Superficies mínimas y problema isoperimétrico

Joaquín Pérez

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(M, ds^2) superficie Riemanniana.

$L^2(M) = L^2(M, ds^2) = \{u \colon M \to \mathbb{R} \text{ medible } : \int_M u^2\, dA < \infty\}$.

- $L^2(M)$ e. Hilbert con $\langle u, v\rangle_{L^2} = \int_M uv\, dA$.

$\mathcal{L}^2(M) = \mathcal{L}^2(M, ds^2) = \{X \colon M \to TM \text{ medible } : |X| \in L^2(M)\}$.

- $\mathcal{L}^2(M)$ e. Hilbert con $\langle X, Y\rangle_{\mathcal{L}^2} = \int_M \langle X, Y\rangle\, dA$.
- $u \in L^2(M)$, $X \in \mathcal{L}^2(M)$. X gradiente débil de u si $\int_M (\langle X, Y\rangle + u \operatorname{div} Y)\, dA = 0, \ \forall Y \in \mathfrak{X}_0(M)$.
- El gradiente débil, si existe, es único $\rightsquigarrow$ $\nabla u \in \mathcal{L}^2(M)$.

Definición

$H^1(M) = H^1(M, ds^2) = \{u \in L^2(M) \mid \exists \nabla u \in \mathcal{L}^2(M)\}$.

- $H^1(M)$ e. Hilbert con $\langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \langle \nabla u, \nabla v\rangle_{\mathcal{L}^2}$.
- $C_0^\infty(M) \subset H^1(M)$ no es denso en $H^1(M)$ (sí en $L^2(M)$).

$H_0^1(M) = H_0^1(M, ds^2) = \overline{C_0^\infty(M)}^{H^1}$.

- $u \colon M \to \mathbb{R}$ lipsch, $\operatorname{sop}(u)$ cpt $\Rightarrow u \in H_0^1(M)$ (ej: f. meseta)

Preliminares: Operadores de Schrödinger

(M, ds^2), $\Omega \subset\subset M$. $\quad L = \Delta + q$, $\quad q \in C^\infty(M)$.

Forma cuadrática asociada: $\quad \forall \varphi \in C_0^\infty(\Omega) \;\; (H_0^1(\Omega))$,

$$Q(\varphi, \varphi) = -\int_\Omega \varphi L \varphi \, dA = -\int_\Omega \varphi(\Delta\varphi + q\varphi)\, dA = \int_\Omega (|\nabla \varphi|^2 - q\varphi^2)\, dA,$$

Definición

$\lambda \in \mathbb{R}$ valor propio de L en Ω sii $\exists u \in H_0^1(\Omega) - \{0\}$ (f. propia) t.q. $Q(u, \varphi) = \lambda \int_\Omega u\varphi \, dA$, $\forall \varphi \in C_0^\infty(\Omega)$.

- $\text{Spec}(L, \Omega) = \{\text{valores propios}\} \subset \mathbb{R}$,
  $V_\lambda = \{\text{f. propias para } \lambda \in \text{Spec}(L, \Omega)\} \leq H_0^1(\Omega) \cap C^\infty(\Omega)$
- $\text{Spec}(L, \Omega) = \{\lambda_1 < \lambda_2, \overset{n_2)}{\ldots}, \lambda_2 < \lambda_3, \overset{n_3)}{\ldots}, \lambda_3, \lambda_4, \ldots\} \nearrow \infty$
  $\{u_1, \;\; u_2, \ldots, u_{1+n_2}, u_{2+n_2}, \ldots\}$ base ort $L^2(M)$.
- $\lambda_{k+1} = \text{ínf}\left\{ \frac{Q(\varphi,\varphi)}{\int_M \varphi^2 \, dA} \mid \varphi \in H_0^1(\Omega) \cap \{u_1, \ldots, u_k\}^\perp, \;\; \varphi \not\equiv 0 \right\}$.
- $u \in V_\lambda - \{0\}$ con signo constante $\;\Leftrightarrow\; \lambda = \lambda_1$.
- $\left\{ \begin{array}{ll} Lu + \lambda u = f & \text{en } \Omega \\ u = 0 & \text{en } \partial\Omega \end{array} \right.$ tiene sol'n([1]) $\Leftrightarrow f \in V_\lambda^\perp$ ($\lambda \notin$Spec)

Preliminares: Indice y estabilidad

(M, ds^2), $\Omega \subset\subset M$. $\quad L = \Delta + q$, $\quad q \in C^\infty(M)$.

Indice(L, Ω)= Indice $\left(Q \text{ en } H_0^1(\Omega)\right) = \#(\text{Spec}(L, \Omega) \cap \mathbb{R}^-)$.

- $\Omega \subseteq \Omega' \Rightarrow$ Indice$(L, \Omega) \leq$ Indice(L, Ω') ($<$ si Vol$(\Omega' - \Omega) > 0$)

Definición (Indice y estabilidad para completas)

Indice(L)=*Indice*$(L, M) = \lim_{n\to\infty}$ *Indice*(L, Ω_n), $\{\Omega_n\}_n \nearrow M$.

Indice$(L) = 0 \iff Q(\varphi, \varphi) \geq 0 \;\; \forall \varphi \in H_0^1(M) \iff -L \geq 0$ *en* M.

- Si $q_1, q_2 \in C^\infty(M)$, $q_1 \leq q_2$ en $M \Rightarrow Q_2(\varphi, \varphi) \leq Q_1(\varphi, \varphi)$.
  Si además $-(\Delta + q_2) \geq 0$ en $M \Rightarrow -(\Delta + q_1) \geq 0$ en M.

Lema (Fischer-Colbrie)

$L = \Delta + q$, $q \in C^\infty(M)$. *Son equivalentes:*

1. $-L \geq 0$ *en* M.
2. $\exists u \in C^\infty(M, \mathbb{R}^+)$ *tal que* $Lu = 0$ *en* M.
3. $\exists u \in C^\infty(M, \mathbb{R}^+)$ *tal que* $Lu \leq 0$ *en* M.

Demostración.

1 $\Rightarrow$ 2. $Q(\varphi,\varphi) \geq 0 \ \forall \varphi \in C_0^\infty(M) \ \Rightarrow$ Dado $\Omega \subset\subset M$,

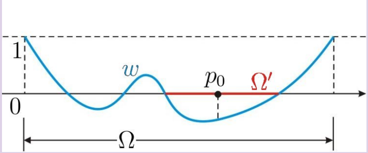
$$\lambda_{1,\text{Dirichlet}}(L,\Omega) = \text{ínf}\left\{ \frac{Q(\varphi,\varphi)}{\int_M \varphi^2 \, dA} \mid \varphi \in C_0^\infty(\Omega), \ \varphi \not\equiv 0 \right\} \geq 0.$$

Monotonía de $\lambda_1 \ \Rightarrow \ \lambda_1(L,\Omega) > \lambda_1(L,\Omega') \geq 0 \ \ \forall \Omega \subset\subset \Omega'$.
$\lambda_1(L,\Omega) > 0 \ \Rightarrow$ existencia, unicidad de solución de

$$\begin{cases} Lv = -q & \text{en } \Omega \\ v = 0 & \text{en } \partial\Omega \qquad (\text{ptualmente si } \partial\Omega \ C^1) \end{cases}$$

Defino $w = v + 1 \Rightarrow \ \Delta w + qw = 0$ en Ω, $w|_{\partial\Omega} = 1$.

- $w \geq 0$ en Ω: Si $w(p_0) < 0$ en $p_0 \in \Omega \ \Rightarrow \ \lambda_1(\Omega') = 0$, donde $\Omega' =$ comp. de p_0 en $w^{-1}(\mathbb{R}^-)$!!



Demostración.

$1 \Rightarrow 2$. $Q(\varphi,\varphi) \geq 0 \ \forall \varphi \in C_0^\infty(M) \ \Rightarrow$ Dado $\Omega \subset\subset M$,

$$\lambda_{1,\text{Dirichlet}}(L,\Omega) = \text{ínf}\left\{ \frac{Q(\varphi,\varphi)}{\int_M \varphi^2 \, dA} \mid \varphi \in C_0^\infty(\Omega), \ \varphi \not\equiv 0 \right\} \geq 0.$$

Monotonía de $\lambda_1 \ \Rightarrow \ \lambda_1(L,\Omega) > \lambda_1(L,\Omega') \geq 0 \ \ \forall \Omega \subset\subset \Omega'$.

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Defino $w = v + 1 \Rightarrow \ \Delta w + qw = 0$ en Ω, $w|_{\partial\Omega} = 1$.

- $w \geq 0$ en Ω.
- $w > 0$ en Ω: Principio del máximo para $\Delta + q$.

Demostración.

$1 \Rightarrow 2$. $Q(\varphi,\varphi) \geq 0 \ \forall \varphi \in C_0^\infty(M) \ \Rightarrow$ Dado $\Omega \subset\subset M$,

$$\lambda_{1,\text{Dirichlet}}(L,\Omega) = \text{ínf}\left\{ \frac{Q(\varphi,\varphi)}{\int_M \varphi^2 \, dA} \mid \varphi \in C_0^\infty(\Omega), \ \varphi \not\equiv 0 \right\} \geq 0.$$

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Defino $w = v + 1 \Rightarrow \ \Delta w + qw = 0$ en Ω, $w|_{\partial\Omega} = 1$.

$w > 0$ en Ω.

Tomo $\{\Omega_n \subset\subset M\}_n$ expansiva, $x_0 \in \Omega_1$.

Harnack ineq $\ \Rightarrow \ u_n := \frac{1}{w_n(x_0)} w_n$ unif acotada en cpts de M.

Schauder estimates $\ \Rightarrow \ \{u_n\}_n$ unif acotada en C^k sobre cpts de M.

Arzelá-Ascolí$\Rightarrow \exists u \in C^\infty(M)$ t.q. u_n (parcial) $\rightarrow u$ en cpts de M.

$u(x_0) = 1$, $u \geq 0$, $\Delta u + qu = 0$ en M, y $u > 0$ por princ. máximo.

Demostración.

$2 \Rightarrow 3$. Trivial.

$3 \Rightarrow 1$. Por hipótesis, $\exists u \in C^\infty(M, \mathbb{R}^+)$ t.q. $\Delta u + qu \leq 0$ in M. Sea $f \in C_0^\infty(M)$. ¿$Q(f,f) \geq 0$? Defino $\varphi = f/u \in C_0^\infty(M)$.

$$\int_M (|\nabla f|^2 - qf^2)\, dA = \int_M (|\nabla(\varphi u)|^2 - q\varphi^2 u^2)\, dA$$

$$\text{(partes)} \qquad = \int_M (-\varphi u \Delta(\varphi u) - q\varphi^2 u^2)\, dA$$

$$= \int_M (-\varphi^2 u\Delta u - 2\langle \nabla\varphi, \nabla u\rangle \varphi u - u^2 \varphi \Delta\varphi - q\varphi^2 u^2)\, dA$$

$$\geq -\int_M \left(\frac{1}{2} \langle \nabla(\varphi^2), \nabla(u^2)\rangle + u^2 \varphi \Delta\varphi \right) dA$$

$$\text{(partes, } u^2\varphi\nabla\varphi) \qquad = \int_M |\nabla\varphi|^2 u^2\, dA \geq 0.$$

Lema (Principio del máximo)

$q \in C^\infty(M)$, $\Omega \subset\subset M$, $v \in C^\infty(\Omega)$ t.q. $\Delta v + qv = 0$ en Ω.
Si $v \geq 0$ en Ω $\Rightarrow$ $v > 0$ ó $v \equiv 0$ en Ω.

Demostración.

Supongo $v(x_0) = 0$ en $x_0 \in \Omega$. Defino
$c := \min\{\inf_\Omega q, 0\} \in (-\infty, 0]$ y $\phi := -v \in C^\infty(\Omega)$.
$\Delta\phi + c\phi = -\Delta v - cv \geq -\Delta v - qv = 0$ en Ω.
$c \leq 0$, ϕ máximo ≥ 0 en $x_0 \overset{\text{Thm 3.5 G-T}}{\Longrightarrow} \phi$ cte $\Rightarrow v$ cte. □

Nota

$\pi : \widetilde{M} \to M$ recubridor Riemanniano.
Si $-(\Delta + q) \geq 0$ en $M \Rightarrow -[\widetilde{\Delta} + (q \circ \pi)] \geq 0$ en $\widetilde{M}$.

El recíproco no es cierto:
(Σ, g) sup cpt orient, género ≥ 2, g curvatura constante -1.
$f \in C^\infty(\mathbb{R}, (0, 1])$, $f(0) = 1$, $-\frac{1}{8} < f''(0) < 0$.

$L := \Delta - 2f''(0)$. $\lambda_1(L, \Sigma) = 2f''(0) < 0 \Rightarrow -L$ no es ≥ 0 en Σ.

Recubridor universal de $\Sigma = \mathbb{D}$ (plano hiperbólico).
$$\text{lím}_{\Omega_n \nearrow \mathbb{D}} \lambda_{1,\text{Dirichlet}}(\widetilde{L}, \Omega_n) = \text{lím}_{\Omega_n \nearrow \mathbb{D}} \lambda_{1,\text{Dirichlet}}(\widetilde{\Delta}, \Omega_n) + 2f''(0)$$
$$= \frac{1}{4} + 2f''(0) > 0 \Rightarrow -\widetilde{L} \geq 0 \text{ en } \mathbb{D}.$$

Nota (Meeks,—,Ros)

Si $\forall \Omega \subset\subset M$, $\pi^{-1}(\Omega)$ subexp area growth $\Rightarrow$ recíproco cierto.

$\Delta - aK$: La técnica de Pogorelov

Observación (Fischer-Colbrie, Schoen, Castillon)

(M, ds^2) *cpl no cpt.* $A(ds^2) := \{a \in \mathbb{R} \mid -(\Delta - aK) \geq 0\}$.
Entonces, $A = [a_0, b_0]$, $-\infty \leq a_0 \leq 0 \leq b_0 \leq +\infty$.

Demostración.

1. $0 \in A$.
$Q(f,f) = \int_M |\nabla f|^2$ si $a = 0$.

□

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Demostración.

1. $0 \in A$.
2. Si $c > 0$ $(c < 0)$ está en A $\Rightarrow$ $[0, c] \subset A$ $([c, 0] \subset A)$:

$c' \neq 0$ t.q. $0 < \frac{c'}{c} < 1$, $f \in C_0^\infty(M)$.

$$0 \leq \int_M (|\nabla f|^2 + cKf^2) = \frac{c}{c'}\left[\int_M (|\nabla f|^2 + c'Kf^2) + \int_M (\frac{c'}{c} - 1)|\nabla f|^2\right]$$

□

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Demostración.

1. $0 \in A$.
2. Si $c > 0$ $\;(c < 0)$ está en $A \Rightarrow [0, c] \subset A \;\; ([c, 0] \subset A)$.
3. Si $\{c_n\}_n \subset A$, $c_n \nearrow c_\infty \in \mathbb{R}^+$ $(c_n \searrow c_\infty \in \mathbb{R}^-)$ $\Rightarrow c_\infty \in A$:

Fijo $f \in C_0^\infty(M)$. $\;-(\Delta - c_n K) \geq 0 \Rightarrow \int_M (|\nabla f|^2 + c_n K f^2) \geq 0$.

Tomar límites.

Otra forma:

$-(\Delta - c_n K) \geq 0 \Rightarrow \exists w_n \in C^\infty(M, \mathbb{R}^+)$ t.q. $\Delta w_n - c_n K w_n = 0$.

$u_n := \frac{1}{w_n(x_0)} w_n \overset{\text{parcial}}{\longrightarrow} u \in C^\infty(M)$, $\Delta u - c_\infty K u = 0$, $u(x_0) = 1$, $u \geq 0$.

Princ. máximo $\Rightarrow$ $u > 0$ en M $\Rightarrow -(\Delta - c_\infty K) \geq 0$. □

$\Delta - aK$: La técnica de Pogorelov

Observación (Fischer-Colbrie, Schoen, Castillon)

(M, ds^2) cpl no cpt. $A(ds^2) := \{a \in \mathbb{R} \mid -(\Delta - aK) \geq 0\}$.
Entonces, $A = [a_0, b_0]$, $-\infty \leq a_0 \leq 0 \leq b_0 \leq +\infty$.

- $(\mathbb{C}, |dz|^2)$, $(\mathbb{C}^*, \frac{|dz|^2}{|z|^2})$: $a_0 = -\infty$, $b_0 = +\infty$.
- $(\mathbb{D}, ds^2_{-1})$: $a_0 = -\infty$, $b_0 = \frac{1}{4}$.
- $K \geq 0 \Leftrightarrow b_0 = +\infty$:

$\boxed{\Rightarrow}$ $0 \leq -\Delta \leq -(\Delta - cK)$ $\forall c \geq 0$.

$\boxed{\Leftarrow}$ Supongo $K(x_0) < 0$ en $x_0 \in M$ $\Rightarrow$ $K < 0$ en $\Omega_{x_0} \subset\subset M$.
Tomo $f \in C_0^\infty(\Omega_{x_0}) - \{0\}$. Dado $a \in A(ds^2)$,

$$\int_{\Omega_{x_0}} |\nabla f|^2 \geq -a \int_{\Omega_{x_0}} K f^2.$$

Pero $\int_{\Omega_{x_0}} (-K) f^2 > 0$ $\Rightarrow$ $A(ds^2)$ acotado superiormente !!

$\Delta - aK$: La técnica de Pogorelov

Observación (Fischer-Colbrie, Schoen, Castillon)

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Entonces, $A = [a_0, b_0]$, $-\infty \leq a_0 \leq 0 \leq b_0 \leq +\infty$.

- $(\mathbb{C}, |dz|^2)$, $(\mathbb{C}^*, \frac{|dz|^2}{|z|^2})$: $a_0 = -\infty$, $b_0 = +\infty$.
- $(\mathbb{D}, ds^2_{-1})$: $a_0 = -\infty$, $b_0 = \frac{1}{4}$.
- $K \geq 0 \Leftrightarrow b_0 = +\infty$:
- $K \leq 0 \Leftrightarrow a_0 = -\infty$ (interés geom: $M \overset{H=0}{\subset} \mathbb{R}^3$ estable)

Q1: ¿Qué relación hay entre la geometría de M y b_0?
Q2: ¿Estimaciones de b_0 para $(\mathbb{D}, e^{2u}|dz|^2)$ completa? (FC-S)

- 1977 Do Carmo-Peng Q2 (no explícito): $b_0 \leq \frac{1}{2}$.
- 1980 Fischer-Colbrie, Schoen Q2: $b_0 < 1$.
- 1981 Pogorelov Q2 (indep, no explícito): $K \leq 0 \Rightarrow b_0 \leq \frac{1}{2}$.
- 1986 Gulliver-Lawson Q1: Si $b_0 > \frac{1}{2} \Rightarrow M$ conform $\mathbb{C}$ ó $\mathbb{C}^*$.
- 1988 Kawai Q1: Si $K \leq 0$, $K \not\equiv 0 \Rightarrow b_0 \leq \frac{1}{4}$.
- 2002 Colding-Minicozzi: result similares $\Delta - aK + q$, $q \geq 0$.

$\Delta - aK$: La técnica de Pogorelov

Teorema (Castillon (2006, $q = 0$), Meeks,—,Ros (2008))

(M, ds^2), $x_0 \in M$, $0 < R' < R < \mathrm{dist}(x_0, \partial M)$.
$a \in (\frac{1}{4}, \infty)$, $q \in C^\infty(M)$, $q \geq 0$ *t.q.* $-(\Delta - aK + q) \geq 0$ *en* M.

$$\Rightarrow \frac{8a^2}{4a-1} \frac{\mathrm{Area}(B(x_0, R'))}{R^2} + \left(1 - \frac{R'}{R}\right)^2 \int_{B(x_0,R')} q \leq 2\pi a \left(1 - \frac{R'}{R}\right)^{\frac{2}{1-4a}}$$

Y si (M, ds^2) *cpl no cpt* $\Rightarrow$ *QAG*, $q \in L^1(M)$, M *conform* $\mathbb{C}$ *ó* $\mathbb{C}^*$.

Corolario

La cota óptima para Q2 es $b_0 = \frac{1}{4}$.

Nota

$$\left.\begin{array}{l} (M, ds^2) \text{ cpl, no cpt, } \Omega \subset\subset M \\ -(\Delta - aK + q) \geq 0 \text{ en } M - \Omega \end{array}\right\} \Rightarrow M \overset{conf.}{\simeq} \overline{M}_k - \{p_1, \ldots, p_r\}.$$

Demostración del teorema.

$$\left.\begin{array}{l} r = d(\cdot, x_0) \colon B(x_0,R) \to [0,R), \\ \phi \in C^\infty([0,R],\mathbb{R}^+) \text{ t.q. } \phi(0)=1,\ \phi(R)=0,\ \phi' \leq 0 \text{ en } [0,R] \end{array}\right\} \rightsquigarrow$$

$$\rightsquigarrow f(q) = \phi(r),\ r = r(q),\ q \in B(x_0,R) \ \Rightarrow f \in H_0^1(B(x_0,R)).$$

$$-(\Delta - aK + q) \geq 0 \Rightarrow \int_{B(x_0,R)} qf^2 \leq \int_{B(x_0,R)} |\nabla f|^2 + a\int_{B(x_0,R)} Kf^2$$

$$\int_{B(x_0,R)} |\nabla f|^2 = \int_{B(x_0,R)} \phi'(r)^2 \overset{\text{coarea}}{=} \int_0^R \phi'(r)^2 l(r)\, dr,$$

$l(r)$: longitud$(\partial B(x_0,r))$

$$\int_{B(x_0,R)} Kf^2 = \int_0^R \phi(r)^2 \int_{\partial B(x_0,r)} K\, ds_r = \int_0^R \phi(r)^2 \widetilde{K}'(r)\, dr \overset{\text{partes}}{=} -\int_0^R (\phi^2)'(r)\widetilde{K}(r)\, dr,$$

$$\widetilde{K}(r) = \int_{B(x_0,r)} K \qquad\qquad \phi(R) = \widetilde{K}(0) = 0$$

$$l'(r) \overset{\text{1 var long}}{=} \int_{\partial B(x_0,r)} \kappa_g(s)\, ds \overset{\text{Gauss-Bonnet}}{\leq} 2\pi\chi(B(x_0,r)) - \int_{B(x_0,r)} K \leq 2\pi - \widetilde{K}(r)$$

$\kappa_g =$ curv geodésica $\qquad (\phi^2)' = 2\phi\phi' \leq 0$

$$\Rightarrow \int_{B(x_0,R)} Kf^2 \leq \int_0^R (\phi^2)'(r)\left[l'(r) - 2\pi\right] dr = \int_0^R (\phi^2)'(r) l'(r)\, dr + 2\pi$$

$$\phi(0) = 1,\ \phi(R) = 0$$

$\Delta - aK$: La técnica de Pogorelov

Demostración del teorema.

$$\left.\begin{array}{l} r = d(\cdot, x_0)\colon B(x_0,R) \to [0,R), \\ \phi \in C^\infty([0,R],\mathbb{R}^+) \text{ t.q. } \phi(0)=1,\ \phi(R)=0,\ \phi' \leq 0 \text{ en } [0,R] \end{array}\right\} \rightsquigarrow$$

$$\rightsquigarrow f(q) = \phi(r),\ r = r(q),\ q \in B(x_0,R) \ \Rightarrow f \in H_0^1(B(x_0,R)).$$

$$\Rightarrow \int_{B(x_0,R)} qf^2 \leq \int_0^R \phi'(r)^2 l(r)\,dr + a\int_0^R (\phi^2)'(r) l'(r)\,dr + 2\pi a$$

Elijo $\phi(r) = \left(1 - \frac{r}{R}\right)^b$, $\ b \geq 1$:

$$\int_{B(x_0,R)} q\left(1-\frac{r}{R}\right)^{2b} \leq \frac{b^2}{R^2}\int_0^R \left(1-\frac{r}{R}\right)^{2b-2} l(r)\,dr$$
$$-\frac{2ab}{R}\int_0^R \left(1-\frac{r}{R}\right)^{2b-1} l'(r)\,dr + 2\pi a.$$

$l(r)$ no C^0 pero sí derivable c.p.d. y: $\forall \psi \in C^\infty([0,R],\mathbb{R})$, $\psi \geq 0$,

$$\int_0^R \left[\psi(r) l'(r) + \psi'(r) l(r)\right] dr \geq \psi(R)l(R) - \psi(0)l(0) = \psi(R)l(R)$$

Elijo $\psi(r) = \left(1-\frac{r}{R}\right)^{2b-1}$ $\qquad (\psi(R)=0)$:

Demostración del teorema.

$$\left.\begin{array}{l} r = d(\cdot, x_0) \colon B(x_0,R) \to [0,R), \\ \phi \in C^\infty([0,R], \mathbb{R}^+) \text{ t.q. } \phi(0)=1,\ \phi(R)=0,\ \phi' \leq 0 \text{ en } [0,R] \end{array}\right\} \rightsquigarrow$$

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$l(r)$ no C^0 pero sí derivable c.p.d. y: $\forall \psi \in C^\infty([0,R],\mathbb{R})$, $\psi \geq 0$,

$$\int_0^R \left(1-\frac{r}{R}\right)^{2b-1} l'(r)\,dr \geq \frac{2b-1}{R}\int_0^R \left(1-\frac{r}{R}\right)^{2b-2} l(r)\,dr$$

Elijo $\psi(r) = \left(1-\frac{r}{R}\right)^{2b-1}$ $\qquad (\psi(R)=0)$:

Demostración del teorema.

$$\int_{B(x_0,R)} q\left(1-\frac{r}{R}\right)^{2b} \leq \frac{b[b(1-4a)+2a]}{R^2}\int_0^R \left(1-\frac{r}{R}\right)^{2b-2} l(r)\,dr + 2\pi a$$

Tomo $b > \frac{2a}{4a-1}$ $\left(\Rightarrow \frac{b[b(1-4a)+2a]}{R^2} < 0\right)$, estimo integral por abajo:

$$\int_0^R \left(1-\frac{r}{R}\right)^{2b-2} l(r)\,dr \geq \min_{[0,R']}\left(1-\frac{r}{R}\right)^{2b-2}\int_0^{R'} l(r)\,dr = \left(1-\frac{R'}{R}\right)^{2b-2}\mathrm{Area}(B(x_0,R'))$$

Estimo la integral de la forma análoga ($q \geq 0$):

$$\int_{B(x_0,R)} q\left(1-\frac{r}{R}\right)^{2b} \geq \min_{[0,R']}\left(1-\frac{r}{R}\right)^{2b}\int_{B(x_0,R')} q = \left(1-\frac{R'}{R}\right)^{2b}\int_{B(x_0,R')} q$$

Luego

$$\frac{b[b(4a-1)-2a]}{R^2}\left(1-\frac{R'}{R}\right)^{2b-2}\mathrm{Area}(B(x_0,R')) + \left(1-\frac{R'}{R}\right)^{2b}\int_{B(x_0,R')} q \leq 2\pi a$$

$$b = \frac{4a}{4a-1} \Rightarrow \frac{8a^2}{4a-1}\frac{\mathrm{Area}(B(x_0,R'))}{R^2} + \left(1-\frac{R'}{R}\right)^2\int_{B(x_0,R')} q \leq 2\pi a\left(1-\frac{R'}{R}\right)^{\frac{2}{1-4a}}$$

Aplico la fla a $\widetilde{M}$ rec universal de M $\Rightarrow \widetilde{M}$ QAG $\Rightarrow \widetilde{M}$ conform $\mathbb{C}$.

Estabilidad de superficies mínimas y de CMC.

$x\colon M^2 \looparrowright N^3$ inm isométrica con normal unitario $\eta\colon M \to UN$.
$\Omega \subset\subset M$, $X\colon (-\varepsilon,\varepsilon)\times M \to N$, $X(0,\cdot)=x$, $\mathrm{sop}(X_t)\subset\Omega$.
$\mathrm{Area}(t) = \mathrm{Area}(X_t)$ $\quad(X_t$ inmersión $\forall t)$
$\mathrm{Vol}(t) = \int_{[0,t]\times\Omega} \mathrm{Jac}(X)\, dV$ $\quad$ (volumen con signo entre x y X_t)

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(t) = -2\int_M Hf\, dA, \qquad \left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(t) = -\int_M f\, dA$$

$H =$ curvatura media de x respecto a η,
$f = \langle \left.\frac{\partial X}{\partial t}\right|_{t=0}, \eta\rangle$ parte normal del campo variacional.
$c\in\mathbb{R}$, $\qquad (\mathrm{Area} - 2c\,\mathrm{Vol})'(0) = 2\int_M (c-H) f\, dA$ luego:

M crítica para $\mathrm{Area} - 2c\,\mathrm{Vol} \;\Leftrightarrow\;$ CMC $H = c$.

$M \looparrowright N^3$, CMC H.

$$(\mathrm{Area} - 2H\,\mathrm{Vol})''(0) = \int_M \left[|\nabla f|^2 - (|\sigma|^2 + \mathrm{Ric}(\eta))f^2\right] dA = Q(f,f),$$

$L = \Delta + |\sigma|^2 + \mathrm{Ric}(\eta)$ oper. Jacobi, $\quad \sigma = 2^a$ ff de M, $\mathrm{Ric} =$ Ricci N

M estable si $-(\Delta + |\sigma|^2 + \mathrm{Ric}(\eta)) \geq 0$ en M. (OJO: Isoperim)

$M \looparrowright N^3$, CMC H.

$$
\begin{aligned}
L &= \Delta + |\sigma|^2 + \mathrm{Ric}(\eta) && (1)\\
&= \Delta - 2K + 4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) && (2)\\
&= \Delta - K + 2H^2 + \frac{1}{2}|\sigma|^2 + \frac{1}{2}S && (3)\\
&= \Delta - K + 3H^2 + \frac{1}{2}S + (H^2 - \det(A)), && (4)
\end{aligned}
$$

A : endomorfismo de Weingarten,
e_1, e_2 : base ortonormal TM,
S : curvatura escalar de N^3 $\quad (S = 6$ en $\mathbb{S}^3(1))$.

Estabilidad de superficies mínimas y de CMC.

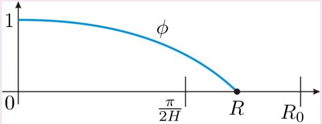
Teorema (Mazet)

$M \subset \mathbb{R}^3$ CMC $H > 0$, estable, $p \in M - \partial M$.
Entonces, $d_M(p, \partial M) \leq \frac{\pi}{2H}$ $\quad$ ("=" $\Leftrightarrow$ $M = (\mathbb{S}^2)^+$)

Demostración. Supongo $D(p, R_0) \subset M - \partial M$, $\frac{\pi}{2H} < R_0 < \frac{\pi}{H}$.

Fijo $R \in (\frac{\pi}{2H}, R_0]$.

$\Delta - aK + q, \qquad a = 2,\ q = 4H^2.$



Técnica de Pogorelov con $\phi(r) = \cos \frac{\pi r}{2R}$

$\int_{B(x_0,R)} q f^2 \leq$

$\int_0^R (\phi')^2 l \, dr + a \int_0^R (\phi^2)' l' \, dr + 2\pi a$

$\Rightarrow \int_0^R [4H^2\phi^2 - (\phi')^2 + 2(\phi^2)''] l(r) \, dr \leq 4\pi$

$$4H^2\phi^2 - (\phi')^2 + 2(\phi^2)'' = (4H^2 - \frac{\pi^2}{R^2}) \cos^2 \frac{\pi r}{2R} + \frac{3\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R} \ (> 0)$$

$$K = H^2 - \frac{1}{4}(k_1 - k_2)^2 \leq H^2 \ \Rightarrow \begin{cases} \text{valores conjugados } \geq \frac{\pi}{H} \\ l(r) \geq \frac{2\pi}{H} \sin(Hr) \qquad \text{(expl)} \end{cases}$$

$$4\pi \geq \int_0^R \left[(4H^2 - \frac{\pi^2}{R^2}) \cos^2 \frac{\pi r}{2R} + \frac{3\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R} \right] \frac{2\pi}{H} \sin(Hr) \, dr := F(R, H)$$

$F(\frac{\pi}{2H}, H) = 4\pi$, $\frac{\partial F}{\partial R}(\frac{\pi}{2H}, H) = 2\pi H > 0$!!

Teorema (Mazet)

$M \subset \mathbb{R}^3$ CMC $H > 0$, estable, $p \in M - \partial M$.
Entonces, $d_M(p, \partial M) \leq \frac{\pi}{2H}$ ("$=$" $\Leftrightarrow$ $M = (\mathbb{S}^2)^+$)

Teorema (versión para operadores)

(M, ds^2), $L = \Delta - 2K + q$, $q \in C^\infty(M)$.

Si $\exists H > 0$ t.q. $K \leq H^2 \leq \frac{1}{4}q$ en M $\Rightarrow d_M(p, \partial M) \leq \frac{\pi}{2H}$.

Teorema (Espacios de c.s.c. $k \in \mathbb{R}$)

$M \looparrowright \mathbb{M}^3(k)$, CMC $H \in \mathbb{R}$, cpl estable.

- $(k = 0) \;\Rightarrow\;$ *M plano.*
- $(k = 1) \;\Rightarrow$ *no existe.*
- $(k = -1)$ *Si* $|H| \geq 1 \;\Rightarrow\;$ *M horosfera* $(|H| = 1,\ K = 0)$.

Demostración. Supongo $H^2 + k \geq 0$.

$$L = \Delta - 2K + \overbrace{4(H^2 + k)}^{q \geq 0} \overset{Thm}{\Rightarrow} M \text{ conform } \mathbb{C} \text{ ó } \mathbb{C}^*.$$

M estable $\Rightarrow\; \exists u \in C^\infty(M, \mathbb{R}^+)$ t.q. $\Delta u = [2K - 4(H^2 + k)]u$.

$$2K - 4(H^2 + k) \overset{K(TM) = K - \det A}{=} 2(\det A - H^2) - 2(H^2 + k)$$

$$= -\tfrac{1}{2}(k_1 - k_2)^2 - 2(H^2 + k) \leq 0$$

$\Rightarrow\; \Delta u \leq 0$ en $M \;\Rightarrow\; \Delta_0 u \leq 0$ en $\mathbb{C}$ ó $\mathbb{C}^* \;\Rightarrow\; u$ cte

$\Rightarrow\; \Delta u = 0 \;\Rightarrow\; 2K - 4(H^2 + k) = 0 \;\Rightarrow M$ umbilical, $H^2 = -k$.

Estabilidad de superficies mínimas y de CMC.

Teorema

$M \looparrowright N^3$, CMC $H \in \mathbb{R}$, cpl estable.

1. $\left.\begin{array}{l} Si\ \mathrm{Ric} \geq -2c \\ y\ H^2 \geq c\ (c \in \mathbb{R}) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} M \text{ totalmente umbilical,} \\ H^2 = c,\ \mathrm{Ric}(\eta) = -2c. \end{array}\right.$
2. *Si* $\mathrm{Ric} \geq 0 \Rightarrow M$ *totalmente geodésica.*

Demostración.

1. $|\sigma|^2 + \mathrm{Ric}(\eta) \geq 0$: $|\sigma|^2 \geq 2H^2 \geq 2c, \quad \mathrm{Ric}(\eta) \geq -2c.$

Estabilidad de superficies mínimas y de CMC.

Teorema

$M \looparrowright N^3$, CMC $H \in \mathbb{R}$, cpl estable.

1. $$\left.\begin{array}{l} \textit{Si } \mathrm{Ric} \geq -2c \\ \textit{y } H^2 \geq c \ (c \in \mathbb{R}) \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} M \textit{ totalmente umbilical,} \\ H^2 = c, \ \mathrm{Ric}(\eta) = -2c. \end{array}\right.$$
2. *Si* $\mathrm{Ric} \geq 0 \ \Rightarrow \ M$ *totalmente geodésica.*

Demostración.

1. $|\sigma|^2 + \mathrm{Ric}(\eta) \geq 0$.
2. M QAG, $4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) \in L^1(M)$:

$$L = \Delta - 2K + \overbrace{4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2)}^{q \,\geq\, 4H^2 - 4c \,\geq\, 0}, \text{ y aplicamos el thm.}$$

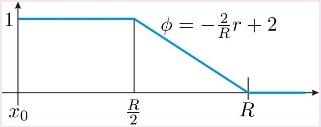
Teorema

$M \looparrowright N^3$, CMC $H \in \mathbb{R}$, cpl estable.

1. $\left.\begin{array}{l} \textit{Si } \mathrm{Ric} \geq -2c \\ \textit{y } H^2 \geq c \ (c \in \mathbb{R}) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} M \textit{ totalmente umbilical,} \\ H^2 = c, \ \mathrm{Ric}(\eta) = -2c. \end{array}\right.$
2. *Si* $\mathrm{Ric} \geq 0 \ \Rightarrow \ M$ *totalmente geodésica.*

Demostración.

1. $|\sigma|^2 + \mathrm{Ric}(\eta) \geq 0$.
2. M QAG, $4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) \in L^1(M)$.
3. $|\sigma|^2 + \mathrm{Ric}(\eta) \in L^1(M)$:



$r = d(\cdot, x_0)$, $f(q) = \phi(r) \in H_0^1(M)$.

$$\int_{B(x_0,\frac{R}{2})} \left(|\sigma|^2 + \mathrm{Ric}(\eta)\right) \overset{(1)}{\leq} \int_M \left(|\sigma|^2 + \mathrm{Ric}(\eta)\right) f^2$$

$$\overset{-L\geq 0}{\leq} \int_M |\nabla f|^2 = \int_{R/2}^{R} \phi'(r)^2 l(r)\, dr = \frac{4}{R^2}\int_{R/2}^{R} l(r)\, dr \overset{(2)}{\leq} \frac{4}{R^2} CR^2 = 4C,$$

$l(r) = \text{length}(\partial B(x_0, R))$.

Teorema

$M \looparrowright N^3$, CMC $H \in \mathbb{R}$, cpl estable.

1. $$\left.\begin{array}{l} \textit{Si } \mathrm{Ric} \geq -2c \\ \textit{y } H^2 \geq c \ (c \in \mathbb{R}) \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} M \textit{ totalmente umbilical,} \\ H^2 = c, \ \mathrm{Ric}(\eta) = -2c. \end{array}\right.$$
2. *Si* $\mathrm{Ric} \geq 0 \ \Rightarrow \ M$ *totalmente geodésica.*

Demostración.

1. $|\sigma|^2 + \mathrm{Ric}(\eta) \geq 0$.
2. M QAG, $4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) \in L^1(M)$.
3. $|\sigma|^2 + \mathrm{Ric}(\eta) \in L^1(M)$.
4. $K \in L^1(M)$:

$-2K + 4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) = |\sigma|^2 + \mathrm{Ric}(\eta)$, y usamos 2, 3.

Teorema

$M \looparrowright N^3$, CMC $H \in \mathbb{R}$, cpl estable.

1. $\left.\begin{array}{l} \textit{Si } \mathrm{Ric} \geq -2c \\ \textit{y } H^2 \geq c \ (c \in \mathbb{R}) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} M \textit{ totalmente umbilical,} \\ H^2 = c, \ \mathrm{Ric}(\eta) = -2c. \end{array}\right.$
2. *Si* $\mathrm{Ric} \geq 0 \ \Rightarrow \ M$ *totalmente geodésica.*

Demostración.

1. $|\sigma|^2 + \mathrm{Ric}(\eta) \geq 0$.
2. M QAG.
3. $|\sigma|^2 + \mathrm{Ric}(\eta) \in L^1(M)$.
4. $K \in L^1(M)$.
5. $l(r) \leq Cr$:

$$l'(r) \overset{1 \text{ var long}}{=} \int_{\partial B(x_0,r)} \kappa_g(s)\, ds \overset{\text{Gauss-Bonnet}}{\leq} 2\pi\chi(B(x_0,r)) - \int_{B(x_0,r)} K,$$

y usamos 4.

Estabilidad de superficies mínimas y de CMC.

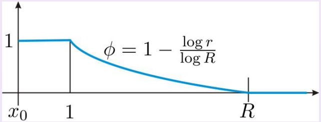
Teorema

$M \looparrowright N^3$, CMC $H \in \mathbb{R}$, cpl estable.

1. $\left.\begin{array}{l} \textit{Si } \mathrm{Ric} \geq -2c \\ y\ H^2 \geq c\ (c \in \mathbb{R}) \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} M \textit{ totalmente umbilical,} \\ H^2 = c,\ \mathrm{Ric}(\eta) = -2c. \end{array}\right.$
2. *Si* $\mathrm{Ric} \geq 0 \Rightarrow M$ *totalmente geodésica.*

Demostración.

1. $|\sigma|^2 + \mathrm{Ric}(\eta) \geq 0$.
2. M QAG.
3. $|\sigma|^2 + \mathrm{Ric}(\eta) \in L^1(M)$.
4. $K \in L^1(M)$.
5. $I(r) \leq Cr$.
6. $|\sigma|^2 + \mathrm{Ric}(\eta) \equiv 0$ en M:



$$\int_M \left(|\sigma|^2 + \mathrm{Ric}(\eta)\right) f^2 \overset{-L\geq 0}{\leq} \int_M |\nabla f|^2 = \int_1^R \phi'(r)^2 I(r)\, dr$$

$$= \frac{1}{(\log R)^2} \int_1^R \frac{I(r)}{r^2}\, dr \overset{(5)}{\leq} \frac{C}{(\log R)^2} \int_1^R \frac{dr}{r} = \frac{C}{\log R} \overset{(R\to\infty)}{\longrightarrow} 0,$$

y usamos 1.

Estabilidad de superficies mínimas y de CMC.

Teorema

$M \looparrowright N^3$, CMC $H \in \mathbb{R}$, cpl estable.

1. $\left.\begin{array}{l} \text{Si } \mathrm{Ric} \geq -2c \\ \text{y } H^2 \geq c \ (c \in \mathbb{R}) \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} M \text{ totalmente umbilical,} \\ H^2 = c, \ \mathrm{Ric}(\eta) = -2c. \end{array}\right.$
2. *Si* $\mathrm{Ric} \geq 0 \ \Rightarrow$ *M totalmente geodésica.*

Demostración.

1. $|\sigma|^2 + \mathrm{Ric}(\eta) \geq 0$.
2. M QAG.
3. $|\sigma|^2 + \mathrm{Ric}(\eta) \in L^1(M)$.
4. $K \in L^1(M)$.
5. $l(r) \leq Cr$.
6. $|\sigma|^2 + \mathrm{Ric}(\eta) \equiv 0$ en M.
7. $0 \overset{(6)}{=} |\sigma|^2 + \mathrm{Ric}(\eta) = 2H^2 + \frac{1}{2}(k_1 - k_2)^2 + \mathrm{Ric}(\eta) \geq 2H^2 + \mathrm{Ric}(\eta) \geq 0$

$$\Rightarrow \left\{ \begin{array}{l} |\sigma|^2 = 2H^2 \ \Rightarrow \ M \text{ totalmente umbilical,} \\ H^2 = c, \ \mathrm{Ric}(\eta) = -2c. \end{array}\right.$$

Estabilidad de superficies mínimas y de CMC.

Teorema

$M \looparrowright N^3$, CMC $H \in \mathbb{R}$, cpl estable.

1. $\left. \begin{array}{l} \textit{Si } \mathrm{Ric} \geq -2c \\ \textit{y } H^2 \geq c \ (c \in \mathbb{R}) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} M \textit{ totalmente umbilical,} \\ H^2 = c, \ \mathrm{Ric}(\eta) = -2c. \end{array} \right.$
2. *Si* $\mathrm{Ric} \geq 0 \ \Rightarrow \ M$ *totalmente geodésica.*

Geometrías de Thurston con $\dim \mathrm{Iso}(N^3) = 4$: $\mathbb{E}^3(\kappa, \tau)$

	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$\tau = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\varnothing$	$\mathbb{H}^2 \times \mathbb{R}$
$\tau \neq 0$	$\mathbb{S}^3_{\mathrm{Berger}}$	Nil_3	$\widetilde{Sl}_2(\mathbb{R})$

$\mathbb{E}^3(\kappa, \tau) \xrightarrow{\pi} \mathbb{M}^2(\kappa)$,
$E_3 = \ker(d\pi)$, $\langle E_1, E_2 \rangle = \langle E_3 \rangle^\perp$
$\mathrm{Ric}(E_1) = \mathrm{Ric}(E_2) = \kappa - 2\tau^2$,
$\mathrm{Ric}(E_3) = 2\tau^2 \Rightarrow S = 2(\kappa - \tau^2)$

$M \looparrowright \mathbb{E}^3(\kappa, \tau)$, CMC $H \in \mathbb{R}$, cpl estable.

1. $\mathbb{S}^2 \times \mathbb{R}$: $M = \mathbb{S}^2 \times \{c\}$.
2. $\mathrm{Nil}_3(\tau = \frac{1}{2})$: $\nexists M$ si $|H| \geq \frac{1}{2}$.
3. $\widetilde{Sl}_2(\mathbb{R})$: $\nexists M$ si $H^2 \geq \tau^2 - \frac{1}{2}\kappa$ (> 0).
4. $\mathbb{H}^2 \times \mathbb{R}$: $\exists \varepsilon > 0$ s.t. $\nexists M$ si $|H| > \frac{1}{\sqrt{3}} - \varepsilon$ (mejora $|H| > \frac{1}{\sqrt{3}}$)
5. $\mathbb{S}^3_{\mathrm{Berger}}$: $\nexists M$ si $3H^2 \geq \tau^2 - \kappa$ (mejora escalar$(\mathbb{S}^3_B) \geq 0$)