ON THE GEOMETRY OF CONSTANT ANGLE SURFACES IN $\text{Sol}_3$

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(Received 11 May 2010 and revised 28 April 2011)

Abstract. In this paper we define and classify all surfaces in the three-dimensional Lie group $\text{Sol}_3$ whose normals make constant angle with a left-invariant vector field.

1. Preliminaries

The space $\text{Sol}_3$ is a simply connected homogeneous three-dimensional manifold whose isometry group has dimension 3 and it is one of the eight models of geometry of Thurston [14]. As for a Riemannian manifold, the space $\text{Sol}_3$ can be represented by $\mathbb{R}^3$ equipped with the metric

$$\tilde{g} = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2,$$

where $(x, y, z)$ are canonical coordinates of $\mathbb{R}^3$. The space $\text{Sol}_3$, with the group operation

$$(x, y, z) * (x', y', z') = (x + e^{-z}x', y + e^z y', z + z'),$$

is a unimodular, solvable but not nilpotent Lie group and the metric $\tilde{g}$ is left-invariant; see, for example, [2, 16]. This group belongs to a wider family of Lie groups, equipped with a left-invariant Riemannian metric, depending on three parameters (see, for example, [8]). Moreover, in [9], Kowalski explains the geometry of $\text{Sol}_3$ where it is realized as the Lie group $E(1, 1)$ of rigid motions of the Minkowski plane $\mathbb{E}_1^2 = (\mathbb{R}^2, dx dy)$, endowed with the metric described above. In the Japanese literature, $\text{Sol}_3$ is known as Takahashi’s B-manifold, see, for example, [13].

With respect to the metric $\tilde{g}$ an orthonormal basis of left-invariant vector fields is given by

$$e_1 = e^{-z} \frac{\partial}{\partial x}, \quad e_2 = e^z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$  \hspace{1cm} (1)

The transformations

$$(x, y, z) \mapsto (y, -x, -z) \quad \text{and} \quad (x, y, z) \mapsto (-x, y, z)$$

span a group of isometries of $(\text{Sol}_3, \tilde{g})$ having the origin as fixed point. This group is isomorphic to the dihedral group (with eight elements) $D_4$. It is, in fact, the complete group of isotropy [16]. The other elements of the group are $(x, y, z) \mapsto (-x, -y, z), (x, y, z) \mapsto (-y, x, -z), (x, y, z) \mapsto (y, x, -z), (x, y, z) \mapsto (y, z, x)$ and $(x, y, z) \mapsto (x, -y, z)$. They

2000 Mathematics Subject Classification: Primary 53B25.

Keywords: constant angle surfaces; homogeneous spaces.
can be unified as follows:

\[(x, y, x) \mapsto (\pm e^{-\epsilon} x + a, \pm e^\epsilon y + b, z + c)\]

\[(x, y, z) \mapsto (\pm e^{-\epsilon} y + a, \pm e^\epsilon x + b, z + c).\]

It is well known that the isometry group of \(Sol_3\) has dimension three.

The Levi-Civita connection \(\nabla\) of \(Sol_3\) with respect to \(\{e_1, e_2, e_3\}\) is given by

\[
\begin{align*}
\tilde{\nabla}_{e_1}e_1 &= -e_3 & \tilde{\nabla}_{e_1}e_2 &= 0 & \tilde{\nabla}_{e_1}e_3 &= e_1 \\
\tilde{\nabla}_{e_2}e_1 &= 0 & \tilde{\nabla}_{e_2}e_2 &= e_3 & \tilde{\nabla}_{e_2}e_3 &= -e_2 \\
\tilde{\nabla}_{e_3}e_1 &= 0 & \tilde{\nabla}_{e_3}e_2 &= 0 & \tilde{\nabla}_{e_3}e_3 &= 0.
\end{align*}
\]

We recall the Gauss and Weingarten formulas:

\[(G)\quad \tilde{\nabla}_XY = \nabla_XY + h(X, Y);\]

\[(W)\quad \tilde{\nabla}_XN = -AX\]

for every \(X\) and \(Y\) tangent to \(M\) and for any \(N\) unitary normal to \(M\). By \(A\) we denote the shape operator on \(M\).

2. Constant angle surfaces in \(Sol_3\)

Constant angle surfaces have been studied in product spaces \(Q_\epsilon \times \mathbb{R}\) [1, 3, 4, 11], where \(Q_\epsilon\) denotes the sphere \(S^2\) (when \(\epsilon = +1\)), the Euclidean plane \(\mathbb{E}^2\) (when \(\epsilon = 0\)), the hyperbolic plane \(\mathbb{H}^2\) (when \(\epsilon = -1\)), respectively. See also [5, 6, 10, 12, 15]. The angle \(\theta\) is considered between the unit normal of the surface \(M\) and the unit tangent vector \(\partial/\partial t\) to the fibers of \(Q_\epsilon \times \mathbb{R}\), where \(t\) is the global parameter on \(\mathbb{R}\). On the other hand, for \(Q_\epsilon \times \mathbb{R}\), the leaves \(\{dt \equiv 0\}\) are orthogonal to \(\partial/\partial t\) and they describe a foliation of the space by totally geodesic surfaces. In particular, the leaves \(Q_\epsilon \times \{t_0\}\) are constant angle surfaces where the angle is \(\theta = 0\).

Motivated by these facts, we consider in \(Sol_3\) the foliations of the ambient space by totally geodesic surfaces. It is known for \(Sol_3\) that there are only two such foliations, namely,

\(\mathcal{H}^1 = \{dx \equiv 0\}\) and \(\mathcal{H}^2 = \{dy \equiv 0\}\).

Each leaf of these foliations is isometric to the hyperbolic plane.

Let us consider \(\mathcal{H}^1\). It follows that the tangent plane to \(\mathcal{H}^1\) (the leaf at each \(x = x_0\)) is spanned by \(\partial/\partial y\) and \(\partial/\partial z\), while the unit normal vector is \(e_1\). Similarly, for the foliation \(\mathcal{H}^2\), the vector \(e_2\) is the unit normal vector to each leaf \(y = y_0\). Due to these reasons, we give the following definition.

**Definition 2.1.** An oriented surface \(M\), isometrically immersed in \(Sol_3\), is called a constant angle surface with respect to \(\mathcal{H}^1\) (respectively \(\mathcal{H}^2\)) if the angle between its unit normal vector and \(e_1\) (respectively \(e_2\)) is constant in each point of the surface \(M\).

Such as it occurs in \(Q_\epsilon \times \mathbb{R}\), the leaves of the foliation of \(Sol_3\) are constant angle surfaces. Exactly, the surfaces \(x = x_0\) (respectively \(y = y_0\)) are constant angle surfaces with respect to \(\mathcal{H}^1\) with \(\theta = 0\) (respectively \(\theta = \pi/2\)) and constant angle surfaces with respect to \(\mathcal{H}^2\) with \(\theta = \pi/2\) (respectively \(\theta = 0\)).

In this article we classify all constant angle surfaces in \(Sol_3\). Because we have two types of constant angle surfaces (with respect to \(\mathcal{H}^1\) or with respect to \(\mathcal{H}^2\)), from now we are only...
going to consider constant angle surfaces with respect to $\mathcal{H}^1$ and from now, we abbreviate by
saying a constant angle surface: the computations of constant angle surfaces with respect to $\mathcal{H}^2$ are similar and since the differences are insignificant we do not give any details for this problem.

Remark 2.2. The vector field $e_3 = \partial/\partial z$ is, together with $e_1$ and $e_2$, the third left-invariant Killing vector field of Sol$_3$ (see (1)). Although the vector field $e_3$ defines in Sol$_3$ a foliation, its leaves are not totally geodesic surfaces but minimal surfaces. Following the motivation of our definition of constant angle surface from what happens in $Q_\epsilon \times \mathbb{R}$, we discard in our study any consideration with the vector field $e_3$.

Let $M$ be a constant angle surface in Sol$_3$. Denote by $\theta \in [0, \pi)$ the angle between the unit normal $N$ and $e_1$. Hence

\[ \tilde{g}(N, e_1) = \cos \theta. \]

Let $T$ be the projection of $e_1$ on the tangent plane $T_p M$ of $M$ in a point $p \in M$. Thus

\[ e_1 = T + \cos \theta N. \]  \hspace{1cm} (2)

3. First computations and some particular cases

First we consider the case $\theta = 0$. Then $N = e_1$ and hence the surface $M$ is isometric to the hyperbolic plane $\mathcal{H}^1 = \{dx = 0\}$.

**Proposition 3.1.** The only constant angle surfaces in Sol$_3$ with $\theta = 0$ are the leaves of the foliation $\mathcal{H}^1$, that is, the surfaces given by $x = x_0, x_0 \in \mathbb{R}$.

We now suppose that $\theta \neq 0$.

**Lemma 3.2.** If $X$ is tangent to $M$ we have:

1. $\tilde{\nabla}_X e_1 = -\tilde{g}(X, e_1)e_3, \tilde{\nabla}_X e_2 = \tilde{g}(X, e_2)e_3, \tilde{\nabla}_X e_3 = \tilde{g}(X, e_1)e_1 - \tilde{g}(X, e_2)e_2$;
2. $AT = -\tilde{g}(N, e_3)T$, hence $T$ is a principal direction on the surface;
3. $g(T, T) = \sin^2 \theta$.

At this point we also have to decompose $e_2$ and $e_3$ into the tangent and the normal parts, respectively. As $\theta \neq 0$, let $E_1 = (1/\sin \theta)T$. Consider $E_2$ tangent to $M$, orthogonal to $E_1$ and such that the basis $\{e_1, e_2, e_3\}$ and $\{E_1, E_2, N\}$ have the same orientation. It follows that

\[
\begin{aligned}
e_1 &= \sin \theta E_1 + \cos \theta N \\
e_2 &= \cos \alpha \cos \theta E_1 + \sin \alpha E_2 - \cos \alpha \sin \theta N \\
e_3 &= -\sin \alpha \cos \theta E_1 + \cos \alpha E_2 + \sin \alpha \sin \theta N
\end{aligned}
\]  \hspace{1cm} (3)

and

\[
\begin{aligned}
E_1 &= \sin \theta e_1 + \cos \theta \cos \alpha e_2 - \cos \theta \sin \alpha e_3 \\
E_2 &= \sin \alpha e_2 + \cos \alpha e_3 \\
N &= \cos \theta e_1 - \sin \theta \cos \alpha e_2 + \sin \theta \sin \alpha e_3
\end{aligned}
\]  \hspace{1cm} (4)

where $\alpha$ is a smooth function on $M$. 
We now study constant angle surfaces in the case that $\theta = \pi/2$. In this case, $e_1$ is tangent to $M$ and $T = E_1$. The metric connection on $M$ is given by

$$\nabla_{E_1} E_1 = -\cos \alpha E_2, \quad \nabla_{E_2} E_1 = 0$$
$$\nabla_{E_1} E_2 = \cos \alpha E_1, \quad \nabla_{E_2} E_2 = 0.$$

The second fundamental form is obtained from

$$h(E_1, E_1) = -\sin \alpha N, \quad h(E_1, E_2) = 0, \quad h(E_2, E_2) = \sigma N,$$

where $\sigma$ is a smooth function on $M$. Writing the Gauss formula (G) for $X = E_1$ and $Y = E_2$, respectively for $X = Y = E_2$, one obtains

$$E_1(\alpha) = 0 \quad \text{and} \quad E_2(\alpha) = \sin \alpha - \sigma.$$

**Remark 3.3.** The surface $M$ is minimal if and only if $\sigma = \sin \alpha$. Since $E_1$ and $E_2$ are linearly independent, it follows that $\alpha$ is constant. Moreover, $M$ is totally geodesic if and only if $\alpha = 0$, in which case $M$ coincides with $H^1$.

Due to the fact that the Lie brackets of $E_1$ and $E_2$ is $[E_1, E_2] = \cos \alpha E_1$, one can choose local coordinates $u$ and $v$ such that

$$E_2 = \frac{\partial}{\partial u} \quad \text{and} \quad E_1 = \beta(u, v) \frac{\partial}{\partial v}.$$

This choice implies $\alpha$ and $\beta$ fulfill the partial differential equation (PDE)

$$\beta_u = -\beta \cos \alpha.$$

Since $\alpha$ depends only on $u$, it follows that

$$\beta(u, v) = \rho(v) e^{-\int_u^v \cos \alpha(\tau) \, d\tau},$$

where $\rho$ is a smooth function depending on $v$.

Denote by

$$F : U \subset \mathbb{R}^2 \rightarrow M \hookrightarrow Sol_3$$

the immersion of the surface $M$ in $Sol_3$. We have:

(i)

$$\frac{\partial}{\partial u} = F_u = (F_{1,u}, F_{2,u}, F_{3,u})$$

$$= E_2 = (\sin \alpha e_2 + \cos \alpha e_3)_{(F(u,v))} = (0, e^{F_3(u,v)} \sin \alpha, \cos \alpha);$$

(ii)

$$\frac{\partial}{\partial v} = F_v = (F_{1,v}, F_{2,v}, F_{3,v})$$

$$= \frac{1}{\beta} E_1 = \frac{1}{\beta} e^{1_{/(u,v)}} = \left( \frac{1}{\beta} e^{-F_3(u,v)}, 0, 0 \right).$$
It follows that

\[ F_1 = F_1(v), \quad \partial_v F_1 = \frac{1}{\beta(u, v)} e^{-F_3(u, v)}, \]

\[ \partial_u F_2 = \sin \alpha(u) e^{F_3(u, v)}, \quad F_2 = F_2(u), \]

\[ \partial_u F_3 = \cos \alpha(u), \quad F_3 = F_3(u). \]

Thus we obtain

\[ F_1(v) = \int^v_1 \frac{1}{\rho(\tau)} d\tau, \]

\[ F_2(u) = \int^u_0 (\sin \alpha(\tau) e^{\int_\tau^u \cos \alpha(s) ds}) d\tau, \]

\[ F_3(u) = \int^u_0 \cos \alpha(\tau) d\tau. \]

Changing the \( v \) parameter, one gets the following.

**PROPOSITION 3.4.** Let \( M \) be a constant angle surface in \( \text{Sol}_3 \) with \( \theta = \pi/2 \). Then it may be parametrized as

\[ F(u, v) = (v, \phi(u), \chi(u)) \]

which represents a cylinder over the planar curve \( \gamma(u) = (0, \phi(u), \chi(u)) \), where

\[ \phi(u) = \int^u_0 (\sin \alpha(\tau) e^{\int_\tau^u \cos \alpha(s) ds}) d\tau \quad \text{and} \quad \chi(u) = \int^u_0 \cos \alpha(\tau) d\tau. \]

Moreover, the surface is the group product between the curve \( v \mapsto (v, 0, 0) \) and the curve \( \gamma \).

In Figure 1 we can see how the curve \( \gamma \) looks for different values of the function \( \alpha \):

(a) \( \alpha \) is a constant:

\[ \gamma(u) = (0, \tan \alpha e^u \cos \alpha, u \cos \alpha); \]

(b) \( \alpha(s) = s \):

\[ \gamma(u) = \left(0, \int^u_0 \sin s e^{\int_0^s \sin \tau \cos \tau d\tau}, \sin u \right); \]

(c) \( \alpha(s) = s^2 \):

\[ \gamma(u) = \left(0, \int^u_0 \sin s e^{\int_0^s \cos \tau^2 d\tau} ds, \int^u_0 \cos s^2 ds \right); \]

(d) \( \alpha(s) = \arccos(s), s \in [-1, 1] \):

\[ \gamma(u) = \left(0, \int^u_0 \sqrt{1 - s^2} e^u ds, u \right); \]

(e) \( \alpha(s) = 2 \arctan e^{2s} \): In this case, the expression of \( \gamma \) involve hypergeometric functions.

The surface \( M \) is totally umbilical but not totally geodesic.
Returning to the general case for $\theta$, we distinguish some particular situations for $\alpha$. Assume that $\sin \alpha = 0$. Then $\cos \alpha = \pm 1$ and the principal curvature corresponding to the principal direction $T$ vanishes. Straightforward computations yield $\theta = \pi/2$, which was discussed before.

We suppose that $\cos \alpha = 0$. Then $\sin \alpha = \pm 1$ and the relations (2) and (4) may be written in an easier way. Assume $\sin \alpha = 1$ (similar results in the case $\sin \alpha = -1$). We have

\[ e_1 = \sin \theta E_1 + \cos \theta N, \quad e_2 = E_2, \quad e_3 = -\cos \theta E_1 + \sin \theta N \]
\[ E_1 = \sin \theta e_1 - \cos \theta e_3, \quad E_2 = e_2, \quad N = \cos \theta e_1 + \sin \theta e_3. \]

The Levi-Civita connection $\nabla$ on the surface $M$ is given by

\[ \nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_2} E_1 = \cos \theta E_2, \quad \nabla_{E_2} E_2 = -\cos \theta E_1. \]

Such a surface is minimal, because the second fundamental form is

\[ h(E_1, E_1) = -\sin \theta N, \quad h(E_1, E_2) = 0, \quad h(E_2, E_2) = \sin \theta N \]

and hence its mean curvature vanishes.

FIGURE 1. The curve $\gamma$ according to items (b), (c), (d) and (e).
In order to obtain explicit embedding equations for the surface $M$ let us choose local coordinates as follows.

Let $u$ be such that $E_1 = \partial/\partial u$ and $v$ such that $E_2$ and $\partial/\partial v$ are collinear. This can be done due to the fact that $[E_1, E_2] = -\cos \theta E_2$. Considering $\partial/\partial v = b(u, v)E_2$, with $b$ a smooth function on $M$, since $[\partial/\partial u, \partial/\partial v] = 0$, it follows that $b$ satisfies $b_u - b \cos \theta = 0$. This PDE has the general solution $b(u, v) = \mu(v)e^u \cos \theta$, with $\mu$ a smooth function defined on a certain interval in $\mathbb{R}$.

Denote by $F = (F_1, F_2, F_3)$ the isometric immersion of the surface $M$ in $Sol_3$. We have:

(i) \[ \frac{\partial}{\partial u} = F_u = (\partial_u F_1, \partial_u F_2, \partial_u F_3) = E_1 = \sin \theta e^1_{|F(u,v)} - \cos \theta e^3_{|F(u,v)} = (\sin \theta e^{-F_3(u,v), 0, -\cos \theta}); \]

(ii) \[ \frac{\partial}{\partial v} = F_v = (\partial_v F_1, \partial_v F_2, \partial_v F_3) = \mu(v)e^u \cos \theta E_2 = \mu(v)e^u \cos \theta e^2_{|F(u,v)} = (0, \mu(v)e^u \cos \theta + F_3(u,v), 0). \]

Looking at (i) we immediately get:

• the third component: $F_3(u, v) = -u \cos \theta + \zeta(v), \quad \zeta \in C^\infty(M)$;
• the second component: $F_2(u, v) = F_2(v)$.

Replacing in (ii) we obtain:

• the third component: $\zeta(v) = \zeta_0 \in \mathbb{R}$;
• the second component: $F_2(v) = e^{\zeta_0} \int^v \mu(\tau) d\tau$;
• the first component: $F_1(u, v) = F_1(u)$.

Going back to (i) and taking the first component one gets

$F_1(u) = e^{-\zeta_0} \tan \theta e^u \cos \theta + \text{constant}.$

Since the map $(x, y, z) \mapsto (x + c, y, z)$ is an isometry for $Sol_3$, we can take the previous constant to be 0. Moreover, the map $(x, y, z) \mapsto (e^{-c}x, e^c y, z + c)$ is also an isometry of the ambient space, so $\zeta_0$ may also be assumed to be 0.

Consequently, one obtains the following parametrization for the surface $M$:

$F(u, v) = \left( \tan \theta e^u \cos \theta, \int^v \mu(\tau) d\tau, -u \cos \theta \right)$.

Finally, we may change the parameter $v$ such that $\mu(v) = 1$. One can state the following.

**Proposition 3.5.** Consider $M$ a constant angle surface in $Sol_3$ with angle $\theta$ and $\theta \neq 0, \pi/2$. If $\cos \alpha = 0$ in (3) and (4), then parametrization of $M$ is

$F(u, v) = (\tan \theta e^u \cos \theta, v, -u \cos \theta).$ (5)

Moreover this surface is minimal and it is the group product between the curve $v \mapsto (0, v, 0)$ and the planar curve $\gamma(u) = (\tan \theta e^u \cos \theta, 0, -u \cos \theta).$
4. The general classification theorem

From now on we will deal with $\theta \neq 0, \pi/2$ and $\cos \alpha \neq 0, \sin \alpha \neq 0$.

**Lemma 4.1.** The Levi-Civita connection $\nabla$ on $M$ and the second fundamental form $h$ are given by

$$
\begin{align*}
\nabla_{E_1} E_1 &= -\cos \alpha E_2, & \nabla_{E_1} E_2 &= \cos \alpha E_1, \\
\nabla_{E_2} E_1 &= \sigma \cot \theta E_2, & \nabla_{E_2} E_2 &= -\sigma \cot \theta E_1,
\end{align*}
$$

(6)

$$
\begin{align*}
h(E_1, E_1) &= -\sin \theta \sin \alpha N, & h(E_1, E_2) &= 0, & h(E_2, E_2) &= \sigma N.
\end{align*}
$$

(7)

The matrix of the Weingarten operator $A$ with respect to the basis $\{E_1, E_2\}$ has the expression

$$
A = \begin{pmatrix}
-\sin \alpha \sin \theta & 0 \\
0 & \sigma
\end{pmatrix}
$$

for a certain function $\sigma \in C^\infty(M)$.

Moreover, the Gauss formula yields

$$
E_1(\alpha) = 2 \cos \theta \cos \alpha,
$$

(8a)

$$
E_2(\alpha) = \sin \alpha - \frac{\sigma}{\sin \theta},
$$

(8b)

and the compatibility condition

$$(\nabla_{E_1} E_2 - \nabla_{E_2} E_1)(\alpha) = [E_1, E_2](\alpha) = E_1(E_2(\alpha)) - E_2(E_1(\alpha))$$

gives rise to the differential equation

$$
E_1(\sigma) + \sigma \cos \theta \sin \alpha + \sigma^2 \cot \theta = 2 \sin \theta \cos \theta \sin^2 \alpha.
$$

(9)

We are looking for a coordinate system $(u, v)$ in order to determine the embedding equations of the surface. Let us take the coordinate $u$ such that $\partial/\partial u = E_1$. We will consider $v$ later.

Let us turn our attention to (8a) which can be re-written as

$$
\delta_u \alpha = 2 \cos \theta \cos \alpha.
$$

Solving this PDE one gets

$$
\sin \alpha = \tanh(2u \cos \theta + \psi(v)),
$$

where $\psi$ is a smooth function on $M$ depending on $v$. Notice that, apparently, the equation also has a second solution $\sin \alpha = \coth(2u \cos \theta + \psi(v))$. This is not valid because $\coth$ takes values in $(-\infty, -1)$ or in $(1, +\infty)$.

Now, let us take $v$ in such way that $\partial \alpha / \partial v = 0$, namely $\psi$ is a constant; denote it by $\psi_0$.

It follows that $\alpha$ is given by

$$
\sin \alpha = \tanh(\bar{u}),
$$

(10)

where $\bar{u} = 2u \cos \theta + \psi_0$. At this point, equation (9) becomes

$$
\sigma \bar{u} + \cot \theta (\sigma + 2 \sin \alpha \sin \theta)(\sigma - \sin \alpha \sin \theta) = 0.
$$

(11)
Since \( \partial/\partial v \) is tangent to \( M \), it can be decomposed in the basis \( \{ E_1, E_2 \} \). Thus, there exist functions \( a = a(u, v) \) and \( b = b(u, v) \) such that
\[
\frac{\partial}{\partial v} = aE_1 + bE_2.
\]

Due to the choice of the coordinate \( v \) we have
\[
0 = \frac{\partial a}{\partial v} = 2a \cos \theta \cos \alpha + b \left( \sin \alpha - \frac{\sigma}{\sin \theta} \right).
\]

If \( b = 0 \), then \( \cos \theta = 0 \) or \( \cos \alpha = 0 \) and both situations were studied separately.

Consider now \( b \neq 0 \). Let us denote \( p(u, v) = a/b \). Hence the equality above yields
\[
\sigma = \sin \theta \sin \alpha + p \sin 2\theta \cos \alpha.
\]  (12)

On the other hand
\[
0 = \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] = a_u E_1 + b_u E_2 + b(\cos \alpha E_1 - \sigma \cot \theta E_2).
\]

Subsequently
\[
\begin{align*}
& a_u + b \cos \alpha = 0, \\
& b_u - b\sigma \cot \theta = 0.
\end{align*}
\]  (13)

If in (12) we take the derivative with respect to \( u \), and combining with (11), it follows that
\[
p_u + \cos \alpha + p \cos \theta \sin \alpha + 2p^2 \cot \theta \cos \alpha = 0.
\]  (14)

Straightforward computations yield the general solution for this equation (see the appendix), namely
\[
p(u, v) = \pm \frac{1}{\cos \theta \sin \bar{u} + \varepsilon \frac{\cosh \frac{1}{2} \bar{u}}{\cosh (u) + \Lambda(v)}},
\]  (15)

where \( \varepsilon = 0, 1 \) and \( \Lambda \) is a certain function depending on \( v \).

Let \( F: U \subset \mathbb{R}^2 \rightarrow M \hookrightarrow \text{Sol}_3 \), \( (u, v) \mapsto (F_1(u, v), F_2(u, v), F_3(u, v)) \) be the immersion of the surface \( M \) in \( \text{Sol}_3 \). We have

I. \( \partial_u = F_u = (F_{1,u}, F_{2,u}, F_{3,u}) = E_1 = \sin \theta e_{1|F(u,v)} + \cos \theta e_{2|F(u,v)} - \cos \alpha e_{3|F(u,v)} = (\sin \theta e^{F_3(u,v)}, \cos \theta \cos \alpha e^{F_3(u,v)}, -\cos \theta \sin \alpha) \),

which implies
\[
\begin{align*}
\partial_u F_1 &= \sin \theta e^{-F_3(u,v)}, \quad (16a) \\
\partial_u F_2 &= \cos \theta \cos \alpha e^{F_3(u,v)}, \quad (16b) \\
\partial_u F_3 &= -\cos \theta \sin \alpha. \quad (16c)
\end{align*}
\]

From the last equation one immediately obtains
\[
F_3(u, v) = -\frac{1}{2} \log \cosh(\bar{u}) + \zeta(v),
\]  (17)
where $\zeta$ is a smooth function. Replacing this expression in (16a) and (16b), one gets

$$F_1 = \sin \theta e^{-\zeta(v)} (I(u) + f_1(v)), \quad (18)$$

$$F_2 = \pm \cos \theta e^{\zeta(v)} (J(u) + f_2(v)), \quad (19)$$

where

$$I(u) = \int_u^\infty \sqrt{\cosh(2\tau \cos \theta + \psi_0)} \, d\tau, \quad J(u) = \int_u^\infty \cosh^{-\frac{3}{2}}(2\tau \cos \theta + \psi_0) \, d\tau \quad (20)$$

and $f_1$, $f_2$ are some smooth functions which will be determined in what follows.

II. $\partial_v F_v = (F_1, v, F_2, v, F_3, v) = a(u, v)E_1 + b(u, v)E_2$

$$= a(u, v)(\sin \theta e_{F_1(u,v)} + \cos \theta \cos \alpha e_{3 \int F_1(u,v)})$$

$$+ b(u, v)(\sin \alpha e_{F_1(u,v)} + \cos \alpha e_{3 \int F_1(u,v)}).$$

It follows that

$$\partial_v F_1 = a(u, v) \sin \theta e^{-F_3(u,v)}, \quad (21a)$$

$$\partial_v F_2 = (a(u, v) \cos \theta \cos \alpha + b(u, v) \sin \alpha) e^{F_3(u,v)}, \quad (21b)$$

$$\partial_v F_3 = -a(u, v) \cos \theta \sin \alpha + b(u, v) \cos \alpha. \quad (21c)$$

From (17) and (21c) we have

$$-a(u, v) \cos \theta \sin \alpha + b(u, v) \cos \alpha = \zeta'(v)$$

and from (18) and (21a) we obtain

$$\zeta'(v)(I(u) + f_1(v)) - f_1'(v) + a(u, v) \sqrt{\cosh(\tilde{u})} = 0. \quad (22)$$

Taking the derivative with respect to $u$, one gets

$$\zeta'(v) + a_{u}(u, v) + a(u, v) \cos \theta \tanh(\tilde{u}) = 0. \quad (23)$$

The equation in $a$ has the solution

$$a(u, v) = \frac{-\zeta'(v)I(u) + \xi(v)}{\sqrt{\cosh(u)}}, \quad (24)$$

yielding

$$b(u, v) = \pm \left( \frac{\cos \theta \sinh(\tilde{u})}{\sqrt{\cosh(u)}} \right) \left( -\zeta'(v)I(u) + \xi(v) + \zeta'(v) \cosh(\tilde{u}) \right). \quad (25)$$

Recall that $p(u, v) = a(u, v)/b(u, v)$. We immediately notice that the general solution given by (15) is obtained with the following identification: $\epsilon = 0 \iff \zeta'(v) = 0$ and $\epsilon = 1 \iff \Lambda(v) = \xi(v)/\zeta'(v)$. It follows that

$$p(u, v) = \pm \frac{1}{\cos \theta \sinh(\tilde{u}) + \frac{\zeta'(v) \cosh^{\frac{3}{2}}(\tilde{u})}{-\zeta'(v)I(u) + \xi(v)}}.$$
At this point we will obtain the parametrization of the surface in the following way.

1. Combining (24) with (22) one gets
   \[ f_1'(v) - \zeta'(v)f_1(v) - \xi(v) = 0 \]
   which has the solution
   \[ f_1(v) = e^{\zeta(v)} \int^v \xi(\tau)e^{-\zeta(\tau)} \, d\tau. \]
   Thus
   \[ F_1(u, v) = \sin \theta \left( e^{-\zeta(v)} I(u) + \int^v \xi(\tau)e^{-\zeta(\tau)} \, d\tau \right). \]

2. Similarly, replace (19) in (21b) and one obtains
   \[
   \cos \theta (f_2'(v) + \zeta'(v)f_2(v)) + \zeta'(v) \left( \cos \theta (I(u) + J(u)) - \frac{\sinh(\bar{u})}{\sqrt{\cosh(u)}} \right) = \cos \theta \xi(v). \tag{26}
   \]
   We have
   \[ a(u, v) \cos \theta \cos \alpha + b(u, v) \sin \alpha = \pm \zeta'(v)(\sinh(\bar{u}) - \cos \theta I(u)\sqrt{\cosh(u)}) \]
   and
   \[ \cos \theta (I(u) + J(u)) - \frac{\sinh(\bar{u})}{\sqrt{\cosh(u)}} = \text{constant} \]
   which can be incorporated in the primitives $I(u)$ or $J(u)$. It follows that $f_2$ satisfies the following ordinary differential equation (ODE)
   \[ f_2'(v) + \zeta'(v)f_2(v) = \xi(v) \]
   which has the solution
   \[ f_2(v) = e^{-\zeta(v)} \int^v \xi(\tau)e^{\zeta(\tau)} \, d\tau. \]
   Thus
   \[ F_2(u, v) = \pm \cos \theta \left( e^{\zeta(v)} J(u) + \int^v \xi(\tau)e^{\zeta(\tau)} \, d\tau \right). \]

We conclude with the following result that classifies the constant angle surfaces in the general case.

**Theorem 4.2.** A general constant angle surface in $\text{Sol}_3$, with $\theta \neq 0, \pi/2$ and $\cos \alpha \neq 0$, can be parameterized as
\[ F(u, v) = \gamma_1(v) \ast \gamma_2(u), \tag{27} \]
where
\[
\gamma_1(v) = \left( \sin \theta \int^v \xi(\tau)e^{-\zeta(\tau)} \, d\tau, \pm \cos \theta \int^v \xi(\tau)e^{\zeta(\tau)} \, d\tau, \zeta(v) \right), \tag{28a}
\]
\[
\gamma_2(u) = (\sin \theta I(u), \pm \cos \theta J(u), -\frac{1}{2} \log \cosh \bar{u}), \tag{28b}
\]
where $I$ and $J$ are defined in (20). Here $\zeta$, $\xi$ are arbitrary functions depending on $v$ and $\bar{u} = 2u \cos \theta + \text{constant}$.

**Corollary 4.3.** The only minimal constant angle surfaces in $\text{Sol}_3$ are the hyperbolic planes $x = x_0$ (for $\theta = 0$), the hyperbolic planes $y = y_0$ (for $\theta = \pi/2$) and the surfaces given in Proposition 3.5.

**Proof.** In the general case when $\theta$ is different from 0 and $\pi/2$ and $\alpha$ is such that $\sin \alpha$ and $\cos \alpha$ do not vanish, the minimality condition can be written as $\sigma = \sin \alpha \sin \theta$. But this relation is impossible due to (12) and (14). \qed
Appendix. Solution of the PDE

We prove that the solution of

\[ p_u + \cos \alpha + p \cos \theta \sin \alpha + 2p^2 \cos^2 \theta \cos \alpha = 0 \]

is given by (15), where \( \theta \neq 0, \pi/2 \) and \( \cos \alpha \neq 0 \). Denote by \( \bar{u} = 2u \cos \theta + \psi_0 \).

Let \( q := 1/p \); it follows that \( q \) satisfies

\[ q_u - q^2 \cos \alpha - q \cos \theta \sin \alpha - 2 \cos^2 \theta \cos \alpha = 0. \]

Let \( A := q - \cos \theta \sinh \bar{u} \). It follows that \( q_u = A_u + 2 \cos^2 \theta \cosh \bar{u} \). Hence, \( A \) satisfies

\[ A_u - 3A \cos \theta \sinh \bar{u} - \frac{1}{\cosh \bar{u}} A^2 = 0. \]

Let \( B := A \cosh^{-1/2} \bar{u} \). It follows that \( A_u = 3B \cos \theta \sinh \bar{u} \cosh^{1/2} \bar{u} + B_u \cosh^{3/2} \bar{u} \). Thus, \( B \) satisfies

\[ B_u - B^2 \cosh^{1/2} \bar{u} = 0. \]

Hence either \( B = 0 \) or \( 1/B(u, v) = -I(u) + \Lambda(v) \), for a smooth \( \Lambda \).

(1) If \( B = 0 \) then \( A = 0 \), \( q = \cos \theta \sinh \bar{u} \). Then \( q \neq 0 \) if and only if \( \theta \neq \pi/2 \) and \( \bar{u} \neq 0 \).

One gets

\[ P = \frac{1}{\cos \theta \sinh \bar{u}}, \]

obtaining (15).

(2) If \( B \neq 0 \) then

\[ q(u, v) = \cos \theta \sinh \bar{u} + \frac{\cosh^{1/2} \bar{u}}{-I(u) + \Lambda(v)}. \]

These solutions correspond to 1. \( \zeta' = 0 \) and 2. \( \Lambda(v) = \xi(v)/\zeta'(v) \).

Acknowledgements. The authors wish to thank the anonymous referee for some useful comments which helped us to improve the quality of the paper. The first author was partially supported by MEC-FEDER grant no. MTM2007-61775 and Junta de Andalucía grant no. P06-FQM-01642. The second author was partially supported by Grant PN II ID 398/2007-2010 (Romania).

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AUTHOR QUERIES

Please reply to these questions on the relevant page of the proof; please do not write on this page.

Q1 (page 1): Changes to sentence ‘As for a Riemannian manifold . . .’ OK?

Q2 (page 2): Please check use of ‘respectively’ as there are four references but only three cases.

Q3 (page 2): Please check sense of ‘Such as it occurs . . .’ and suggest alternative text if necessary.

Q4 (page 3): Please check sense of ‘. . . any consideration with . . .’ and suggest alternative text if necessary.

Q5 (page 5): Please confirm figure labels are correct.

Q6 (page 9): Please check sense of ‘. . . were studied separately’ and suggest alternative text if necessary.

Q7 (page 13): Please update details for [10].