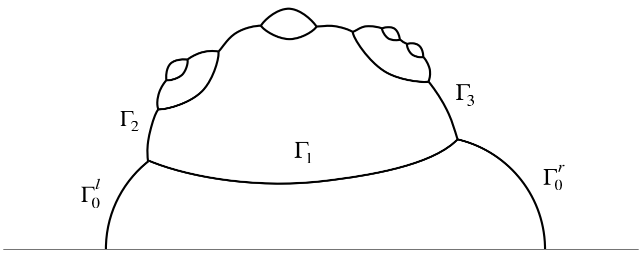

Figure 4. Generating curve of nonstandard area-minimizing double bubble in $\mathbb{R}^{n+1}$.

Now we review some facts about hypersurfaces of revolution with constant mean curvature in $\mathbb{R}^{n+1}$, known as Delaunay hypersurfaces (see [D], [E], [Ke] on $\mathbb{R}^3$ and [Hs] on $\mathbb{R}^{n+1}$). Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface and assume that Σ is invariant under the action of the group $O(n)$ of isometries of $\mathbb{R}^{n+1}$ fixing the x_1-axis. The hypersurface Σ is generated by a curve Γ contained in the x_1x_2-plane. The coordinates x_1, x_2, will be denoted by x, y, respectively. We parametrize the curve $\Gamma = (x, y)$ by arc-length s. If α is the angle between the tangent to Γ and the positive x-direction we shall always choose the normal vector field $N = (\sin\alpha, -\cos\alpha)$. Then we have

LEMMA 4.2. *The generating curve* Γ *of an* $O(n)$*-invariant hypersurface* $\Sigma \subset \mathbb{R}^{n+1}$ *with mean curvature* H *with respect to the normal vector* $N = (\sin\alpha, -\cos\alpha)$ *satisfies the following system of ordinary differential equations*

$$\begin{aligned} x' &= \cos\alpha, \\ y' &= \sin\alpha, \\ \alpha' &= -nH + (n-1)\,\frac{\cos\alpha}{y}. \end{aligned} \tag{4.1}$$

Moreover, if H *is constant then the above system has the first integral*

$$y^{n-1}\cos\alpha - Hy^n = F. \tag{4.2}$$

The constant F in (4.2) is called the *force* of the curve Γ. Existence of the first integral is standard in the Calculus of Variations (see [GH, §3.4] and the references therein). For constant mean curvature surfaces see [P, pp. 138–139], with earlier reference to Beer and [KKS, §3].

From Lemma 4.2 we can obtain the following known properties.

PROPOSITION 4.3. *Any local solution of the system* (4.1) *is a part of a complete solution* Γ, *which generates a hypersurface* Σ *with constant mean curvature of several possible types* (*see Figure* 5).

(i) *If* $FH > 0$ *then* Γ *is a periodic graph over the* x*-axis. It generates a periodic embedded unduloid, or a cylinder.*

(ii) *If* $FH < 0$ *then* Γ *is a locally convex curve and* Σ *is a nodoid, which has self-intersections.*

(iii) *If* $F = 0$ *and* $H \neq 0$ *then* Σ *is a sphere.*

(iv) *If* $H = 0$ *and* $F \neq 0$ *we obtain a catenary which generates an embedded catenoid* Σ *with* $F > 0$ *if the normal points down and* $F < 0$ *if the normal points up.*

(v) *If* $H = 0$ *and* $F = 0$ *then* Γ *is a straight line orthogonal to the* x*-axis which generates a hyperplane.*

(vi) *If* Σ *touches the* x*-axis, then it must be a sphere or a hyperplane.*

(vii) *The curve* Γ *is determined, up to translation along the* x*-axis, by the pair* (H, F).

The generating curves of nodoids and unduloids are called *nodaries* and *undularies.* Since we shall often identify the curves and the generated hypersurfaces we shall refer to them as nodoids and unduloids.

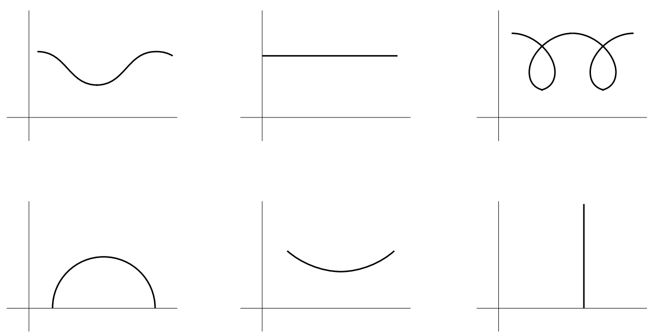

Figure 5. The generating curves for Delaunay hypersurfaces: unduloid, cylinder, nodoid, sphere, catenoid, hyperplane.

Remark 4.4. Henceforth we shall use the following properties of generating curves of Delaunay hypersurfaces.

(i) Unduloids and nodoids have positive mean curvature with respect to the normal which points downward at the maximum of the y-coordinate.

(ii) The nodoid is convex in the sense that the normal vector rotates monotonically. This follows from equations (4.1) and (4.2).

LEMMA 4.5 (Force balancing [KKS, (3.9)]). *Assume that three generating curves* Γ_i, $i = 0, 1, 2$ *of hypersurfaces with constant mean curvature* H_i *and forces* F_i *meet at some point. If* $-H_1 + H_2 + H_0 = 0$ *and* $-N_1 + N_2 + N_0 = 0$ *at this point, then*

$$-F_1 + F_2 + F_0 = 0. \tag{4.3}$$

The lemma follows directly from (4.2).

LEMMA 4.6. *Let* Σ *be a nonstandard minimizing double bubble in* $\mathbb{R}^{n+1}$, *as in Figure* 4, *and let* R_1 *be the region of larger or equal pressure. Assume that the spherical component* Ω *is contained in* R_1. *Let* Γ_1 *be the generating curve of* $M_0 = \Sigma_0 \cap \partial\Omega$.

Then the force of Γ_1 *is positive and* Γ_1 *is an unduloid or catenoid and in particular a graph.*

Proof. Let Γ_0^l, Γ_0^r be the left and right circles in $\partial\Omega$. Consider the embedded curve Γ determined by Γ_0^r, Γ_0^l and Γ_1. Let Γ_2 be the third generating curve meeting $\Gamma_0^l \cap \Gamma_1$ and Γ_3 the one meeting $\Gamma_0^r \cap \Gamma_1$.

If the force of Γ_1 is negative then Γ_1 is a nodoid with positive mean curvature since $H_0 = H_1 - H_2 \geqslant 0$. The graph Γ is convex and, since Γ meets L orthogonally, its total curvature equals π. At each one of the vertices $\Gamma_0^r \cap \Gamma_1$, $\Gamma_0^l \cap \Gamma_1$ the inner angle of Γ is exactly $\pi/3$. By force balancing 4.5, both Γ_2 and Γ_3 have positive force and they are unduloids. Since R_2 has positive pressure both Γ_2 and Γ_3 are inward graphs with respect to Γ_1 (i.e., the exterior region lies above Γ_1 and above Γ_2). Hence the two circular arcs Γ_0^l, Γ_0^r have angular measure larger than $\pi/3$. So the total curvature of Γ is larger than $4\pi/3$, which is a contradiction.

If the force of Γ_1 is 0 then Γ_1 is part of a circle or of a line orthogonal to the axis of revolution L. The former possibility is discarded by the same argument used for nodoids. The latter possibility is clearly not possible.

Hence the force of Γ_1 is positive and Γ_1 is an unduloid or catenoid and graph. □

By a similar argument to the one used in Lemma 4.6 we obtain

LEMMA 4.7. *Let* Σ *be a double bubble of revolution such that both regions have positive pressure. Then it is not possible that* Σ *contains pieces of spheres* Γ_0^l, Γ_0^r, Γ_1, Γ_2, Γ_3 *as in Figure* 6, *with* $\Gamma_1 \subset \Sigma_0$.

LEMMA 4.8. *Let* Σ *be a nonstandard minimizing double bubble in* $\mathbb{R}^3$. *Let* θ_i *be the subtending angle of the spherical caps* D_i *as in Figure* 7.

(i) *If* θ_1, $\theta_2 \leqslant \pi/6$ *then* $\theta_1 = \theta_2$ *and* M_0 *is symmetric with respect to a plane orthogonal to the axis of revolution.*

(ii) *If* $\theta_1 \leqslant \pi/6 < \theta_2 \leqslant \pi/3$ *then* $\theta_2 > \dfrac{\pi}{3} - \theta_1$.

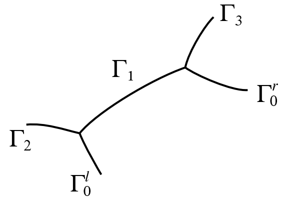

Figure 6.

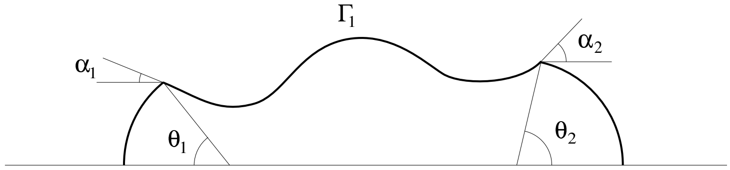

Figure 7. The spherical component Ω

Proof. Assume that the spherical component Ω is contained in R_1. By scaling, normalize $H_1 = 1$.

Let Γ_1 be the generating curve of M_0 parametrized from the left to the right. As $\alpha_i = \theta_i - \pi/6$ we get from (4.2) that the force F_0 of Γ_1 is given by $g(\theta_1) = g(\theta_2)$, where

$$g(\theta) = \left(\frac{1}{2} - H_0\right) \sin^2\theta + \frac{\sqrt{3}}{2} \cos\theta \sin\theta.$$

Since $H_0 = 1 - H_2 < 1$ by Proposition 4.1, we get $g'(\theta) \geqslant -\frac{1}{2}\sin 2\theta + \frac{\sqrt{3}}{2}\cos 2\theta > 0$, and so the function g is strictly increasing in $[0, \pi/6]$. Hence θ_1, $\theta_2 \leqslant \pi/6$ implies $\theta_1 = \theta_2$. Moreover the endpoints of Γ_1 have the same height y_i and the same angle α_i. This proves (i) by the uniqueness for solutions to (4.1) with respect to the initial conditions.

To see (ii) let

$$h(\theta) = -\frac{1}{2}\sin^2\theta + \frac{\sqrt{3}}{2}\sin\theta\cos\theta = g(\theta) - (1 - H_0)\sin^2\theta.$$

It is easily proved that the function h is symmetric with respect to $\pi/6$ and increasing in $[0, \pi/6]$. Thus we have

$$\begin{aligned} g\left(\frac{\pi}{3} - \theta_2\right) &= (1 - H_0)\sin^2\left(\frac{\pi}{3} - \theta_2\right) + h\left(\frac{\pi}{3} - \theta_2\right) \\ &< (1 - H_0)\sin^2\theta_2 + h(\theta_2) = g(\theta_2) = g(\theta_1) \end{aligned}$$

and, as g is increasing in $[0, \pi/6]$, we get $(\pi/3) - \theta_2 < \theta_1$, as we wished. $\square$

Remark 4.9. If Σ were n-dimensional, then the force of Γ_1 would be given by $F_0 = g(\theta)\,\sin^{n-2}\theta$ and so Lemma 4.8 works in arbitrary dimension.

5. Separation and instability

Let $\Sigma \subset \mathbb{R}^{n+1}$ a stationary double bubble of revolution whose axis L is the x_1-axis with generating curve $\Gamma \subset \{(x_1, x_2)|\ x_2 \geqslant 0\}$ consisting of circular arcs $\overline{\Gamma}_0$ meeting the axis and other arcs $\overline{\Gamma}_i$ meeting in threes, with interiors Γ_i (see Figure 9). The bubble Σ is invariant under the action of the group $O(n)$ of orthogonal transformations in $\mathbb{R}^{n+1}$ which fix the x_1-axis. We consider the map $f : \Gamma - L \longrightarrow L \cup \{\infty\}$ which maps each $p \in \Gamma - L$ to the point $L(p) \cap L$, where $L(p)$ denotes the normal line to Γ at p. If $L(p)$ does not meet L, we define the image of p as $f(p) = \infty$. Note that f is multivalued at the endpoints of the arcs Γ_i, where three of them meet. We will use the notation iA and iB for the image under f of the endpoints of $\overline{\Gamma}_i$.

Remark 5.1. Using (4.1) and (4.2), we find that the derivative of f with respect to arc length is given by $f' = \frac{nF}{y^{n-1}\cos^2\alpha}$. In particular, f is increasing as we move to the right along an unduloid or the concave up portion of a nodoid, decreasing as we move to the right along the concave down portion of a nodoid, and of course constant on spheres and vertical hyperplanes. Hence f is locally injective on any Delaunay curve with nonzero force.

Proposition 5.2. *Consider a stable double bubble of revolution $\Sigma \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, with axis L. Assume there is a finite number of points $\{p_1, \ldots, p_k\}$ in $\bigcup_i \Gamma_i$ with $x = f(p_1) = \cdots = f(p_k)$ which separates Γ. Assume further that $\{p_1, \ldots, p_k\}$ is a minimal set with this property.*

Then every connected component of Σ_0, Σ_1 and Σ_2, which contains one of the points p_i is part of a sphere centered at x (if $x \in L$) or a part of a hyperplane orthogonal to L (if $x = \infty$).

Proof. First suppose that $x \in L$ and take, after translation, $x = 0$. The 1-parameter group of rotations

$$\varphi_\theta(x_1, \ldots, x_{n+1}) = (\cos\theta\, x_1 + \sin\theta\, x_2, -\sin\theta\, x_1 + \cos\theta\, x_2, x_3, \ldots, x_{n+1}),$$

$\theta \in \mathbb{R}$, determines a Jacobi function on the bubble, $u : \Sigma \to \mathbb{R}$, given by

$$u(p) = \left\langle \left.\frac{d}{d\theta}\right|_{\theta=0} \varphi_\theta(p), N(p) \right\rangle = -\det(p, N(p), e_3, \ldots, e_{n+1}),$$

where $N(p)$ is the unit normal vector of Σ at p, $\{e_1, \ldots, e_{n+1}\}$ is the standard orthonormal basis of $\mathbb{R}^{n+1}$ and det denotes the Euclidean volume element. We define here $M_0 = \Sigma \cap \{x_2 = 0\}$. By the symmetry of Σ, if $p \in M_0$, then the vector $N(p)$ also lies in the hyperplane $x_2 = 0$ and therefore u vanishes on M_0.

On the other hand, if we take p in $f^{-1}\{0\}$, then the vectors $N(p)$ and p are collinear. Using again the invariance of Σ with respect to $O(n)$, we get that u vanishes on the orbit $M(p)$ of p under the action of $O(n)$ (note that $M(p)$ is a hypersurface of Σ).

As the points $p_1, \ldots, p_k$ separate the curve Γ, the set $M(p_1)\cup\ldots\cup M(p_k)\cup M_0$ is a hypersurface of the bubble contained in $u^{-1}\{0\}$ which separates Σ in at least four connected components. In fact, as the set $\{p_1, \ldots, p_k\}$ is minimal among the subsets of $f^{-1}\{0\}$ satisfying the separation property, it follows that $\Sigma - [M(p_1) \cup \ldots, M(p_k) \cup M_0]$ has exactly four components $\Lambda_1, \ldots, \Lambda_4$ and that each one of the sets $M(p_1), \ldots, M(p_k), M_0$ meets the boundary of each one of these four components.

Consider the functions $v^{(i)}$, $i = 1, \ldots, 4$, on Σ given by

$$v^{(i)} = \begin{cases} u, & \text{on } \Lambda_i, \\ 0, & \text{on } \Sigma - \Lambda_i. \end{cases}$$

Then $v^{(i)}$ are admissible and we can find scalars a_1, a_2, a_3, not all equal to zero, such that $v = \sum_{i=1}^{3} a_i v^{(i)}$ verifies the mean value conditions (3.11), so that $v \in \mathcal{F}(\Sigma)$. By Lemma 3.9,

$$Q(v,v) = \sum_{i=1}^{3} a_i^2\, Q(v^{(i)}, v^{(i)}) = 0.$$

Since u is a Jacobi function,

$$\Delta v + |\sigma|^2 v = 0 \tag{5.1}$$

on $\Sigma \setminus [M(p_1)\cup\cdots\cup M(p_k)\cup M_0]$. By our stability hypothesis and Lemma 3.8, equation (5.1) holds on all of Σ.

Fix i and let S be the connected component of a smooth piece of Σ which contains the point p_i. As p_i lies in the interior of S, the four domains Λ_i meet the interior of S. As v vanishes on $S \cap \Lambda_4$, from the unique continuation property, we conclude that $v = 0$ on S. Hence $u = 0$ on $S \cap \Lambda_j$, for any $j \in \{1, 2, 3\}$ such that $a_j \neq 0$. As such j exists we conclude that $u = 0$ on S again from the unique continuation property. Thus the 1-parameter group of rotations φ_θ preserves S. Since S is rotationally symmetric around the x_1-axis, we conclude that this component is a part of a sphere centered at the origin.

This finishes the proof of the proposition if x is a finite point of the axis L.

It remains to prove the result when $x = \infty$. In order to prove it we repeat the argument by considering, instead of the rotations φ_θ, the 1-parameter group of translations $T_\theta(x_1, \ldots, x_{n+1}) = (x_1, x_2 + \theta, \ldots, x_{n+1})$ and its associated Jacobi function $u(p) = \langle N(p), e_2\rangle$. □

Corollary 5.3. *There is no stable double bubble of revolution in* $\mathbb{R}^{n+1}$ *in which the graph structure is the one in Figure* 8.

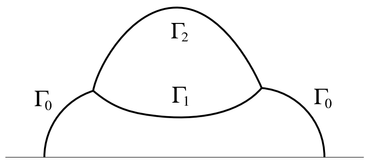

Figure 8. There is no stable nonstandard double bubble with connected regions.

Proof. Assume first that the line equidistant from the two vertices intersects the axis L in a point p. Then Γ_1 and Γ_2 each has an interior point farthest from or closest to p, so that $p \in f(\Gamma_1) \cap f(\Gamma_2)$. By Proposition 5.2, Γ_1 and Γ_2 are both spherical (centered on the axis), which is impossible.

If the equidistant line is horizontal Γ_1, Γ_2 each has an interior point farthest left or right, so that $\infty \in f(\Gamma_1) \cap f(\Gamma_2)$. By Proposition 5.2, Γ_1 and Γ_2 are both vertical, which is impossible. □

Remark 5.4. When $n = 2$ and the volumes are equal, Hutchings [Hu, Thm. 5.1, Cor. 4.4] showed, as described in our Section 6, that any nonstandard minimizing bubble satisfies the hypotheses of Corollary 5.3. Therefore in this case the minimizing solution is the standard bubble. This fact was first proved by computer analysis by Hass and Schlafly [HS2].

COROLLARY 5.5. *Consider a stable double bubble of revolution in* $\mathbb{R}^{n+1}$ *in which both regions have positive pressure. Assume that one of the regions* R_2 *is connected, that the other one* R_1 *has two components and that the graph structure is the one in Figure* 9.

Then there is no $x \in L$ *such that* $f^{-1}(x) - \Gamma_0$ *contains points in the interiors of distinct* Γ_j *which separate* Γ.

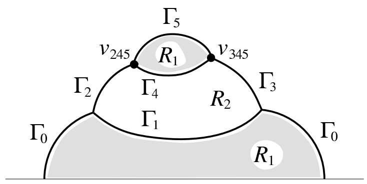

Figure 9. A candidate double bubble of three components.

Proof. There must be points on Γ_1, Γ_2, or Γ_3. By Proposition 5.2, one of them is spherical. By force balancing 4.5, all three are spherical, which is impossible by Lemma 4.7. □

We give the following version of Corollary 5.5, although we will not use it here.

COROLLARY 5.6. *For a stable double bubble of revolution, if* f *is not injective on the interior of* Γ_i, *then* Γ_i *is a circular arc or vertical line.*

Proposition 5.7. *Consider a minimizing nonstandard double bubble in* $\mathbb{R}^{n+1}$, *which is necessarily rotationally symmetric around an axis* L.

Then there exists no $x \in L$ *such that* $f^{-1}(x) - \Gamma_0$ *contains points in the interiors of distinct* Γ_j *which separate* Γ.

Proof. Since every component borders the exterior, a separating set must cut the outer boundary of some component. By force balancing 4.5, every piece of the outer boundary of this component is spherical. If $f^{-1}(x)$ cuts two pieces of the outer boundary, then these are pieces of spheres with the same center and the same mean curvature, and hence the same sphere. The portion of the bubble between these two pieces can then be rolled around the sphere, without changing perimeter or enclosed volume, until it touches some other part of the bubble, resulting in a bubble which is not regular, and hence not minimizing. So it cuts an inner boundary (part of Σ_0). By force balancing 4.5, each end of the inner boundary meets two other spheres, which contradicts Lemma 4.7. □

Proposition 5.8. *There is no stable double bubble of revolution in* $\mathbb{R}^{n+1}$ *in which both regions have positive pressure, the region of smaller or equal pressure* R_2 *is connected, the other region* R_1 *has two components, and the graph structure is the one in Figure* 9.

Proof. Suppose there were. Γ_0 are spherical. Γ_1 is an unduloid or catenoid and graph by Lemma 4.6. By force balancing 4.5, Γ_2 and Γ_3 are (convex) nodoids. Since the top, third component has larger pressure, Γ_4 must be a (convex) nodoid, catenoid, or vertical line, unless it is upside down (which cannot occur in the principal cases of Figure 14). (Here by "convex" we just mean that the tangent vector rotates monotonically.)

We focus on the third component and its two vertices v_{245} and v_{345}. For the simplest case when all the curves are graphs as in Figure 10A, then the images iA and iB under f of the left and right endpoints of Γ_i satisfy

$$4A < 2B < 5A \qquad \text{and} \qquad 5B < 3A < 4B.$$

This remains true as a vertex rotates until one of the three tangent vectors goes vertical. (The borderline position with $5A = \infty$ may be considered an extreme position of either case; in the proof we consider it part of the second case eliminated.) Rotating v_{245} counterclockwise one notch as in Figure 10B yields instead $5A < 4A < 2B$. To avoid giving Γ_4 or Γ_5 two vertical tangents contrary to Corollary 5.6, the two vertices must be rotated in the same direction, say counterclockwise. Suppose that v_{245} is rotated m_1 notches counterclockwise and that v_{345} is rotated m_2 notches counterclockwise. Then $m_1 \leqslant 2$, or Γ_2 (where R_2 is a convex region by positive pressure) could not meet the circle Γ_0 at 120 degrees (see Figure 11). Also $m_2 \leqslant 3$, or Γ_4 would go vertical twice (see Figure 12), contrary to Corollary 5.6.

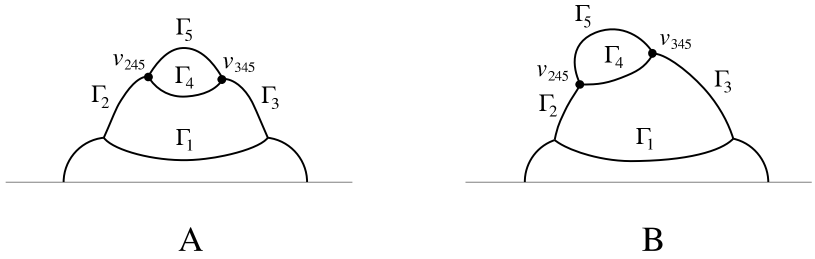

Figure 10. Rotating vertex v_{245} one "notch" (counterclockwise) means turning one tangent (here Γ_5) past vertical. (Rotating another notch would turn Γ_2 past vertical.)

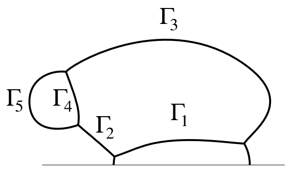

Figure 11. If v_{245} turns three notches, Γ_2 cannot be convex.

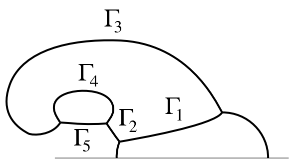

Figure 12. If v_{345} turns four notches, Γ_4 goes vertical twice.

Next consider cases with $m_2 = 2$ or $m_2 = 3$, as in Figure 13. Γ_3 is not a graph, $f(\Gamma_3) = [\infty, 3A) \cup (3B, \infty]$, and by stability Proposition 5.2 gives that, for Γ_3, $3A \leqslant 3B$. We then have $3B < 1B$, or else Γ_3 would go vertical a second time near $3B$, contradicting Corollary 5.6. Therefore $1B$ is contained in $f(\Gamma_3)$, which contradicts Corollary 5.5 for Γ_1 and Γ_3.

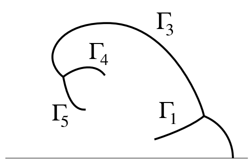

Figure 13. If $m_2 = 2$ or $m_2 = 3$, then Γ_3 is not a graph and $f(\Gamma_3) = [\infty, 3A) \cup (3B, \infty]$.

Now remain only the cases $0 \leqslant m_1 \leqslant 2$, $0 \leqslant m_2 \leqslant 1$ of Figure 14. (It is easy to see that Γ_4 cannot be a vertical line.) We claim that

$$3A < f(\Gamma_1). \tag{5.2}$$

This is easy if Γ_3 is a graph, since consideration of v_{13} shows that $3B < 1B$, and then $f(\Gamma_3) < f(\Gamma_1)$ by Corollary 5.5 for Γ_1 and Γ_3. v_{13} can rotate only clockwise one notch to keep the stem part of a circle and Γ_1 a graph. Now Γ_3 is not a graph and $f(\Gamma_3)$ includes $[\infty, 3A)$. By Corollary 5.5 for Γ_1 and Γ_3, $3A < f(\Gamma_1)$ as claimed.

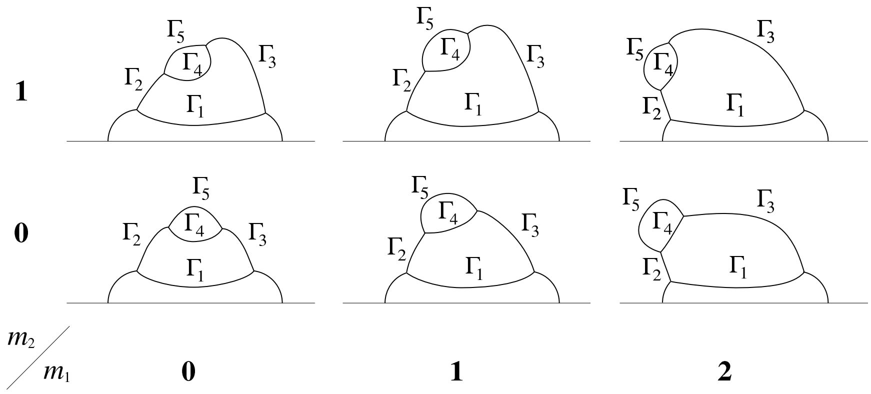

Figure 14. The six principal cases to be eliminated

For the cases $(0,0)$, $(0,1)$, a similar argument shows that $f(\Gamma_1) < 2B$. Consideration of the vertices v_{245} and v_{345} leads to the conclusion

$$5B < 3A < f(\Gamma_1) < 2B < 5A. \tag{5.3}$$

Since the net angle θ_5 through which Γ_5 turns satisfies $\theta_5 \leqslant 180$ degrees, obviously $4A < 3A$. For the case $(0,1)$, where $f(\Gamma_4)$ contains $(4A, \infty]$, this puts $3A$ in $f(\Gamma_4) \cap f(\Gamma_5)$, a contradiction of Corollary 5.5 for Γ_3, Γ_4, Γ_5. For the case $(0,0)$, consideration of v_{345} shows that $3A < 4B$ and leads to the same contradiction.

Next we consider the cases $(1,0)$, $(2,0)$. Since $5B < 3A$, $3A$ is contained in $f(\Gamma_5)$. Since $3A < 4B$, by Corollary 5.5 for Γ_3, Γ_4, Γ_5, we must have $3A \leqslant 4A$. In particular, $\theta_5 > 180$ degrees. Of course by Corollary 5.6 for Γ_5, $5A \leqslant 5B$. Moreover Γ_5 leaves v_{345} above the horizontal. Now Corollary 5.10 implies that $3A > 4A$, a contradiction.

Similarly for the final cases $(1,1)$ and $(2,1)$, $3A$ is contained in $f(\Gamma_5)$. Since $f(\Gamma_4)$ includes $(4A, \infty]$, by Corollary 5.5 for Γ_3, Γ_4, Γ_5, we must have $3A \leqslant 4A$, an immediate contradiction in case $(1,1)$. In particular, $\theta_5 > 180$ degrees, and $5A \leqslant 5B$. If Γ_5 leaves v_{345} at or above the horizontal, Corollary 5.10 yields the contradiction $3A > 4A$. If on the other hand Γ_5 leaves v_{345} below the horizontal, then the downward normal n to Γ_3 at v_{345} is counterclockwise from the downward tangent to Γ_2 at v_{12} (and hence from every downward tangent to Γ_2) and hence counterclockwise from the downward normal to Γ_1 at v_{12}. Since Γ_4 is convex, n stays to the right of Γ_2 and $1A < 3A$, a contradiction of (5.2). □

LEMMA 5.9. *Given points A and B, consider two points D, E on the same side of AB subtending the same angle θ as in Figure 15. Then $\angle CDE = \angle ABC$.*

Proof. Since $\angle BCA = \angle DCE$ and $ACE \sim BCD$, $ABC \sim CDE$. □

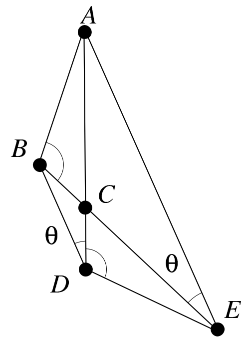

Figure 15. $\angle CDE = \angle ABC$

COROLLARY 5.10. *In cases $(1,0)$, $(2,0)$, and $(2,1)$ of Figure 14, suppose that the net angle θ_5 through which Γ_5 turns exceeds 180 degrees, that Γ_5 leaves v_{345} at or above the horizontal, and that $5A \leqslant 5B$. Then $4A < 3A$.*

Proof. Let $\theta = \theta_5 - 180 > 0$. Apply Lemma 5.9 with $A = v_{345}$, $B = v_{245}$, $AD \perp \Gamma_5$ and $AE \perp \Gamma_3$ at v_{345}, $BD \perp \Gamma_5$ and $BE \perp \Gamma_4$ at v_{245}; then $\angle ADB = \angle AEB = \theta$. Since Γ_4 is strictly convex (it cannot be a vertical line because $5A \leqslant 5B$), $\angle ABC > 90$. By Lemma 5.9, $\angle CDE = \angle ABC > 90$. Since by hypothesis Γ_5 leaves v_{345} at or above the horizontal, DE heads downward. (In these cases, Figure 15, in which AD is vertical, is rotated clockwise by an amount less than 90 degrees, strictly less by Corollary 5.6 since $\theta_5 > 180$.) Since $5A \leqslant 5B$, D lies on or below the horizontal axis. Hence E lies below the horizontal axis and $4A < 3A$. □

6. Estimates on the number of components

In this section we prove that for a minimizing double bubble in $\mathbb{R}^3$, the larger region is connected, and the smaller region has at most two components.

We begin by recalling a version of the Hutchings Basic Estimate. Let $A(v)$ denote the volume of a sphere in $\mathbb{R}^3$ enclosing volume v, and let $A(v_1, v_2)$ denote the area of the standard double bubble enclosing volumes v_1 and v_2 in $\mathbb{R}^3$. We then have:

Proposition 6.1 ([Hu, Thm. 4.2]). *In an area-minimizing double bubble enclosing volumes v_1 and v_2 in $\mathbb{R}^3$, suppose that R_2 contains a component of volume λv_2. Then*

$$2A(v_1, v_2) \geqslant \lambda^{-1/3} A(v_2) + A(v_1) + A(v_1 + v_2).$$

This inequality places a lower bound on λ in terms of v_1 and v_2. Hutchings [Hu, 4.4, 4.5] calculated that when $v_1 = v_2$, the lower bound is greater than 1/2, so both regions are connected; and when $v_1 >> v_2$, the lower bound approaches 2/5, so the smaller region has at most 2 components. More recently, Heilmann et al. [HLRS, Fig. 8] ([M2, Fig. 14.11.1]) used a computer to plot the lower bound on λ as a function of the ratio v_2/v_1, and found that it is apparently increasing. This would imply that the larger region is always connected and the smaller region always has at most two components. However this observation has not been rigorously proved, because the function $A(v_1, v_2)$ is difficult to work with. Thus we will use different methods to prove the above connectedness results.

Proposition 6.2. *In a minimizing double bubble in $\mathbb{R}^3$, the region with larger or equal volume is connected.*

Proof. By rescaling, we may assume that the two volumes are $1-w$ and w. Hutchings [Hu, Thm. 3.5] showed that if $w < 1/3$, then the larger region is connected. (This is true in higher dimensions as well.)

For $w \geqslant 1/3$, to prove that the region of volume $1 - w$ is connected, it is enough to show that the inequality of Proposition 6.1 fails for $\lambda = 1/2$; i.e.

$$2A(w, 1 - w) < 2^{1/3} A(1 - w) + A(w) + A(1).$$

We observe that by Lemma 3.1, $dA(w, 1 - w)/dw > 0$ for $w < 1/2$, because we can continuously deform one standard double bubble to another, and the smaller region has larger pressure. Thus $A(w, 1-w) \leqslant A(1/2, 1/2)$. It is straightforward to compute that $A(1/2, 1/2) = 2^{-4/3}3A(1)$, and $A(v) = v^{2/3}A(1)$. Thus it will suffice to show that

$$2^{-1/3}3 < 2^{1/3}(1-w)^{2/3} + w^{2/3} + 1.$$

Since this holds at $w = 0.10$ and $w = 0.63$ and the right-hand side is concave, it holds for $0.10 \leqslant w \leqslant 0.63$. In particular, it holds for $1/3 \leqslant w \leqslant 1/2$, as desired. □

Remark 6.3. An alternative proof is given by Heilmann et al. [HLRS, Prop. 2.5]. Actually as in the proof above, any region with at least 37% of the total volume is connected.

LEMMA 6.4. *In a minimizing double bubble in* $\mathbb{R}^{n+1}$ *enclosing two unequal volumes, the smaller region has larger pressure.*

Proof. Consider the function $A(v, 1-v)$ giving the least area enclosing and separating regions of volume v, $1-v$. By Hutchings [Hu, Thm. 3.2], A is strictly concave and of course symmetric about $v = 1/2$. Moving the separating surface (of mean curvature H_0) of an area-minimizing bubble we have $dA/dv = nH_0$, and the left and right derivatives of A must satisfy

$$A'_R \leqslant nH_0 \leqslant A'_L.$$

Consequently H_0 is positive for $v < 1/2$ and negative for $v > 1/2$. In other words, the smaller region has larger pressure. □

The following Proposition 6.5 shows that small region with three components is unstable, by using a volume-preserving linear combination of (non-volume-preserving) variations of each of the components.

PROPOSITION 6.5. *In a minimizing double bubble in* $\mathbb{R}^3$, *the region with smaller or equal volume has at most two connected components.*

Proof. Assume that the volume of R_1 is less than or equal to the volume of R_2. By Proposition 4.1 we obtain H_1, $H_2 > 0$. By Lemma 6.4 we get $H_0 \geqslant 0$.

Recall that $\kappa_i = \sigma_i(\nu_i, \nu_i)$, and let $c_i = \sigma_i(T, T)$, where T is the unit tangent vector to the singular curve C. So $2H_i = \kappa_i + c_i$.

We consider an admissible function u invariant by the one-parameter group of rotations of Σ. The functions u_i are locally constant over C. If we apply (3.12) to u, adding and subtracting $c_i u_j^2$ in the boundary term, we see that the second variation form satisfies

$$\begin{aligned} Q(u,u) &= \sum_i \int_{\Sigma_i} \{|\nabla u_i|^2 - |\sigma_i|^2 u_i^2\} \\ &\quad - \frac{2}{\sqrt{3}} \int_C (H_0 - H_2)\, u_1^2 + (-H_1 - H_0)\, u_2^2 + (H_1 + H_2)\, u_0^2 \\ &\quad + \frac{1}{\sqrt{3}} \int_C (c_0 - c_2)\, u_1^2 + (-c_1 - c_0)\, u_2^2 + (c_1 + c_2)\, u_0^2. \end{aligned}$$

Taking the scalar product with $D_T T$ in the formulae (3.9) we have

$$\kappa_g^1 = \frac{1}{\sqrt{3}}\,(c_0 - c_2), \qquad \kappa_g^2 = \frac{1}{\sqrt{3}}\,(-c_1 - c_0), \qquad \kappa_g^0 = \frac{1}{\sqrt{3}}\,(c_1 + c_2),$$

where κ_g^i stands for the geodesic curvature of C inside Σ_i (with respect to the conormal ν_i). So we have

$$\begin{aligned} (6.1)\quad Q(u,u) &= \sum_i \int_{\Sigma_i} \left\{|\nabla u_i|^2 - |\sigma_i|^2 u_i^2\right\} \\ &\quad - \frac{2}{\sqrt{3}} \int_C (H_0 - H_2)\, u_1^2 + (-H_1 - H_0)\, u_2^2 + (H_1 + H_2)\, u_0^2 \\ &\quad + \int_C \kappa_g^1\, u_1^2 + \kappa_g^2\, u_2^2 + \kappa_g^0\, u_0^2. \end{aligned}$$

Consider a connected component Ω of R_1. Let $M_i = \Sigma_i \cap \partial\Omega$, and let $C^* = C \cap \partial\Omega$. We want to find an admissible function u such that $Q(u,u) < 0$ with support inside $\partial\Omega$. Then if R_1 had three connected components, some nonzero linear combination would preserve the volumes and yield a contradiction.

We define the function

$$v = \begin{cases} 1, & \text{on } M_0 \cup M_1, \\ 0, & \text{elsewhere in } \Sigma. \end{cases}$$

Then (6.1) gives

$$(6.2)\qquad Q(v,v) = -\sum_{i=0,1} \int_{M_i} |\sigma_i|^2 - \frac{2}{\sqrt{3}} \int_{C^*} (H_0 + H_1) + \int_{C^*} (\kappa_g^0 + \kappa_g^1).$$

Since $|\sigma_i|^2 = 4H_i^2 - 2K_i$, from (6.2) and Gauss-Bonnet $\int_{M_i} K_i = 2\pi\chi(M_i) - \int_{\partial M_i} \kappa_g^i$ we obtain

$$(6.3)\quad Q(v,v) = \sum_{i=0,1} \left\{4\pi\chi(M_i) - \int_{M_i} 4H_i^2\right\} - \frac{2}{\sqrt{3}} \int_{C^*} (H_0 + H_1) - \int_{C^*} (\kappa_g^0 + \kappa_g^1).$$

Assume first that Ω is a torus component, so that its boundary is a union of annuli. Adding (6.2) and (6.3) and taking into account that $\chi(M_i) = 0$, we eliminate the geodesic curvature to obtain

$$2Q(v,v) = -\sum_{i=0,1}\int_{M_i}\left\{|\sigma_i|^2+4H_i^2\right\} - \frac{4}{\sqrt{3}}\int_{C^*}(H_0+H_1) < 0,$$

as desired.

We now assume that Ω is the spherical component, so that M_1 is the union of two spherical caps D_1, D_2 and an annulus M_0, as in Figure 7. As M_0 is a graph by Lemma 4.6 we conclude $0 < \theta_i \leqslant \frac{2\pi}{3}$, where θ_i is the angle determined by D_i. By scaling we may assume that the spherical caps have mean curvature $H_1 = 1$. Using Gauss-Bonnet we get that

$$A(M_1) = \int_{M_1} K_1 = 4\pi - \int_{C^*}\kappa_g^1. \tag{6.4}$$

Since $\nu_0 = (-1/2)\,\nu_1 + (\sqrt{3}/2)\,N_1$, taking the scalar product with $D_T T$ we have

$$\kappa_g^0 = -\frac{1}{2}\kappa_g^1 + \frac{\sqrt{3}}{2}. \tag{6.5}$$

From (6.3), (6.4) and (6.5) we obtain, taking into account that $\chi(M_1) = 2$ and discarding the summands containing H_0,

$$Q(v,v) \leqslant 6\pi + \frac{7}{2}\,A(M_1) - \frac{7}{2\sqrt{3}}\,L(C^*).$$

As $A(D_i) = 2\pi\,(1-\cos\theta_i)$ and $L(\partial D_i) = 2\pi\sin\theta_i$, we have

$$Q(v,v) \leqslant 2\pi\left\{-4 + h(\theta_1) + h(\theta_2)\right\}, \tag{6.6}$$

where $h(\theta) = \frac{7}{2}\left(\cos\theta - \frac{1}{\sqrt{3}}\sin\theta\right)$, which is decreasing in $[0, 2\pi/3]$. Thus if θ_1 or θ_2 is greater than or equal to $\pi/2$, we have

$$Q(v,v) \leqslant 2\pi\{h(0) + h(\pi/2)\} < 0.$$

Assume now that both θ_1, $\theta_2 < \pi/2$, and consider the function

$$w = \begin{cases} \dfrac{\cos\theta}{\cos\theta_i}, & \text{in } D_i, \\ 1, & \text{in } M_0, \\ 0, & \text{elsewhere in } \Sigma. \end{cases}$$

As v and w differ only on M_1 we obtain from (3.12)

$$Q(w,w) = \int_{M_1}\left(|\nabla w|^2 - 2w^2\right) + 2\int_{M_1} 1 + Q(v,v). \tag{6.7}$$

By direct computation we get

$$\int_{M_1}\left(|\nabla w|^2 - 2w^2\right) = -2\pi\sum_{i=1,2}\frac{\sin^2\theta_i}{\cos\theta_i}, \qquad \int_{M_1} 1 = 2\pi\sum_{i=1,2}(1-\cos\theta_i),$$

which combined with (6.6) and (6.7) yield

$$Q(w,w) \leqslant 2\pi\,\{g(\theta_1)+g(\theta_2)\}\,, \tag{6.8}$$

where g is given by

$$g(\theta) = \frac{3}{2}\cos\theta - \frac{7}{2\sqrt{3}}\sin\theta - \frac{\sin^2\theta}{\cos\theta}.$$

The function g is strictly decreasing in $[0,\pi/2]$ since it is the sum of three decreasing functions. As $g(0) = \frac{3}{2} > 0$, $g(\pi/6) = 0$, and $g(\pi/3) = -5/2$ we conclude

$$Q(w,w) < 0 \quad \text{if either both} \quad \theta_1,\theta_2 > \frac{\pi}{6} \quad \text{or some} \quad \theta_i \geqslant \frac{\pi}{3}.$$

We finally consider the remaining cases in which at least one of the angles θ_i is less than or equal to $\pi/6$ and both are less than $\pi/3$.

Case 1. $\theta_1 \leqslant \frac{\pi}{6} < \theta_2 < \frac{\pi}{3}$.

Observe that g is concave in $[0,\pi/3]$ since

$$g''(\theta) = -\frac{7}{2}\left(\cos\theta - \frac{1}{\sqrt{3}}\sin\theta\right) - \left(3\,\frac{\sin^2\theta}{\cos\theta} + 2\,\frac{\sin^3\theta}{\cos^2\theta}\right) < 0.$$

By Lemma 4.8 we know that $\frac{\pi}{3} - \theta_1 < \theta_2$. As g is decreasing and concave

$$\frac{1}{2\pi}\,Q(w,w) \leqslant g(\theta_1)+g(\theta_2) < g(\theta_1) + g\left(\frac{\pi}{3}-\theta_1\right) \leqslant 2\,g\left(\frac{\pi}{6}\right) = 0.$$

Case 2. $\theta_1,\, \theta_2 \leqslant \frac{\pi}{6}$.

By Lemma 4.8 M_0 is symmetric with respect to a plane orthogonal to the line of symmetry. So if $\kappa_1 = \sigma_1(\nu_1,\nu_1) \geqslant 0$ we get from (3.12)

$$Q(v,v) = -\int_{M_0\cup M_1} |\sigma|^2 - \int_{C^*} (\kappa_1+\kappa_0) < 0.$$

If $\kappa_1 = \sigma_1(\nu_1,\nu_1) < 0$ then the Gauss curvature of M_0 along C is negative. By Lemma 4.6 M_0 is an unduloid or a catenoid. As $\theta_i \leqslant \pi/6$ the vectors ν_1, which are tangent to the generating curve Γ_1 of M_0 at their endpoints, are either horizontal or upper pointing. Therefore M_0 contains a nodal region of the Gauss curvature in its interior, which implies that M_0 is unstable [RR, Thm. 3], [PR, Prop. 4.1].

So for any component of R_1 we have an admissible function u such that $Q(u,u) < 0$ with support inside the boundary of the component. If we had three connected components in R_1 then we could get an admissible function satisfying the mean value zero property (3.11), which gives instability of the considered double bubble, a contradiction. ◻

Remark 6.6.

(a) An alternative, computational proof of Proposition 6.5, using Proposition 6.1, is outlined by Heilmann et al. [HLRS, Prop. 4.6] (see [M2, 14.11, 14.13]).

(b) Reichardt et al. [RHLS] extended the arguments of Section 5 to prove the double bubble conjecture assuming only that one region is connected, thus providing an alternative to proving Proposition 6.5.

7. Proof of the double bubble conjecture

THEOREM 7.1. *The standard double bubble in* $\mathbb{R}^3$ *is the unique area-minimizing double bubble for prescribed volumes.*

Proof. Let Σ be an area-minimizing double bubble. By Propositions 6.2 and 6.5, either both regions are connected, or the region of larger volume and smaller pressure is connected and the one of smaller volume and larger pressure has two components. By Proposition 4.1, Σ is either the standard double bubble or a bubble like the ones in Figures 8 or 9. As Σ is stable, by Corollary 5.3 and Proposition 5.8 we conclude that it must be the standard double bubble. □

7.1. *Immiscible fluid clusters.* The methods of this paper extend to double clusters in which the three interfaces carry different costs, so-called immiscible fluid clusters (see M2, Chapt. 16]). We assume that the costs a_{01}, a_{02}, a_{12} satisfy strict triangle inequalities, such as

$$\varepsilon_{02} = a_{01} + a_{12} - a_{02} > 0.$$

THEOREM 7.2. *For nearly unit costs, if the smaller region has at least* 37% *of the volume, then the standard double cluster minimizes energy.*

Proof sketch. Proposition 6.1 has the following generalization to least energy:

$$2E(v_1, v_2) \geqslant \lambda^{-1/3}\varepsilon_{01}A(v_2) + \varepsilon_{02}A(v_1) + \varepsilon_{12}A(v_1 + v_2). \tag{7.1}$$

When the costs a_{ij} and hence the ε_{ij} are all 1, (7.1) reduces to Proposition 6.1. When they are near 1, the proof of Proposition 6.2 still shows that both regions are connected. Now the simple plane geometry of Corollary 5.3 shows that the cluster must be standard. □

Remark 7.3. For general costs, even for equal volumes, it remains an open question whether the standard double immiscible fluid cluster minimizes energy. The above proof applies whenever we know that both regions are connected. (The more complicated plane geometry of Proposition 5.8 (or [RHLS]), for the case when the larger region is connected but the smaller region has two (or more) components, does not immediately generalize, because the generating curves no longer meet at 120 degrees.) Unfortunately, for general costs, even for equal volumes, (7.1) does not imply both regions connected.

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