Constant angle surfaces in Minkowski space

Rafael López* Marian Ioan Munteanu†

Abstract

A constant angle surface in Minkowski space is a spacelike surface whose unit normal vector field makes a constant hyperbolic angle with a fixed timelike vector. In this work we study and classify these surfaces. In particular, we show that they are flat. Next we prove that a tangent developable surface (resp. cylinder, cone) is a constant angle surface if and only if the generating curve is a helix (resp. a straight line, a circle).

1 Introduction and statement of results

A constant angle surface in Euclidean three-dimensional space $E^3$ is a surface whose tangent planes make a constant angle with a fixed constant vector field of the ambient space [1, 9]. These surfaces generalize the concept of helix, that is, curves whose tangent lines make a constant angle with a fixed vector of $E^3$. This kind of surfaces are models to describe some phenomena in physics of interfaces in liquids crystals and of layered fluids [1]. Constant angle surfaces have been studied for arbitrary dimension in Euclidean space $E^n$ [3, 12] and in different ambient spaces, e.g. $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$ and $Nil_3$ [2, 4, 5].

In this work we extend the concept of constant angle surfaces to a Lorentzian ambient space. Let $E^3_1$ denote the three-dimensional Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle , \rangle = (dx_1)^2 + (dx_2)^2 - (dx_3)^2$$

---

*Partially supported by MEC-FEDER grant no. MTM2007-61775 and Junta de Andalucía grant no. P06-FQM-01642.

†The author was supported by grant PN-II ID 398/2007-2010 (Romania)

Received by the editors March 2010.

Communicated by L. Vanhecke.

2000 Mathematics Subject Classification : 53B25.

Key words and phrases : constant angle surfaces, Minkowski space, helix.

where \((x_1, x_2, x_3)\) are the canonical coordinates in \(\mathbb{R}^3\). In Minkowski space \(E_1^3\) and due to the variety of causal character of a vector, there is not a natural concept of angle between two arbitrary vectors and only it is possible to define the angle between timelike vectors.

Consider a (connected) surface \(M\) and a smooth immersion \(x : M \to E_1^3\). We say that \(x\) is a spacelike immersion if the induced metric on \(M\) via \(x\) is a Riemannian metric. This is equivalent to saying that any unit normal vector field \(\xi\) of \(M\) is timelike at each point. In particular, if \(x : M \to E_1^3\) is a spacelike immersion, then the surface \(M\) is orientable.

**Definition 1.1.** Let \(x : M \to E_1^3\) be a spacelike immersion and let \(\xi\) be a unit normal vector field on \(M\). We say that \(M\) is a constant angle surface if there is a fixed timelike vector \(U\) such that \(\xi\) makes a constant hyperbolic angle with \(U\).

In Theorem 3.4 we give a local description of any constant angle spacelike surface. As a consequence, we prove that they are ruled and flat surfaces (Corollary 3.6). Thus they must be tangent developable surfaces, cylinders and cones. In Section 4 we deal with tangent surfaces showing in Theorem 4.1 that

\[
\text{A tangent developable surface is a constant angle surface if and only if the generating curve is a helix.}
\]

Finally we consider in Section 5 cylinders and cones. We show (see Theorems 5.1 and 5.3)

\[
\text{The only spacelike cylinders that are constant angle surfaces are planes. A cone is a constant angle surface if and only if the generating curve is a circle contained in a spacelike plane.}
\]

## 2 Preliminaries

Most of the following definitions can be found in O’Neill’s book [11]. Let \(E_1^3\) be the three-dimensional Minkowski space. A vector \(v \in E_1^3\) is said spacelike if \(\langle v, v \rangle > 0\) or \(v = 0\), timelike if \(\langle v, v \rangle < 0\), and lightlike if \(\langle v, v \rangle = 0\) and \(v \neq 0\). The norm (length) of a vector \(v\) is given by \(|v| = \sqrt{|\langle v, v \rangle|}\).

In Minkowski space \(E_1^3\) one can define the angle between two vectors only if both are timelike. We describe this fact. If \(u, v \in E_1^3\) are two timelike vectors, then \(\langle u, v \rangle \neq 0\). We say that \(u\) and \(v\) lie in the same timelike cone if \(\langle u, v \rangle < 0\). This defines an equivalence binary relation with exactly two equivalence classes. If \(v\) lies in the same timelike cone than \(E_3 := (0, 0, 1)\), we say that \(v\) is future-directed. For timelike vectors, we have the Cauchy-Schwarz inequality given by

\[
|\langle u, v \rangle| \geq \sqrt{-\langle u, u \rangle} \sqrt{-\langle v, v \rangle}
\]

and the equality holds if and only if \(u\) and \(v\) are two proportional vectors. In the case that both vectors lie in the same timelike cone, there exists a unique number \(\theta \geq 0\) such that

\[
\langle u, v \rangle = -|u||v| \cosh(\theta).
\]

This number \(\theta\) is called the hyperbolic angle between \(u\) and \(v\).
Remark 2.1. We point out that the above reasoning cannot work for other pairs of vectors, even if they are spacelike. For example, the vectors \( u = (\cosh(t), 0, \sinh(t)) \) and \( v = (0, \cosh(t), \sinh(t)) \) are spacelike vectors with \( |u| = |v| = 1 \) for any \( t \). However the number \( \langle u, v \rangle = -\sinh(t)^2 \) takes arbitrary values from 0 to \( -\infty \). Thus, there is no \( \theta \in \mathbb{R} \) such that \( \cos(\theta) = \langle u, v \rangle \).

We also need to recall the notion of Lorentzian cross-product \( \times : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{E}^3 \). If \( u, v \in \mathbb{E}^3 \), the vector \( u \times v \) is defined as the unique one that satisfies \( \langle u \times v, w \rangle = \det(u, v, w) \), where \( \det(u, v, w) \) is the determinant of the matrix whose columns are the vectors \( u, v \) and \( w \) with respect to the usual coordinates. An easy computation gives

\[
u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\.
\]

As the cross-product in Euclidean 3-space, the Lorentzian cross-product in Minkowski space has similar algebraic and geometric properties, such as the anti-symmetry or the orthogonality on both factors.

Let \( x : M \to \mathbb{E}^3 \) be an immersion of a surface \( M \) into \( \mathbb{E}^3 \). We say that \( x \) is spacelike (resp. timelike, lightlike) if the induced metric on \( M \) via \( x \) is Riemannian (resp. Lorentzian, degenerated). This is equivalent to assert that a (local) normal vector \( \xi \) is timelike (resp. spacelike, lightlike). As the concept of angle is given only for timelike vectors, we have to consider those immersions whose unit normal vector is timelike, that is, spacelike immersions. Let \( x \) be a spacelike immersion. At any point \( p \in M \), it is possible to choose a unit normal vector \( \xi(p) \) such that \( \xi(p) \) is future-directed, i.e. \( \langle \xi(p), E_3 \rangle < 0 \). This shows that if \( x \) is a spacelike immersion, the surface \( M \) is orientable.

Denote \( \mathfrak{X}(M) \) the space of tangent vector fields on \( M \). Let \( X, Y \in \mathfrak{X}(M) \). We write by \( \widetilde{\nabla} \) and \( \nabla \) the Levi-Civita connections of \( \mathbb{E}^3 \) and \( M \) respectively. Moreover,

\[
\nabla_X Y = (\widetilde{\nabla}_X Y)^\top
\]

where the superscript \( \top \) denotes the tangent part of the vector field \( \widetilde{\nabla}_X Y \). We define the second fundamental form of \( x \) as the tensorial, symmetric map \( \sigma : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \) given by

\[
\sigma(X, Y) = (\widetilde{\nabla}_X Y)^\perp
\]

where by \( \perp \) we mean the normal part. The Gauss formula can be written as

\[
\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y).
\]  \( (1) \)

We denote by \( A_\xi(X) = A(X) \) the tangent component of \( -\widetilde{\nabla}_X \xi \), that is, \( A_\xi(X) = -(\widetilde{\nabla}_X \xi)^\top \). Because \( \langle \widetilde{\nabla}_X \xi, \xi \rangle = 0 \), we have the so-called Weingarten formula

\[
\widetilde{\nabla}_X \xi = -A_\xi(X).
\]  \( (2) \)

The map \( A : \mathfrak{X}(M) \to \mathfrak{X}(M) \) is called the Weingarten endomorphism of the immersion \( x \). We have then \( \langle AX, Y \rangle = \langle X, AY \rangle \). As a consequence, the Weingarten endomorphism is diagonalizable, that is, if \( p \in M \), the map \( A_p : T_p M \to T_p M \) defined...
by $A_p(v) = (AX)_p$ is diagonalizable, where $X \in \mathfrak{x}(M)$ is a vector field that extends $v$. The eigenvalues of $A_p$ are called the principal curvatures and they will be denoted by $\lambda_i(p)$. Moreover, if $X, Y \in \mathfrak{x}(M)$, we have $\langle A(X), Y \rangle = \langle \sigma(X, Y), \xi \rangle$ and

$$
\sigma(X, Y) = -\langle \sigma(X, Y), \xi \rangle \xi = -\langle A(X), Y \rangle \xi.
$$

Let $\{v_1, v_2\}$ be a basis in the tangent plane $T_pM$ and denote

$$
\sigma_{ij} = \langle \sigma(v_i, v_j), \xi \rangle = \langle A(v_i), v_j \rangle.
$$

If we assume that this basis is orthonormal, we have from (1) and (2)

$$
\tilde{\nabla}_{v_i} V_j = \nabla_{v_i} V_j - \sigma_{ij} \xi. \quad (3)
$$

$$
\tilde{\nabla}_{v_i} \xi = \sigma_{i1} v_1 + \sigma_{i2} v_2. \quad (4)
$$

where $V_i$ is a tangent vector field that extends $v_i$.

### 3 Classification of constant angle surfaces in $E^3_1$

Let $M$ be a constant angle spacelike surface in $E^3_1$ whose unit normal vector field $\xi$ is assumed to be future-directed. Without loss of generality, we assume that $U$ is a unitary vector and after an isometry of the ambient space, we can take $U$ as $E_3$. Denote by $\theta$ the hyperbolic angle between $\xi$ and $U$, that is, $\cosh(\theta) = -\langle \xi, U \rangle$. If $\theta = 0$, then $\xi = U$ on $M$. This means that $x$ describes the immersion of an affine plane parallel to $O x^1 x^2$. Throughout this work, we discard the trivial case that $\theta = 0$.

We decompose $U$ as

$$
U = U^\top + \cosh(\theta) \xi
$$

where $U^\top$ is the projection of $U$ on the tangent plane of $M$. Let

$$
e_1 = \frac{U^\top}{|U^\top|},
$$

which defines a unit tangent vector field on $M$ and consider $e_2$ a unit vector field on $M$ orthogonal to $e_1$ in such a way that $\{e_1, e_2, \xi\}$ defines a positively oriented unit orthonormal basis for every point of $M$. We write now the vector $U$ in the following form

$$
U = \sinh(\theta) e_1 + \cosh(\theta) \xi. \quad (5)
$$

As $U$ is a constant vector field, $\tilde{\nabla}_{e_2} U = 0$ and (5) gives

$$
\sinh(\theta) \tilde{\nabla}_{e_2} e_1 + \cosh(\theta) \tilde{\nabla}_{e_2} \xi = 0. \quad (6)
$$

Taking the normal component and using (3), we obtain

$$
\sinh(\theta) \langle \tilde{\nabla}_{e_2} e_1, \xi \rangle = -\sinh(\theta) \sigma_{21} = 0.
$$
Since $\theta \neq 0$, we conclude $\sigma_{21} = \sigma_{12} = 0$. By combining (4) and (6), it follows that
\[
\nabla_{e_2} e_1 = - \coth(\theta) \sigma_{22} e_2.
\]

Analogously, we have $\nabla_{e_1} U = 0$ and (5) yields
\[
\sinh(\theta) \nabla_{e_1} e_1 + \cosh(\theta) \nabla_{e_1} \xi = 0.
\]

The normal component of the above expression together with (3) gives $\sigma_{11} \sinh(\theta) = 0$, that is, $\sigma_{11} = 0$. We can summarize the above computations with a description of $\nabla$ as follows:

**Theorem 3.1.** With the above notations, the Levi-Civita connection $\nabla$ for a constant angle spacelike surface in $E_3^1$ is given by
\[
\begin{align*}
\nabla_{e_1} e_1 &= 0, \\
\nabla_{e_1} e_2 &= 0, \quad \nabla_{e_2} e_1 = - \coth(\theta) \sigma_{22} e_2, \\
\nabla_{e_2} e_2 &= \coth(\theta) \sigma_{22} e_1.
\end{align*}
\]

Moreover, with respect to $\{e_1, e_2\}$, the Weingarten map takes the form
\[
\begin{pmatrix}
0 & 0 \\
0 & -\sigma_{22}
\end{pmatrix}.
\]

At this moment one can choose coordinates $u$ and $v$ such that $\frac{\partial}{\partial u} = e_1$ and $\frac{\partial}{\partial v} = \beta e_2$, where $\beta$ is a certain smooth function on the surface.

**Corollary 3.2.** Given a constant angle spacelike surface $M$ in $E_3^1$, there exist local coordinates $u$ and $v$ such that the metric on $M$ writes as $\langle \cdot, \cdot \rangle = du^2 + \beta^2 dv^2$, where $\beta = \beta(u, v)$ is a smooth function on $M$, i.e. the coefficients of the first fundamental form are $E = 1$, $F = 0$ and $G = \beta^2$.

Now, we will consider that the parametrization $x(u, v)$ given by the above Corollary. We know that $A(x_u) = 0$ and $\sigma_{11} = \sigma_{12} = 0$. From Theorem 3.1 one obtains
\[
\begin{align*}
x_{uu} &= 0, \\
x_{uv} &= \frac{\beta_u}{\beta} x_v, \\
x_{vv} &= -\beta \beta_u x_u + \frac{\beta_v}{\beta} x_v + \beta^2 \sigma_{22} \xi.
\end{align*}
\]

On the other hand, we have
\[
\begin{align*}
\xi_u &= \nabla_{x_u} \xi = 0, \\
\xi_v &= \nabla_{x_v} \xi = \beta \sigma_{22} e_2 = \sigma_{22} x_v.
\end{align*}
\]
As $\xi_{uv} = \xi_{vu} = 0$, it follows $\tilde{\nabla}_{x_u}(\sigma_{22}x_v) = 0$. Using the fact that $\sigma_{12} = 0$, $\tilde{\nabla}_{x_u}x_v = \tilde{\nabla}_{x_v}x_u$ and Theorem 3.1, we obtain

$$0 = (\sigma_{22})_ux_v + \sigma_{22}\tilde{\nabla}_{x_u}x_v = (\sigma_{22})_ux_v - \coth(\theta)\sigma_{22}^2x_v.$$ 

Therefore

$$(\sigma_{22})_u - \coth(\theta)\sigma_{22}^2 = 0. \quad (7)$$

Also, we use the expression of $x_{uv}$ to conclude that

$$(\sigma_{22})_u + \sigma_{22}\frac{\beta_u}{\beta} = 0$$

that is, $(\beta\sigma_{22})_u = 0$ and then, there exists a smooth function $\varphi = \varphi(v)$ depending only on $v$ such that

$$\beta\sigma_{22} = \varphi(v). \quad (8)$$

Moreover, by combining (7) and (8), we have

$$\frac{\beta_u}{\beta} = -\coth(\theta)\sigma_{22}.$$ 

**Proposition 3.3.** Consider a constant angle spacelike surface $x = x(u, v)$ in $E_3^1$ where $(u, v)$ are the coordinates given in Corollary 3.2. If $\sigma_{22} = 0$ on $M$, then $x$ describes an affine plane.

**Proof.** We know that $\beta_u = 0$ on $M$. Thus $x_{uv} = 0$ and hence, $x_u$ is a constant vector. From (5), $\xi$ is a constant vector field along $M$, and so, $x$ parameterizes a (spacelike) plane. 

Here and in the rest of the work, we will assume that $\sigma_{22} \neq 0$. By solving equation (7), we obtain a function $\alpha = \alpha(v)$ such that

$$\sigma_{22}(u, v) = \frac{1}{-\coth(\theta)u + \alpha(v)}.$$ 

Then (8) yields

$$\beta(u, v) = \varphi(v)\left(-\coth(\theta)u + \alpha(v)\right).$$

Consequently,

$$x_{uu} = 0 \quad (9)$$

$$x_{uv} = \frac{\coth(\theta)}{\coth(\theta)u - \alpha(v)}x_v \quad (10)$$

$$x_{vv} = \varphi^2\coth(\theta)(-\coth(\theta)u + \alpha)x_u$$

$$+ \left(\frac{\varphi'}{\varphi} + \frac{\alpha'}{-\coth(\theta)u + \alpha}\right)x_v + \varphi^2(-\coth(\theta)u + \alpha)\xi. \quad (11)$$

From (5) we have

$$\langle x_u, U \rangle = \sinh(\theta), \quad \langle x_v, U \rangle = 0.$$
or equivalently
\[ \langle x, U \rangle_u = \sinh(\theta), \quad \langle x, U \rangle_v = 0. \]

Then
\[ \langle x, U \rangle = \sinh(\theta)u + \mu, \quad \mu \in \mathbb{R}. \]

The parametrization of \( x \) is now (up to vertical translations)
\[ x(u, v) = (x_1(u, v), x_2(u, v), -\sinh(\theta)u). \]

As \( E = 1 \), there exists a function \( \phi : M \to \mathbb{R} \) such that
\[ x_u = (\cosh(\theta) \cos(\phi(u, v)), \cosh(\theta) \sin(\phi(u, v)), -\sinh(\theta)). \]

Since \( x_{uu} = 0 \), one obtains \( \phi_u = 0 \), that is, \( \phi = \phi(v) \) and hence
\[ x_u = (\cosh(\theta) \cos(\phi(v)), \cosh(\theta) \sin(\phi(v)), -\sinh(\theta)) = \cosh(\theta)(\cos(\phi(v)), \sin(\phi(v)), 0) - \sinh(\theta)(0, 0, 1). \]

Denoting by \( f(v) = (\cos(\phi(v)), \sin(\phi(v))) \) we can rewrite \( x_u \) as
\[ x_u = \cosh(\theta)(f(v), 0) - \sinh(\theta)(0, 0, 1). \]

We compute \( x_{uv} \):
\[ x_{uv} = \cosh(\theta)(f'(v), 0). \]  
(12)

An integration with respect to \( u \) leads to
\[ x_v = \cosh(\theta)(uf'(v) + h(v), 0) \]  
(13)

where \( h = h(v) \) is a smooth curve in \( \mathbb{R}^2 \). From (10) and (13)
\[ x_{uv} = \frac{1}{\coth(\theta)u - \alpha(v)} \cosh^2(\theta) \sinh(\theta)(uf'(v) + h(v), 0). \]

Comparing with (12) one gets
\[ h = -\tanh(\theta)\alpha(v)f'(v) \]

and so,
\[ x_v = \cosh(\theta)(u - \tanh(\theta)\alpha(v))(f'(v), 0). \]

The value of \( x_{vv} \) is now
\[ x_{vv} = \cosh(\theta)(u - \tanh(\theta)\alpha(v))(f''(v), 0) - \sinh(\theta)\alpha'(v)(f'(v), 0). \]  
(14)

Multiplying the two expressions of \( x_{vv} \) in (11) and (14) by \( x_u \), we conclude
\[ \phi'(v) = \frac{1}{\sinh(\theta)}\phi(v). \]
We do a change in the variable $v$ to get $\phi' = 1$ for any $v$, that is, $\phi(v) = v$. It is not difficult to see that this does not change the second derivatives of $x$ in (9), (10) and (11). Then

\begin{align*}
x_u &= \cosh(\theta)(\cos(v), \sin(v), 0) - \sinh(\theta)(0, 0, 1), \\
x_v &= (\cosh(\theta)u - \sin(\theta)\alpha(v))(-\sin(v), \cos(v), 0).
\end{align*}

The above reasoning can be written in the following

**Theorem 3.4.** Let $M$ be a constant angle spacelike surface in Minkowski space $E^3_1$ which is not totally geodesic. Up to a rigid motion of the ambient space, there exist local coordinates $u$ and $v$ such that $M$ is given by the parametrization

\begin{equation}
x(u, v) = (u \cosh(\theta)(\cos(v), \sin(v)) + \psi(v), -u \sinh(\theta))
\tag{15}
\end{equation}

with

\begin{equation}
\psi(v) = \sinh(\theta) \left( \int \alpha(v) \sin(v), -\int \alpha(v) \cos(v) \right)
\tag{16}
\end{equation}

where $\alpha$ is a smooth function on a certain interval $I$. Here $\theta$ is the hyperbolic angle between the unit normal of $M$ and the fixed direction $U = (0, 0, 1)$.

**Proposition 3.5.** A constant angle spacelike surface is flat.

**Proof.** At each point $p \in M$, we consider $\{v_1(p), v_2(p)\}$ a basis of eigenvectors of the Weingarten endomorphism $A_p$. In particular, $\lambda_i(p) = -\sigma_i(p)$. As the function $\langle \xi, U \rangle$ is constant, a differentiation along $v_i(p)$ yields $\langle \nabla_{v_i(p)}\xi, U \rangle = 0$, $i = 1, 2$. Using (4), we obtain

$$\lambda_1(p)\langle v_1(p), U \rangle = \lambda_2(p)\langle v_2(p), U \rangle = 0.$$ 

Assume that at the point $p$, $\lambda_1(p)\lambda_2(p) \neq 0$. By using the continuity of the principal curvature functions, we have $\langle v_1(q), U \rangle = \langle v_2(q), U \rangle = 0$ for every point $q$ in a neighborhood $N_p$ of $p$. This means that $U$ is a normal vector in $N_p$ and hence it follows $\theta = 0$: contradiction. Thus $\lambda_1(p)\lambda_2(p) = 0$ for any $p$, that is, $K = 0$ on $M$. 

As in Euclidean space, all flat surfaces are characterized to be locally isometric to planes, cones, cylinders or tangent developable surfaces.

**Corollary 3.6.** Any constant angle spacelike surface is isometric to a plane, a cone, a cylinder or a tangent developable surface.

The fact that a constant angle (spacelike) surface is a ruled surface appears in Theorem 3.4. Exactly, the parametrization (15) writes as

\begin{equation}
x(u, v) = (\psi(v), 0) + u\left( \cosh(\theta)(\cos(v), \sin(v)), -\sinh(\theta) \right),
\end{equation}

which proves that our surfaces are ruled. Next we present some examples of surfaces obtained in Theorem 3.4.
**Example 1.** We take different choices of the function $a$ in (16).

1. Let $a(v) = 0$. After a change of variables, $\psi(v) = (0, 0)$ and
   \[ x(u, v) = u(\cosh(\theta)(\cos(v), \sin(v)), -\sinh(\theta)). \]

   This surface is a cone with the vertex the origin and whose basis curve is a circle in a horizontal plane. See Figure 1, left.

2. Let $a(v) = 1$. Then $\psi(v) = -\sinh(\theta)(\cos(v), \sin(v))$ and
   \[ x(u, v) = -\sinh(\theta)(\cos(v), \sin(v), 0) + u(\cosh(\theta)(\cos(v), \sin(v)), -\sinh(\theta)). \]

   Again, this surface is a cone based in a horizontal circle.

3. Consider $a(v) = 1/ \sin(v)$. Then $\psi(v) = \sinh(\theta)(v, -\log(|\sin(v)|))$ and
   \[ x(u, v) = \sinh(\theta)(v, -\log(|\sin(v)|), 0) + u(\cosh(\theta)(\cos(v), \sin(v)), -\sinh(\theta)). \]

   See Figure 1, right.

![Figure 1: Constant angle surfaces corresponding to several choices of $a$ in Theorem 3.4: $a(v) = 0$ (left) and $a(v) = 1/ \sin(v)$ (right).](image)

4 **Tangent developable constant angle surfaces**

In this section we study tangent developable surfaces that are constant angle surfaces (see [10] for the Euclidean ambient space). Given a regular curve $\gamma : I \to \mathbb{E}^3_1$, we define the tangent surface $M$ generated by $\gamma$ as the surface parameterized by

\[ x(s, t) = \gamma(s) + t\gamma'(s), \quad (s, t) \in I \times \mathbb{R}. \]

The tangent plane at a point $(s, t)$ of $M$ is spanned by $\{x_s, x_t\}$, where

\[ x_s = \gamma'(s) + t\gamma''(s), \quad x_t = \gamma'(s). \]
The surface is regular at those points where \( t(\gamma'(s) \times \gamma''(s)) \neq 0 \). Without loss of generality, we will assume that \( t > 0 \).

On the other hand, since \( M \) is a spacelike surface and \( \gamma(s) \in M \), the curve \( \gamma \) must be spacelike. We parameterize \( \gamma \) such that \( s \) is the arc-length parameter, that is, \( \langle \gamma'(s), \gamma'(s) \rangle = 1 \) for every \( s \). As a consequence, \( \gamma''(s) \) is orthogonal to \( \gamma'(s) \). We point out that although \( \gamma \) is a spacelike curve, the acceleration vector \( \gamma''(s) \) can be of any causal character, that is, spacelike, timelike or lightlike. However, the surface \( M \) is spacelike, which implies that \( \gamma \) is not an arbitrary curve. Indeed, by computing the first fundamental form \( \{E, G, F\} \) of \( M \) with respect to basis \( \{x_s, x_t\} \), we obtain

\[
\begin{pmatrix}
E & F \\
F & G \\
\end{pmatrix}(s, t) = \begin{pmatrix}
1 + t^2 \langle \gamma''(s), \gamma''(s) \rangle & 1 \\
1 & 1 \\
\end{pmatrix}.
\]

\( M \) is spacelike if and only if \( EG - F^2 > 0 \). This is equivalent to \( \langle \gamma''(s), \gamma''(s) \rangle > 0 \), that is, \( \gamma''(s) \) is spacelike for any \( s \).

The tangent vector \( T(s) \) and the normal vector \( N(s) \) are defined by \( T(s) = \gamma'(s) \), \( N(s) = \gamma''(s)/\kappa(s) \), respectively, where \( \kappa(s) = |\gamma''(s)| > 0 \) is the curvature of \( \gamma \) at \( s \). The Frenet Serret frame of \( \gamma \) at each point \( s \) associates an orthonormal basis \( \{T(s), N(s), B(s)\} \), where \( B(s) = T(s) \times N(s) \) is called the binormal vector ([6, 8]). We remark that \( B(s) \) is a unit timelike vector. The corresponding Frenet equations are

\[
\begin{align*}
T' &= \kappa N \\
N' &= -\kappa T + \tau B \\
B' &= \tau N.
\end{align*}
\]

The function \( \tau(s) = -\langle N'(s), B(s) \rangle \) is called the torsion of \( \gamma \) at \( s \). For tangent surfaces \( x \), the unit normal vector field \( \xi \) to \( M \) is \( \xi = (x_s \times x_t)/\sqrt{EG - F^2} = -B(s) \).

In order to give the next result, recall the concept of a helix in Minkowski space. A spacelike (or timelike) curve \( \gamma = \gamma(s) \) parameterized by the arc-length is called a helix if there exists a vector \( v \in E^3_1 \) such that the function \( \langle \gamma'(s), v \rangle \) is constant. This is equivalent to saying that the function \( \tau/\kappa \) is constant.

**Theorem 4.1.** Let \( M \) be a tangent developable spacelike surface generated by \( \gamma \). Then \( M \) is a constant angle surface if and only if \( \gamma \) is a helix with \( \tau^2 < \kappa^2 \). Moreover the direction \( U \) with which \( M \) makes a constant hyperbolic angle \( \theta \) can be taken such that

\[
U = \frac{1}{\sqrt{\kappa^2 - \tau^2}} \left( -\tau(s)T(s) + \kappa(s)B(s) \right)
\]

and the angle \( \theta \) is determined by the relation

\[
cosh(\theta) = \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}.
\]

**Proof.** 1. Assume that \( M \) makes a constant angle with a fixed direction \( U \), with \( \langle U, U \rangle = -1 \). Then \( \langle B(s), U \rangle \) is a constant function \( c \) with \( c < 0 \). By differentiation with respect to \( s \), and using the Frenet equation, we have
\( \tau \langle \mathbf{N}(s), U \rangle = 0 \) for any \( s \). If \( \langle \mathbf{N}(s_0), U \rangle \neq 0 \) at some point \( s_0 \), then \( \tau = 0 \) in a neighborhood of \( s_0 \). This means that the binormal \( \mathbf{B}(s) \) is a constant vector \( V \), \( \gamma \) is a planar curve and \( \zeta = -V \) is constant on \( M \). Thus, \( \zeta \) makes constant angle not only with the vector \( U \) (which is fix from the beginning), but with any timelike vector. Hence \( U \) could be replaced by other vector, for example by \( V \). Equations (17) and (18) are trivial. Finally, \( \gamma \) is a helix with \( \tau^2 < \kappa^2 \) and the surface is a (spacelike) affine plane.

If \( \langle \mathbf{N}(s), U \rangle = 0 \) on \( I \), and because \( \langle U, U \rangle = -1 = \langle \mathbf{T}(s), U \rangle^2 - c^2 \), the function \( \langle \mathbf{T}(s), U \rangle \) is a constant function. Therefore \( \gamma \) is a helix in \( E_3^1 \) again. A differentiation of \( \langle \mathbf{N}(s), U \rangle = 0 \) gives \( \langle \mathbf{T}(s), U \rangle = c\tau / \kappa \). Thus \( -1 = c^2\tau^2 / \kappa^2 - c^2 \), which shows that \( \tau^2 < \kappa^2 \). Moreover, \( c = -\kappa / \sqrt{\kappa^2 - \tau^2} \). As \( U = \langle \mathbf{T}(s), U \rangle \mathbf{T}(s) - c\mathbf{B}(s) \), we get the expression (17). Finally (18) is trivial.

2. Conversely, let \( \gamma = \gamma(s) \) be a helix and let \( x = x(s, t) \) be the corresponding tangent surface. We know that \( \tau / \kappa \) is a constant function. If \( \tau = 0 \), \( \gamma \) is a planar curve. Then the tangent surface generated by \( \gamma \) is a plane, which is a constant angle surface. If \( \tau \neq 0 \), let us define

\[
U(s) = -\frac{\tau}{\kappa} \mathbf{T}(s) + \mathbf{B}(s).
\]

Using the Frenet equations, we have \( dU/ds = 0 \), that is, \( U \) is a constant vector. Moreover, \( \langle \zeta, U \rangle = -\langle \mathbf{B}(s), U \rangle = 1 \). Thus \( M \) is a constant angle surface. The hyperbolic angle \( \theta \) is given by

\[
\cosh(\theta) = \frac{\langle \zeta, U \rangle}{\sqrt{-\langle U, U \rangle}} = \frac{\kappa}{\sqrt{\kappa^2 - \tau^2}}.
\]

We present two examples of constant angle surfaces that are tangent surfaces. After an isometry of the ambient space, we assume that \( U = E_3 \). From (18) if \( \tau / \kappa = a \), with \( |a| < 1 \), then \( \cosh(\theta) = 1 / \sqrt{1 - a^2} \). Moreover \( \langle \mathbf{T}(s), U \rangle = -\sinh(\theta) \) and \( \langle \gamma(s), E_3 \rangle = -\sinh(\theta)s + b \), with \( b \in \mathbb{R} \). After an appropriate change of variables, we take \( b = 0 \) and we write

\[
\gamma(s) = (\gamma_1(s), \gamma_2(s), \sinh(\theta)s).
\]

Because \( s \) is the arc-length parameter, there exists a smooth function \( \lambda(s) \) such that \( \gamma'(s) = (\cosh(\theta) \cos(\lambda(s)), \cosh(\theta) \sin(\lambda(s)), \sinh(\theta)) \). An easy computation leads to

\[
\mathbf{N}(s) = (-\sin(\lambda(s)), \cos(\lambda(s)), 0)
\]
\[
\mathbf{B}(s) = (-\sinh(\theta) \cos(\lambda(s)), -\sinh(\theta) \sin(\lambda(s)), -\cosh(\theta)).
\]

The curvature is \( \kappa(s) = \cosh(\theta) \lambda'(s) \) and the torsion is \( \tau(s) = -\sinh(\theta) \lambda'(s) \).

**Example 2.** We take \( \lambda(s) = s \). An integration yields

\[
\gamma(s) = (\cosh(\theta) \sin(s), -\cosh(\theta) \cos(s), \sinh(\theta)s).
\]
Figure 2: A constant angle tangent developable surface with \( \kappa(s) = \cosh(\theta) \) and \( \tau(s) = -\sinh(\theta) \). Here \( \theta = 2 \) and \( U = (0, 0, 1) \).

Here \( \kappa(s) = \cosh(\theta) \) and \( \tau(s) = -\sinh(\theta) \) and \( \gamma \) is a helix where both the curvature and torsion functions are constant. A picture of the curve \( \gamma \) and the corresponding tangent surface appears in Figure 2.

**Example 3.** We take \( \lambda(s) = s^2 \). Recall that the Fresnel functions are defined as

\[
\operatorname{FrS}(x) = \int_0^x \sin \left( \frac{\pi t^2}{2} \right) dt \quad \operatorname{FrC}(x) = \int_0^x \cos \left( \frac{\pi t^2}{2} \right) dt.
\]

Then

\[
\gamma(s) = \left( \sqrt{\frac{2}{\pi}} \cosh(\theta) \operatorname{FrC}(\sqrt{\frac{2}{\pi}} s), \sqrt{\frac{2}{\pi}} \cosh(\theta) \operatorname{FrS}(\sqrt{\frac{2}{\pi}} s), \sinh(\theta) s \right)
\]

is a helix where \( \kappa(s) = 2 \cosh(\theta)s \) and \( \tau(s) = -2 \sinh(\theta)s \). Figure 3 shows the curve \( \gamma \) and the generated tangent surface.

**Remark.** We can extend the concept of constant angle surfaces for tangent developable *timelike* surfaces. Let \( M \) be a tangent surface generated by a curve \( \gamma \) such that \( M \) is timelike. Then \( \gamma \) is a spacelike curve (with \( \gamma'' \) timelike) or \( \gamma \) is a timelike curve (with \( \gamma'' \) spacelike). Assume that \( \gamma \) is parameterized by the arc-length \( s \). Denote by \( \{T, N, B\} \) the Frenet frame of \( \gamma \), that is, \( T(s) = \gamma'(s), N(s) = \gamma''(s)/\kappa(s), \) with \( \kappa(s) = |\gamma''(s)| \) and \( B(s) = T(s) \times N(s) \). The Frenet equations are

\[
\begin{align*}
T' &= \kappa N \\
N' &= \kappa T + \tau B \\
B' &= \epsilon \tau N
\end{align*}
\]
where $\tau(s) = \langle N'(s), B(s) \rangle$ and $\langle T(s), T(s) \rangle = \epsilon = -\langle N(s), N(s) \rangle$, $\epsilon \in \{1, -1\}$. Anyway, $B$ is always spacelike. We assume that there exists a fixed vector $U \in \mathbb{E}_3^1$ such that the function $\langle \xi, U \rangle$ is constant. Then it is not difficult to show that this condition is equivalent to saying that $\gamma$ is a planar curve ($\tau = 0$, and $M$ is an affine plane), or $\langle N(s), U \rangle = 0$ for any $s$. In this case, the first Frenet equation yields $\langle T'(s), U \rangle = 0$ and thus, $\langle T(s), U \rangle$ is a constant function. This means that $\gamma$ is a helix of $\mathbb{E}_3^1$. This generalizes Theorem 4.1 for tangent timelike surfaces.

We point out that our parametrization of $M$, $x(s,t) = \gamma(s) + t\gamma'(s)$ where $\gamma$ is a helix given by

$$\gamma(s) = \left( \cosh(\theta) \int \cos(\lambda(s)), \cosh(\theta) \int \sin(\lambda(s)), \sinh(\theta)s \right)$$

does not satisfy the conditions of Corollary 3.2 since $F \neq 0$. In order to obtain the parametrization given in Theorem 3.4, we do a change of parameters given by

$$u = -(s + t) \quad v = \pi + \lambda(s).$$

Now we obtain $x_s = -x_u + \lambda'x_v$ and $x_t = -x_u$.

But $x_t = (\cosh(\theta) \cos(\lambda(s)), \cosh(\theta) \sin(\lambda(s)), \sinh(\theta))$ or, in terms of $u$ and $v$

$$x_u = (\cosh(\theta) \cos(v), \cosh(\theta) \sin(v), -\sinh(\theta)).$$

Similarly $x_s = x_t + t\lambda'(s) \left( -\cosh(\theta) \sin(\lambda(s)), \cosh(\theta) \cos(\lambda(s)), 0 \right)$. It follows

$$x_v = (u + \lambda^{-1}(v - \pi)) \cosh(\theta) \left( -\sin(v), \cos(v), 0 \right).$$

Consequently, the function $a$ involved in the general formula can be expressed as

$$a(v) = -\coth(\theta) \lambda^{-1}(v - \pi).$$
5 Constant angle cylinders and cones

In this section we consider cylinders and cones that are constant angle (spacelike) surfaces. A ruled surface is called a cylinder if it can be parameterized by \( x(s,t) = \gamma(s) + tv \), where \( \gamma \) is a regular curve and \( v \) is a fixed vector. The regularity of the cylinder is given by the fact that \( \gamma'(s) \times v \neq 0 \). A cone is a ruled surface that can be parameterized by \( x(s,t) = t\gamma(s) \), where \( \gamma \) is a regular curve. The vertex of the cone is the origin and the surface is regular whenever \( t(\gamma(s) \times \gamma'(s)) \neq 0 \).

**Theorem 5.1.** The only constant angle (spacelike) cylinders are planes.

*Proof.* Let \( M \) be a spacelike cylinder generated by a curve \( \gamma \) and a fixed direction \( v \). As the surface is spacelike, \( v \) is a spacelike vector, for which we will assume \(|v| = 1\). We can suppose that \( \gamma \) is contained in a plane \( \Pi \) such that \( v \) is orthogonal to \( \Pi \). In particular, \( \Pi \) is a timelike plane. The unit normal vector is \( \xi(s,t) = \gamma'(s) \times v \).

By contradiction, we assume that \( \gamma \) is not a straight line, that is, \( \kappa(s) \neq 0 \) at some interval. We consider \( \{T, N, B\} \) the Frenet frame of \( \gamma \). As \( \gamma \) is a planar curve, \( B(s) = \pm v \) and so, \( \xi(s) = \pm N(s) := \gamma''(s)/\kappa(s) \). Let \( U \) be the unit (timelike) vector such that the function \( \langle \xi(s), U \rangle \) is constant, that is, \( \langle N(s), U \rangle \) is constant. By differentiation with respect to \( s \), using the Frenet equations and since \( \gamma \) is a planar curve, we obtain \( \langle T(s), U \rangle = 0 \) for any \( s \). A new differentiation gives \( \kappa(s) \langle N(s), U \rangle = 0 \) for any \( s \). As \( \kappa(s) \neq 0 \), we have \( \langle N(s), U \rangle = 0 \), for any \( s \). However, \( N(s) \) and \( U \) are both timelike vectors and thus, the product \( \langle N(s), U \rangle \) can never vanish: contradiction. Consequently, \( \kappa(s) = 0 \) for any \( s \), that is, \( \gamma \) is a straight line and then \( M \) is a (spacelike) plane.

**Remark 5.2.** We point out that this result is more restrictive than the corresponding in Euclidean space \( \mathbb{E}^3 \). In \( \mathbb{E}^3 \), any cylinder is a constant angle surface: it suffices to take \( U \) as the vector that defines the rulings of the cylinder. The difference in Lorentzian ambient is that our surfaces are spacelike and the vector \( U \) is timelike, which imposes extra conditions.

For the next result concerning cones, we recall that a (spacelike) circle in Minkowski space is a planar curve with constant curvature [7, 8]. We also point out that the plane \( \Pi \) containing the circle can be of any causal character. Indeed, after a rigid motion of \( \mathbb{E}^3 \), a spacelike circle can be viewed as follows: a Euclidean circle in a horizontal plane (if \( \Pi \) is spacelike), a hyperbola in a vertical plane (if \( \Pi \) is timelike) and a parabola in a \( \pi/4 \)-inclined plane (if \( \Pi \) is lightlike).

**Theorem 5.3.** Let \( M \) be a (spacelike) cone. Then \( M \) is a constant angle surface if and only if the generating curve is a circle in a spacelike plane or it is a straight line (and \( M \) is a plane).

*Proof.* Let \( M \) be a cone, for which one can assume that its vertex is the origin of \( \mathbb{R}^3 \). Let \( x(s,t) = t\gamma(s) \) be a parametrization of \( M \), where \( t \neq 0 \) and \( \gamma(s) \neq 0 \), \( s \in I \). As \( x_s = t\gamma'(s) \) is spacelike, \( \langle \gamma'(s), \gamma'(s) \rangle > 0 \). On the other hand, \( x_t \) must be spacelike, this means that \( \langle \gamma(s), \gamma(s) \rangle > 0 \). We can change \( \gamma(s) \) by a proportional vector and suppose that \( \gamma \) lies in the unit Minkowski sphere of \( \mathbb{E}^3 \).
that is, in the de Sitter space $S^2_1 = \{x \in E^3_1; x_1^2 + x_2^2 - x_3^2 = 1\}$. Thus, $|\gamma(s)| = 1$ for any $s \in I$. Without loss of generality, we suppose that $\gamma = \gamma(s)$ is parameterized by the arc-length. Then $\gamma(s)$ and $\gamma''(s)$ are orthogonal to $\gamma'(s)$. The unit normal vector field $\xi = T(s) \times \gamma(s)$. In particular,

$$\gamma''(s) = -\gamma(s) - \langle \gamma''(s), \xi(s) \rangle \xi(s).$$

(19)

Assume that $M$ is a constant angle surface and let $U$ be the unit timelike vector such that $\langle \xi(s), U \rangle$ is constant. By differentiation with respect to $s$, we have

$$\langle \gamma''(s) \times \gamma(s), U \rangle = 0$$

(20)

for any $s$. Substituting in (20) the value of $\gamma''(s)$ obtained in (19), we get

$$\langle \gamma'(s), \gamma'(s) \times \gamma(s) \rangle \langle \gamma'(s), U \rangle = 0.$$

We discuss the two possibilities:

1. If $\langle \gamma''(s), \gamma'(s) \times \gamma(s) \rangle \neq 0$ at some point, then $\langle \gamma'(s), U \rangle = 0$ for any $s$. This means that $\gamma(s)$ lies in a plane orthogonal to $U$ and so, this plane must be spacelike. Thus the acceleration $\gamma''(s)$ is a spacelike vector. Then we can take the Frenet frame of $\gamma$, namely $\{T, N, B\}$, where $B = T \times N$ is a timelike vector. Moreover, $B(s) = \pm U$. If $\kappa(s) = 0$ for any $s$, then $\gamma$ is a straight line and the surface is a plane. On the contrary, since $\langle T(s), \gamma(s) \rangle = 0$, by taking the derivative, one obtains $\kappa(s) \langle N(s), \gamma(s) \rangle + 1 = 0$. On the other hand, because $\gamma$ is a planar curve ($\tau = 0$), the derivative of the function $\langle N(s), \gamma(s) \rangle$ vanishes. This means that $\langle N(s), \gamma(s) \rangle$ is constant and so, $\kappa(s)$ is constant.

2. Assume $\langle \gamma''(s), \gamma'(s) \times \gamma(s) \rangle = 0$ for any $s$. As $\gamma(s)$ and $\gamma'(s)$ are orthogonal spacelike vectors, then $\gamma''(s)$ is a spacelike vector. Again, we consider the Frenet frame $\{T, N, B\}$ where $B$ is a timelike vector. The above equation writes now as $\kappa(s) \langle B(s), \gamma(s) \rangle = 0$. If $\kappa(s) = 0$ for any $s$, then $\gamma$ is a straight line again. Suppose now $\langle B(s), \gamma(s) \rangle = 0$. Similar to the previous case, because $\gamma(s) \in S^2_1$, it follows $\langle T(s), \gamma(s) \rangle = 0$ and $\kappa(s) \langle N(s), \gamma(s) \rangle + 1 = 0$. In particular, $\langle N(s), \gamma(s) \rangle \neq 0$ and then, the derivative of $\langle B(s), \gamma(s) \rangle$ implies $\tau = 0$, that is, $\gamma$ is a planar curve. Finally, the derivative of $\langle N(s), \gamma(s) \rangle$ is zero, namely $\langle N(s), \gamma(s) \rangle$ is constant, and then, $\kappa(s)$ is constant too.

As an example of constant angle cones, Figure 1 (left) shows a cone based on a circle contained in a (horizontal) spacelike plane.
References


Departamento de Geometría y Topología
Universidad de Granada
18071 Granada, Spain
email: rcamino @ ugr.es

Universitatea ‘Al.I.Cuza’ Iaşi,
Facultatea de Matematică
Bd. Carol I, n.11, 700506 Iaşi, Romania
email: marian.ioan.munteanu @ gmail.com