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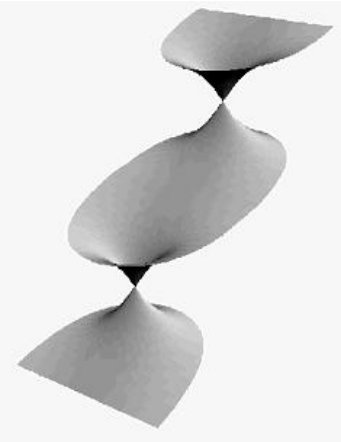

TITLES AND ABSTRACTS

Shintaro AKAMINE
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Behavior of the Gaussian curvature of timelike minimal surfaces with singularities

Abstract. A timelike minimal surface in the 3-dimensional Lorentz-Minkowski space is a surface with a Lorentzian metric whose mean curvature vanishes identically. One of the most important differences between spacelike surfaces (or surfaces in the Euclidean space) and timelike surfaces is the diagonalizability of the shape operator of a surface. For the minimal case, the diagonalizability of the shape operator corresponds to the sign of the Gaussian curvature of a timelike minimal surface away from flat points. More precisely the shape operator is diagonalizable over the real number field on points with negative Gaussian curvature and diagonalizable over the complex number field on points with positive Gaussian curvature. Flat points consist of umbilic points and quasi-umbilic points. The aim of this talk is to investigate the behavior of the Gaussian curvature near regular points and singular points on timelike minimal surfaces. First we prove that the sign of the Gaussian curvature of any timelike minimal surface is de- termined only by the orientations of two null curves that generate the surface, and flat points are characterized by the degeneracies of these null curves. Next we also discuss the behavior of the Gaussian curvature near singular points on timelike minimal surfaces. In particular, about relations between the Gaussian curvature and singular points, we prove that near a non-degenerate singular point of a timelike minimal surface which is not a cuspidal edge, there is no flat points. Moreover in this case, the sign of the Gaussian curvature is negative (resp. positive) near a singular point if and only if the surface has (resp. has not) the structure of a wave front at the singular point. On the other hand, near cuspidal edges, we can not determine the sign of the Gaussian curvature in general, but we can determine the behavior of the Gaussian curvature if cuspidal edges accumulate to another non-degenerate singular point. We also determine when flat points accumulate to a non-degenerate singular point of a timelike minimal surface. In this talk we try to give detailed proofs of results and related ideas. This talk is based on the preprint arXiv:1701.00238.

Busra ATKAS
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Planar mechanisms in Lorentz space

Abstract. In Euclidean space, structure equations of planar, spherical and spatial mechanisms are investigated. In this paper, we obtain structure equations of a chain by using Lorentz matrix and inner product in Lorentz plane. Then, we show rotation and translation motion of the chain in this plane.

Eva M. ALARCÓN
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Spacelike hypersurfaces in the Lorentz-Minkowski space with the same Riemannian and Lorentzian mean curvature

Abstract. Spacelike hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ can be endowed with two Riemannian metrics, the one inherited from $\mathbb{L}^{n+1}$ and the one induced by the Euclidean metric of $\mathbb{R}^{n+1}$. As a direct consequence of the classical theorems of Bernstein and Calabi-Bernstein, and of the genera- lization of the last one to arbitrary dimension, we can deduce that the only entire graphs in $\mathbb{L}^{n+1}$ that are simultaneously minimal and maximal are the spacelike hyperplanes. Using a theorem of Heinz, Chern and Flanders, we can extend this result to entire spacelike graphs with the same constant mean curvature functions H_R and H_L. We consider the general case of spacelike hypersurfaces with the same Riemannian and Lorentzian mean curvature functions not necessarily cons- tant, and we study some of their geometric properties. Specifically, we prove that a spacelike hypersurface in $\mathbb{L}^{n+1}$ such that $H_R = H_L$ does not have any elliptic points. As an application of this result jointly with a well- known result by Osserman about the non-existence of ellictic points for a certain class of compact hypersurfaces in $\mathbb{R}^{n+1}$, we give some interesting consequences about the geometry of such hypersurfaces. This is a joint work with Alma L. Albujer and Magdalena Caballero.

References

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Magdalena CABALLERO
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On the surfaces with the same mean curvature in the Euclidean 3-space and the Lorentz-Minkowski 3-space, and the $H_R = H_L$ surface equation

Abstract. Spacelike surfaces in the Lorentz-Minkowski 3-dimensional space $\mathbb{L}^3$ can be endowed with another Riemannian metric, the one induced by the Euclidean space $\mathbb{R}^3$. Those surfaces are locally the graph of a smooth function $u(x, y)$ satisfying $|Du| < 1$. If in addition they have the same mean curvature with respect to both metrics, they are the solutions to a certain partial differential equation, the $H_R = H_L$ surface equation.

It is well known that the only surfaces that are simultaneously minimal in $\mathbb{R}^3$ and maximal in $\mathbb{L}^3$ are open pieces of helicoids and of spacelike planes, [3]. The proof of this result consists in proving that those surfaces are ruled surfaces. And finishes using that the only ruled surfaces which are maximal and minimal at the same time are the plane and the helicoid.

In this talk we consider the general case of spacelike surfaces with the same mean curvature with respect to both metrics. Firstly, we prove the existence of examples with non-zero mean curvature. Afterwards, we show that our surfaces do not have any elliptic points. As an application of this result, jointly with a classical argument on the existence of elliptic points due to Osserman [4], we present several geometric consequences for the surfaces we are considering. Finally, we focus on the $H_R = H_L$ equation. Its character is studied, some uniqueness results for its Dirichlet problem are given, as well as some uniqueness and non-existence results for entire solutions.

References

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Marginally trapped submanifolds into a null hypersurface of de Sitter space

Abstract. We study codimension two trapped submanifolds contained into one of the two following null hypersurfaces of de Sitter spacetime: (i) the future component of the light cone, and (ii) the past infinite of the steady state space.

For codimension two compact spacelike submanifolds in the light cone we show that they are conformally diffeomorphic to the round sphere. This fact enables us to deduce that the problem of characterizing compact marginally trapped submanifolds into the light cone is equivalent to solving the Yamabe problem on the round sphere, allowing us to obtain our main classification result for such submanifolds.

We also fully describe the codimension two compact marginally trapped submanifolds contained into the past infinite of the steady state space and characterize those having parallel mean curvature field. Finally, we consider the more general case of codimension two complete, non-compact, weakly trapped spacelike submanifolds contained into the light cone. This is a joint work with Luis J. Alías and Marco Rigoli.

References

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Existence and extendibility of rotationally symmetric spacelike graphs with prescribed higher mean curvature function in Minkowski space

Abstract. In this talk I investigate the existence of rotationally symmetric entire spacelike graphs with prescribed k-th mean curvature function in Minkowski space

$\mathbb{L}^{n+1}$. As a previous step, I analyse the associated homogeneous Dirichlet problem on a ball, which is not elliptic for $k > 1$, and then I prove that it is possible to extend the solutions. Moreover, a sufficient condition for uniqueness of these graphs is given. Finally, a brief comment is done for the same problems in the a little more difficult case of the Euclidean space $\mathbb{R}^{n+1}$.

These results are contained in [1].

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Double rotational surfaces in Lorentz-Minkowski 4-space

Abstract. One of the most basic examples of surfaces in 3-dimensional Euclidean space is a rotational surface or a surface of revolution, that is, a surface which is the trace of a planar curve that is rotated about an axis in its supporting plane. Its simple construction makes that it is appealing to geometers, but also is open to alteration. One possible generalization is to subject a planar curve to two simultaneous rotations. The resulting surface is called a twisted surface and studied in [3, 4] (see also the references therein). Another possibility is to extend the concept of a rotational surface to higher dimensional ambient spaces, see for instance [5, 6].

Combining these two points of view leads to the construction of a double rotational surface in 4-space: perform on a planar curve in 4-space two simultaneous rotations, possibly at different rotation speeds. In [1], double rotational surfaces in 4-dimensional Euclidean space are defined and curvature properties on it are examined. The relation of flat double rotational surfaces with newly defined Clelia curves in Euclidean 4-space is also highlighted there. These results turn out to be analogous to the 3-dimensional case (for an overview of properties of Clelia curves in 3-space and their relation with flat twisted surfaces, see [2] and the references therein).

In this contribution, double rotational surfaces in 4-dimensional Lorentz-Minkowski space are defined. Two problems arise when transferring the construction of sub-

jecting a planar curve, the profile curve, to two simultaneous rotations in Lorentz-Minkowski 4-space. Firstly, the supporting plane of the profile curve has one of three causal characters, namely spacelike, timelike or lightlike (null). Secondly, because of the existence of these different causal characters, there exist different rotations, depending on the causal character of the planes that are left invariant. Therefore, different possibilities for the construction must be considered. Since constructions involving only rotations which leave invariant spacelike or timelike planes are very similar to the construction in Euclidean 4-space (see [1]), most attention will be paid to the cases in which so-called 'null-rotations' are involved. As to be expected, incorporating rotations which leave invariant lightlike planes, leads to results deviating from the ones in Euclidean 4-space. Focus will be on obtaining explicit parameterizations of these surfaces. To conclude, some interesting curvature properties (flatness, minimality) for these double rotational surfaces are studied and illustrated with examples.

References

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Seher KAYA
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Björling Problem and Weierstrass- Enneper representation of Maximal Surface in Lorentz-Minkowski Space

Abstract. Minimal surface has zero mean curvature at every point in Euclidean space. Many studies have been carried out to obtain minimal surfaces. Björling problem is one of them and we can get minimal surface on a given curve with the help of complex variables. Also one can express a minimal surface with holomorphic forms with the help of Weierstrass-Enneper representation. In this talk we consider the Björling problem and Weierstrass- Enneper representation in Lorentz-Minkowski space for maximal surface which is a spacelike surface with zero mean curvature in $\mathbb{L}^3$. Then we get new examples of maximal surfaces which based on circle and helix. Then we obtain 1-holomorphic forms of these surfaces. Also we deal with the relation between minimal and maximal surfaces in terms of Weierstrass-Enneper representation. Especially we investigate duality of rotational and helicoidal surfaces.

Rahul KUMAR

Maximal surfaces, Born-Infeld solitons and Ramanujan?s identities

Abstract. In the first part of these lectures we make an observation that the maximal surface equation and Born-Infeld equation (which arises in physics in the context of nonlinear electrodynamics) are related by a wick rotation. Using this observation we present a method to construct a one parameter family of complex Born-Infeld solitons (solutions of Born-Infeld equation) from a given one parameter family of maximal surfaces. We shall also show that a Born-Infeld soliton can be realised either as a spacelike min- imal graph or timelike minimal graph over a timelike plane or a combination of both away from singular points. In the next part we discuss a different formulation for describing maximal surfaces in Lorentz-Minkowski space, $\mathbb{L}^3$, using the identification of $\mathbb{L}^3$ with $\mathbb{C} \times \mathbb{R}$. This description of maximal sur- faces will help us to give a different proof of the singular Björling problem for the case of

closed real analytic null curve. As an application, we show the existence of maximal surface which contains a given spacelike closed curve and has a special singularity. Finally in the last part of these lectures we show the connection of maximal surfaces to analytic number theory through certain Ramanujan's identities.

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Constant Mean Curvature Invariant Surfaces in $\mathbb{L}^3$ and a Blaschke's Variational Problem

Abstract. n 1930, in [2], Blaschke studied the solutions of the variational problem for the energy $\Theta(\gamma) = \int_\gamma \sqrt{\kappa}$ acting on certain spaces of curves in the Euclidean 3-space $\mathbb{R}^3$. In particular, in $\mathbb{R}^2$, he obtained the catenaries.

In this talk, for a fixed $\mu \in \mathbb{R}$, we are going to extend this problem and we will consider curves in L3 which are extremal for the action

$$\Theta(\gamma) = \int_\gamma \sqrt{\kappa - \mu} \quad (1)$$

We are going to get all solutions of the Euler-Lagrange equations of (1) in Minkowski 3-space $\mathbb{L}^3$, [1].

Finally, making critical curves evolve under their associated Killing vector field ([3] and [4]), these solutions are going to be related with profile curves of constant mean curvature invariant surfaces of $\mathbb{L}^3$; showing that a invariant surface of $\mathbb{L}^3$ has constant mean curvature, if and only if, it is geodesically foliated by critical curves of (1), [1]. This leads to another description of the well-known families of constant mean curvature surfaces in $\mathbb{L}^3$, ([5] and [6]).

Furthermore, our results can be extended to any Riemannian and Lorentzian 3-space form, [1].

References

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Minding isometries of B-scrolls in Minkowski space

Abstract. Ruled surfaces in 3-dimensional Minkowski space $\mathbb{R}^3_1$ are surfaces that admit parametrization of the form

$$f(u,v) = c(u) + ve(u),\ u \in I \subset \mathbb{R}, v \in \mathbb{R}, \quad (1)$$

where c is the base curve and the e(u) is a non-vanishing vector field along c which generates the rulings. Ruled surfaces in $\mathbb{R}^3_1$ are classified with respect to the causal character of their base curve and their rulings (spacelike, timelike and null (lightlike, isotropic)). Among surfaces with null rulings, so called class M_0, B-scrolls of null Frenet curves are of special interest. In this work we study local isometries of ruled surfaces that preserve rulings, so called the Minding isometries. We investigate conditions on invariants of B-scrolls to obtain such isometries and show that if two B-scrolls are locally isometric, then the local isometry preserves their rulings, unless they are B-scrolls with constant Gaussian curvature.

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José Miguel MANZANO
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On the correspondence between CMC spacelike graphs in Minkowski space and minimal graphs in Heisenberg group

Abstract. H. Lee [1] discovered an interesting correspondence between spacelike graphs with constant mean curvature τ in Minkowski space $\mathbb{L}^3$ and minimal graphs in Heisenberg group $Nil_3(\tau)$, which generalises the classical Calabi correspondence between maximal spacelike graphs in $\mathbb{L}^3$ and minimal graphs in Euclidean space $\mathbb{R}^3$.

In this talk, we shall introduce this conformal duality of graphs from the more general point of view of Killing submersions, which allows us to transform the prescribed mean curvature equation for spacelike graphs $\mathbb{L}^3$ into a minimal surface equation in certain 3-manifolds endowed with a unit Killing vector field. We will use several results of Cheng, Yau and Treibergs for spacelike graphs with constant mean curvature in $\mathbb{L}^3$ to obtain sharp geometric estimates on the height, area and curvature of entire minimal graphs in Heisenberg space.

This talk is based on joint works with H. Lee [2], A. Lerma [3], and B. Nelli [4].

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Spacelike surfaces of constant Gaussian curvature in (2+1)-Minkowski

space

Abstract. We will discuss the problem of existence and uniqueness of spacelike surfaces of negative constant (or prescribed) Gaussian curvature K in $(2+1)$-dimensional Minkowski space. The simplest example, for $K = 1$, is the well-known embedding of hyperbolic plane as the one-sheeted hyperboloid; however, as a striking difference with the sphere in Euclidean space, in Minkowski space there are many non-equivalent isometric embeddings of the hyperbolic plane. This problem is related to solutions of the Monge-Ampère equation

$$detD^2u(z) = \frac{1}{|K|}(1-|z|^2)^{-2}$$

on the unit disc. We will prove the existence of surfaces with the condition $u = f$ on the boundary of the disc, for f a bounded lower semicontinuous function. If the curvature $K = K(z)$ depends smoothly on the point z, this gives a solution to the so-called Minkowski problem. On the other hand, we will prove that, for K constant, the principal curvatures of a K-surface are bounded if and only if the corresponding function f is in the Zygmund class.

Francisco TORRALBO
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A geometrical correspondence between maximal surfaces in anti-De Sitter space-time and minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Abstract. A geometrical correspondence between maximal surfaces in anti-De Sitter space-time and minimal surfaces in the Riemannian product of the hyperbolic plane and the real line is established. New examples of maximal surfaces in anti-De Sitter space-time are obtained in order to illustrate this correspondence.

This work is based on [1].

References

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Riemannian extensions and quasi-Einstein metrics

Abstract. A Walker manifold (M, g) is a pseudo-Riemannian manifold that admits a null parallel distribution. Riemannian extensions are Walker metrics realized on the cotangent bundle of an an affine manifold (Σ, D) and defined in terms of the torsion-free connection D. Different modifications of these metrics give rise to Walker manifolds with special curvature properties (see [3] and references therein).

A pseudo-Riemannian manifold (M, g) is called generalized quasi-Einstein if there is a solution $f \in C^\infty(M)$ of the equation

$$\mathrm{H}essf + \rho\mu df \otimes df = \lambda g \quad (1)$$

for some $\mu \in \mathbb{R}$ and $\lambda \in C^\infty(M)$ [1]. This definition includes as particular cases important geometric structures such as Einstein metrics, gradient Ricci solitons, gradient Ricci almost solitons and quasi-Einstein metrics [2, 4]. The aim of the talk is to present a classification of self dual generalized quasi-Einstein manifolds in dimension 4, showing the importance of Riemannian extensions when the level set hypersurfaces of the solution f of Equation (1) are degenerate [1].

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