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# A criterion on instability of cylindrical rotating surfaces

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**Abstract.** We consider a column of a stationary rotating surface in Euclidean space. In this paper we obtain a value $l_0 > 0$ in such a way that if the length $l$ of the column satisfies $l > l_0$, then the surface is instable. This extends previous results due to Plateau and Rayleigh for columns of surfaces with constant mean curvature.

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**1. Statement of the problem and results.** Consider a column of a circular cylinder of radius $r$. One asks which lengths $l$ of columns determine stable cylinders (as surfaces with constant mean curvature). Rayleigh [13] proved that the column is stable if and only if $l < 2\pi r$ (see also [3]). This was experimentally observed by Plateau [12]. In this work, we give two similar results for rotating cylindrical surfaces. First, we present the physical framework necessary for the statements of our results.

We consider the steady rigid rotation of a homogeneous incompressible fluid drop in absence of gravity, which is surrounded by a rigidly rotating incompressible fluid. In mechanical equilibrium, we say that the drop is a stationary rotating drop, or simply, a rotating drop. Rotating drops have been the subject of intense study beginning from the work of Plateau [12], because they could be models in other areas of physics, such as astrophysics, nuclear physics, or fluid dynamics. In the case of our interest, the surfaces are of cylindrical type, that is, ruled surfaces whose rulings are parallel to the axis of rotation. These surfaces have appeared in a number of physical settings, usually referred in the literature as the problem of the shape of a Hele-Shaw cell. For example, if the fluid rotates with respect to a vertical line, we consider that the liquid lies

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confined in a narrow slab determined by two parallel horizontal plates. Then we can suppose that the shape of the drop is invariant by vertical displacements. We refer the reader to [5,6,15] and references therein for further details and a more precise statement of the physical problem.

We assume the liquid drop surrounded for a fluid rotates about the $x_3$-axis with a constant angular velocity $\omega$, where $(x_1, x_2, x_3)$ are the usual coordinates of Euclidean space $\mathbb{R}^3$. Let $\rho_1$ and $\rho_2$ be the constant densities of the liquid drop and its surrounding, respectively, and denote by $W$ the bounded open set in $\mathbb{R}^3$ which is the region occupied by the rotating drop. We set $S = \partial W$ as the free interface between the drop and the ambient liquid, which we suppose to be a smooth boundary surface. The energy of this mechanical system is given by

$$E = \tau|S| - \frac{1}{2}(\rho_1 - \rho_2)\omega^2 \int_W r^2 dx,$$

where $\tau$ stands for the surface tension on $S$, $|S|$ is the surface area of $S$ and $r = r(x) = \sqrt{x_1^2 + x_2^2}$ is the distance from a point $x$ to the $x_3$-axis. The term $\tau|S|$ is the surface energy of the drop and $\frac{1}{2}(\rho_1 - \rho_2)\omega^2 \int_W r^2 dx$ is the potential energy associated with the centrifugal force. We assume that the volume $V$ of the drop remains constant while rotating. If we assume that the liquid drop is denser than its surrounding, that is, $\rho_1 > \rho_2$, the physical setting corresponds to *heavy liquid drops*, whereas if $\rho_1 < \rho_2$, we have *air bubbles*. In equilibrium, the interface $S$ is governed by the Laplacian equation

$$2\tau H(x) = \frac{1}{2}(\rho_1 - \rho_2)\omega^2 r^2 + \lambda \quad x \in S,$$

where the mean curvature $H$ is calculated with the inward direction. The constant $\lambda$ depends on the volume constraint. As a consequence, the mean curvature of the interface $S$ satisfies an equation of type

$$2H(x_1, x_2, x_3) = ar^2 + b,$$

where $a, b \in \mathbb{R}$. We then say that $S$ is a *stationary rotating surface*. If the liquid does not rotate ($a = 0$, or $\omega = 0$ in the Laplace equation), the interface is modeled by a surface with constant mean curvature, namely, $H = b/2$.

Consider a cylindrical surface $M$ whose generating curve $C$ is a closed embedded curve that lies in the $x_1x_2$-plane and the rulings of the surface are straight-lines parallel to the $x_3$-axis. We parametrize $M$ as $x(t, s) = \alpha(s) + t(0, 0, 1)$, $t, s \in \mathbb{R}$, where $\alpha$ is a parametrization of $C$. We always orient $M$ so the Gauss map points inside (this means that $\alpha$ runs in counterclockwise direction). We consider *columns* of cylindrical surfaces, that is, the range of $t$ is some interval $I$ and we call the length of the column the length of that interval.

It is important to point out that there exist cylindrical rotating surfaces whose generating curve is *closed*. For the case that $ab < 0$, the variety of shapes of closed curves is as follows. We begin with a circle corresponding to zero angular velocity. As we increase the angular velocity, the curve changes through a sequence of shapes which evolved from non-self-intersecting curves with fingered morphology and high symmetry for slow rotation to self-intersecting curves with a richness of symmetric properties for fast angular velocity. A complete description of these curves can be seen in [10].

Recall that a stationary rotating surface is called *stable* if the second variation of the energy of the surface is non-negative for every smooth deformation of the surface which fixes the enclosed volume and the boundary of the surface. Stability of rotating drops has been widely studied by many authors. In physics the literature is big and we omit references. We only point out that Brown and Scriven performed numerical analysis of stability of rotating drops [4] and that Wang et al. [17] did experiments in low gravity environments in order to confirm the numerical results. See also Chapter 5 in [9]. Mathematically, the problem has been studied in different articles: [1,2,7,11,14,20].

In this paper we will prove two results about the stability of columns of cylindrical surfaces.

**Theorem 1.1.** *Let $M$ be a cylindrical rotating surface of column length $l$, let $C$ be the embedded planar generating curve of $M$ in the $x_1x_2$-plane and let $\Omega$ be the domain bounded by $C$. Denote by $L$ and $|\Omega|$ the length of $C$ and the area of $\Omega$, respectively. We orient $M$ so the Gauss map points inside the solid cylinder $\Omega\times\mathbb{R}$. Assume that $M$ satisfies the Laplace equation $2H(x) = ar^2+b$. If $a > 0$ and $M$ is stable, then*

$$l \leq \frac{\pi}{2}\sqrt{\frac{L}{a|\Omega|}}.$$

**Theorem 1.2.** *Consider a cylindrical rotating surface $M$ with the same hypothesis and notation as in Theorem 1.1. If $a, b \geq 0$, with $a^2 + b^2 \neq 0$, and $M$ is stable, then*

$$l \leq 2\pi\sqrt{\frac{L}{4a|\Omega| + Lb^2}}.$$

**2. Preliminaries about rotating surfaces.** In this section, we give some preliminaries and definitions and we derive the formula of second variation of the energy. Let $x : M \to \mathbb{R}^3$ be an immersed oriented compact surface and let $N$ be the Gauss map. Denote by $\{E_1, E_2, E_3\}$ the canonical orthonormal base in $\mathbb{R}^3$, that is, $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$ and $E_3 = (0, 0, 1)$. We write $N_i = \langle N, E_i\rangle$ and $x_i = \langle x, E_i\rangle$ for $1 \leq i \leq 3$. Given a constant $a \in \mathbb{R}$, we define the energy functional (associated to $a$) as

$$E(x) = \int_M dM - a\int_M r^2x_3N_3\; dM, \quad r = \sqrt{x_1^2 + x_2^2}$$

where $dM$ is the area element on $M$. Here $\int_M r^2x_3N_3\; dM$ represents the centrifugal force of the surface with respect to the $x_3$-axis. Consider a smooth variation $X$ of $x$, that is, a smooth map $X : (-\epsilon, \epsilon) \times M : M \to \mathbb{R}^3$ such that, by setting $x_t = X(t, -)$, we have $x_0 = x$ and $x_t - x \in C_0^\infty(M)$. Let $u \in C^\infty(M)$ be the normal component of the variational vector field of $x_t$,

$$u = \left\langle \frac{\partial x_t}{\partial t}\bigg|_{t=0}, N \right\rangle.$$

We take $E(t) := E(x_t)$ the value of the energy for each immersion $x_t$. The first variation of $E$ at $t = 0$ as well as the volume functional are given by

$$E'(0) = -\int_M \left(2H - ar^2\right) u \, dM \quad V'(0) = \int_M u \, dM.$$

Here $H$ is the mean curvature of the immersion $x$. We require that the volume $V(t)$ of each immersion $x_t$ remains constant throughout the variation. By the method of Lagrange multipliers, the first variation of $E$ at $t = 0$ has to be zero relative to all volume preserving variations if there is a constant $b$ so that $E'(0) + bV'(0) = 0$. This yields the condition (see [19] for details):

$$2H(x) = ar^2 + b, \quad a, b \in \mathbb{R}. \tag{2.1}$$

Now we derive the formula of the second variation of the area. As far as I know, this formula is not explicitly written in the literature, although a general formula was obtained by Wente [18]; see also [11]. We present a derivation of the second variation of the energy for a rotating surface following ideas of Koiso and Palmer [8]. Assume that $x : M \to \mathbb{R}^3$ is a critical point of $E$, and we write the second variation of the functional $E$ in the form

$$E''(0) = -\int_M u \cdot L[u] \, dM,$$

where $L$ is a linear differential operator acting on the normal component, which we want to find. Since the translations in the $E_3$ are symmetries of the energy functional $E$, then $L[N_3] = 0$. The same occurs for the rotation with respect to the $x_3$-axis, $L[\psi] = 0$, where $\psi = \langle x \wedge N, E_3\rangle$ and $\wedge$ is the vector cross product of $\mathbb{R}^3$. We now compute $L[N_i]$ and $L[\psi]$. The tension field of the Gauss map satisfies

$$\Delta N + |\sigma|^2 N = -2\nabla H,$$

where $\Delta$ is the Laplacian in the metric induced by $x$, $\sigma$ is the second fundamental of the immersion and $\nabla$ is the covariant differentiation. Since $2H = ar^2 + b$, we have

$$2\nabla H = 2a(x_1 \nabla x_1 + x_2 \nabla x_2) = 2a\left(x_1 E_1 + x_2 E_2 - (h - x_3 N_3) N\right). \tag{2.2}$$

Here $h := \langle N, x\rangle$ stands for the support function of $M$. In particular,

$$\Delta N_3 + \left(|\sigma|^2 - 2a(h - x_3 N_3)\right) N_3 = 0 \tag{2.3}$$

We now take the function $\psi$. In general, we have

$$\Delta\psi + |\sigma|^2 \psi = -2\langle \nabla H, E_3 \wedge x\rangle.$$

It follows from (2.2) that

$$-2\langle \nabla H, E_3 \wedge x\rangle = 2a(h - x_3 N_3)\langle N, E_3 \wedge x\rangle = 2a(h - x_3 N_3)\psi.$$

Therefore,

$$\Delta\psi + \left(|\sigma|^2 - 2a(h - x_3 N_3)\right)\psi = 0. \tag{2.4}$$

As a consequence of (2.3) and (2.4), we conclude

$$L = \Delta + |\sigma|^2 - 2a(h - x_3 N_3). \tag{2.5}$$

**Definition 2.1.** Let $x : M \to \mathbb{R}^3$ be a smooth immersion that satisfies the Laplace equation (2.1). We say that $x$ is stable if

$$\Phi(u) := -\int_M u \left(\Delta u + \left(|\sigma|^2 - 2a(h - x_3 N_3)\right) u\right) \, dM \geq 0 \tag{2.6}$$

for all $u \in C_0^\infty(M)$ such that

$$\int_M u \, dM = 0.$$

**Remark 2.2.** In the literature there are different notions about the stability of rotating surfaces according to the physical problem that lies behind the mathematical formulation. Therefore the admissible deformations of the surface change with each problem (see [9]). Of course, a volume constraint is always assumed and this has been the reason of our definition. Some authors consider variations with extra assumptions. For example, in [14] it is assumed that the center of mass of any surface of the variation lies in the $x_3$-axis. In such case, the functions $u$ in the above definition must satisfy the extra conditions $\int_M x_1 u \, dM = \int_M x_2 u \, dM = 0$. On the other hand, in the mentioned article of Brown and Scriven [4], the authors study the stability for axisymmetric liquid drops, assuming axisymmetric variations of the surface.

**3. Proof of theorems.** Assume that $M$ is a column of a cylindrical rotating surface of length $l$ whose generating curve $\alpha = \alpha(s)$ is a closed curve in the $x_1x_2$-plane with $s$ the arc-length parameter. We denote by $\mathbf{n}(s)$ the unit normal vector to $\alpha$, that is, $\mathbf{n}(s) = J\alpha'(s)$, where $J$ is the $\pi/2$-rotation of the $x_1x_2$-plane to counterclockwise direction. In particular, $\{\alpha'(s), \mathbf{n}(s), E_3\}$ is an oriented orthonormal basis of $\mathbb{R}^3$. Moreover, $\alpha''(s) = \kappa(s)\mathbf{n}(s)$, where $\kappa$ is the curvature of $\alpha$.

We parametrize $M$ as $x(t,s) = \alpha(s) + tE_3$, $t \in [0,l]$ and $s \in [0,L]$, where $L$ denotes the length of $\alpha$. The first fundamental form of $M$ is $I = dt^2 + ds^2$ and the Gauss map is $N(t,s) = E_3 \wedge \alpha'(s)$. We remark that $N$ is the orientation that points towards $\Omega \times \mathbb{R}$ due to the chosen parametrization of $\alpha$, and thus, $N$ is the orientation of the statement of Theorem. A computation of the mean curvature $H$ gives

$$2H(t,s) = \langle E_3 \wedge \alpha'(s), \alpha''(s)\rangle = \kappa(s).$$

Before the proof of our theorems, we extend a result that holds for rotating liquid drops: if $M$ is an embedded closed surface that satisfies (2.1), the center of mass lies in the $x_3$-axis [16]. For cylindrical rotating surfaces we prove

**Theorem 3.1.** *The center of mass of a column of a cylindrical rotating surface lies in the $x_3$-axis.*

*Proof.* Since the surface is invariant in the $x_3$-axis and the parameter $t$ ranges in a fix interval independent on $s$, it suffices to show that the center of mass of any horizontal cross-section $\Omega$ of the surface is the origin. Let $\alpha$ be the embedded closed curve of $M$ at the level $x_3 = 0$. We know that the curvature $\kappa$ of $\alpha$ is $\kappa(s) = ar(s)^2 + b$, where $s$ denote the arc-length parameter of $\alpha$. In $\Omega$ we define the vector field $X(x_1, x_2) = (x_1^2, 0)$. Then $\operatorname{div}(X) = 2x_1$ and the divergence theorem assures $2\int_\Omega x_1 \, d\Omega = -\int_\alpha \langle X, \mathbf{n}\rangle ds$. Here $\mathbf{n}$ denotes the inward unit normal vector along $\alpha$. If $\alpha(s) = (x_1(s), x_2(s))$, then $\mathbf{n}(s) = (-x_2'(s), x_1'(s))$. On the other hand, $x_1'' = -\kappa x_2'$ and so

$$\begin{aligned} 0 &= \int_\alpha x_1'' \, ds = -a\int_\alpha r^2 x_2' \, ds - b\int_\alpha x_2' \, ds = -a\int_\alpha r^2 x_2' \, ds \\ &= -a\int_\alpha x_1^2 x_2' \, ds - a\int_\alpha x_2^2 x_2' \, ds = -a\int_\alpha x_1^2 x_2' \, ds = a\int_\alpha \langle X, \mathbf{n}\rangle ds. \end{aligned}$$

Because $a \neq 0$, we conclude that $\int_\Omega x_1 \, d\Omega = 0$, which proves that the first coordinate of the center of mass is 0. Similarly, one shows that the second coordinate of the center mass is 0, proving the result. □

We proceed with the proof of the theorems. We calculate each one of the terms of (2.5). Since the first fundamental form of $M$ is the Euclidean metric, $\Delta$ coincides with the Euclidean Laplacian operator $\Delta_0 = \partial_{tt}^2 + \partial_{ss}^2$. The term $|\sigma|^2$ is $\kappa(s)^2$ and

$$h - N_3 x_3 = \langle N(t,s), x(t,s)\rangle = -\langle \alpha(s) \wedge \alpha'(s), E_3\rangle.$$

Consider smooth functions on $M$ written in the form $u(t,s) = f(t)g(s)$. Then $u$ is a test-function for $\Phi$ if $f(0) = f(l) = 0$, $g(0) = g(L) = g'(0) = g'(L)$ and

$$\int_0^l f(t) \, dt = 0 \quad \text{or} \quad \int_0^L g(s) \, ds = 0.$$

From the above computations, $\Phi(u)$ in (2.6) writes as

$$\begin{aligned} \Phi(u) = -\int_0^L \int_0^l & f(t)g(s) \left(f''(t)g(s) + f(t)g''(s)\right. \\ & \left. + \left(\kappa(s)^2 + 2a\langle \alpha(s) \wedge \alpha'(s), E_3\rangle\right) f(t)g(s)\right) \, dt \, ds. \end{aligned}$$

**3.1. Proof of Theorem 1.1.** Let us take $f(t) = \sin\left(\frac{\pi t}{l}\right)$ and $g_i(s) = \langle \alpha'(s), E_i\rangle$, $i = 1, 2$. The functions $g_i$ correspond, up to sign, to the functions $N_i$, $i = 1, 2$. Then $\int_0^L g_i(s) \, ds = 0$. Let $u_i(t,s) = f(t)g_i(s)$, $i = 1, 2$. If $M$ is stable, then $\Phi(u_i) \geq 0$ for $i = 1, 2$. Let us do the computations of $\Phi(u_i)$. As $f''(t) = -\frac{\pi^2}{l^2} f(t)$, we have that $\Phi(u_i) \geq 0$ becomes

$$\frac{\pi^2}{l^2}\int_0^L g_i(s)^2 \, ds - \int_0^L g_i(s) g_i''(s) \, ds \geq \int_0^L \left(\kappa(s)^2 + 2a\langle \alpha(s) \wedge \alpha'(s), E_3\rangle\right) g_i(s)^2 \, ds. \tag{3.1}$$

Integration by parts leads to $\int_0^L g_i(s)g_i''(s)\ ds = -\int_0^L g_i'(s)^2\ ds$. We note that $g_i'(s) = \kappa(s)\langle \mathbf{n}(s), E_i\rangle$. Therefore,

$$g_1(s)^2 + g_2(s)^2 = 1, \quad g_1'(s)^2 + g_2'(s)^2 = \kappa(s)^2.$$

Summing the two expressions of (3.1) for $i = 1, 2$, we have

$$\frac{\pi^2}{l^2} L \geq 2a \int_0^L \langle \alpha(s) \wedge \alpha'(s), E_3\rangle\ ds.$$

As the curve $\alpha$ runs in counterclockwise direction, then

$$\int_0^L \langle \alpha'(s) \wedge \alpha'(s), E_3\rangle\ ds = 2|\Omega|,$$

where $\Omega$ is the bounded domain determined by $\alpha$ in the $x_1x_2$-plane. Therefore

$$l^2 \leq \frac{\pi^2 L}{4a|\Omega|}$$

and this proves the theorem. □

Taking into account Remark 2.2, we point that the variation of the surface $M$ associated to the functions $u_i$ does not preserve the center mass in the $x_3$-axis. Let us see for $u = fg_1$. Since $\int_0^l f(t)\ dt \neq 0$, we consider the function $g_1 = x_1'$. In such case, $\int_0^L x_1g_1\ ds = \int_0^L x_1(s)x_1'(s)\ ds = 0$, but

$$\int_0^L x_2g_1\ ds = \int_0^L x_2(s)x_1'(s)\ ds = |\Omega| \neq 0.$$

**3.2. Proof of Theorem 1.2.** One can check that $u(t, s) = \sin(\frac{2\pi t}{l})$ is a test-function for the functional $\Phi$ since $\int_0^l f(t)\ dt = 0$ and $f(0) = f(l) = 0$. Doing similar computations as in the proof above, we obtain that if $M$ is stable then

$$\frac{4\pi^2}{l^2} L \geq \int_0^L \kappa(s)^2\ ds + 4a|\Omega| = \int_0^L (ar^2 + b)^2\ ds + 4a|\Omega| \geq bL^2 + 4a|\Omega|.$$

Since some of the values $a$ of $b$ does not vanish, we have

$$l^2 \leq \frac{4\pi^2 L}{Lb^2 + 4a|\Omega|}.$$

This concludes the proof of theorem. □

In this result, the variation associated with the function $u$ preserves the fact that the center of mass of the surface lies in the $x_3$-axis, since now $\int_0^l f(t)\ dt = 0$.

**Remark 3.2.** Consider a circular cylinder $M$ of radius $r$ and length $l$. According with our choice of the orientation of the generating curve, $M$ is a surface with constant mean curvature if in the Laplace equation (2.1) we put $a = 0$ and $b = 1/r$. The estimate of the length $l$ for stable columns of cylinders given in Theorem 1.2 reads now as $l \leq 2\pi r$, re-discovering the value obtained by Rayleigh.

**Remark 3.3.** Consider again a circular cylinder $M$ of radius $r$. Then $M$ can be viewed as a rotating surface, with $a = 1/r^3$ and $b = 0$. Then Theorem 1.1 says that if the cylinder is stable, then $l \leq \frac{\pi r}{\sqrt{2}}$. In the case of Theorem 1.2, $l \leq \sqrt{2}\pi r$. As we have pointed out, this difference appears by the different types of admissible variations. Anyway, both estimates differ if we view $M$ as a constant mean curvature surface, where the bound for $l$ was $2\pi r$. As a consequence, there exist columns of circular cylinders that are instable viewed as rotating surfaces but are stable as surfaces with constant mean curvature.

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