Overdetermined problems, rigidity results and applications

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Question: *When is the tangential stress the same at each point of a cross section of the wall of the pipe?* $\Longrightarrow$ OVERDETERMINED ELLIPTIC PROBLEM

The same situation, but in capillarity

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We denote by $u = u(x, y)$ the height, with respect to the level of Ω, to which the liquid rises at coordinate (x, y). We have

$$\begin{cases} div \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} - \frac{\rho g}{\sigma} u & = & k & \text{in} & \Omega \\ \frac{\partial u}{\partial \nu} & = & \cos\alpha \sqrt{1+|\nabla u|^2} & \text{on} & \partial\Omega \end{cases}$$

where ρ is the density, g the gravity, σ the surface tension, α the contact angle between the liquid surface and the wall of the tube.

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$$\begin{cases} div \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} - \frac{\rho g}{\sigma} u & = & k & \text{in} & \mathbb{R}^2 \setminus \Omega \\ u & \geq & 0 & \text{in} & \mathbb{R}^2 \setminus \Omega \\ u & = & a > 0 & \text{on} & \partial\Omega \\ u & \to & 0 & \text{if} & |(x,y)| \to \infty \\ \frac{\partial u}{\partial \nu} & = & -\cot\alpha & \text{on} & \partial\Omega \end{cases}$$

A question raised by Berestycki-Caffarelli-Nirenberg

Problem: to classify domains $\Omega \in \mathbb{R}^n$ that support a positive solution of the overdetermined elliptic system

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Question (1997). Under the assumption that $\mathbb{R}^n \backslash \overline{\Omega}$ is connected and u is bounded, is it true that Ω must be a ball, or a half space, or a cylinder $\mathbb{R}^j \times B$ (where B is a ball) or the complement of one of these three domains?

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Farina-Valdinoci (*ARMA*, 2009)
Rigidity results for epigraphs in $\mathbb{R}^2$ for all functions f, and in $\mathbb{R}^3$ for some classes of functions f.

Constant mean curvature surfaces

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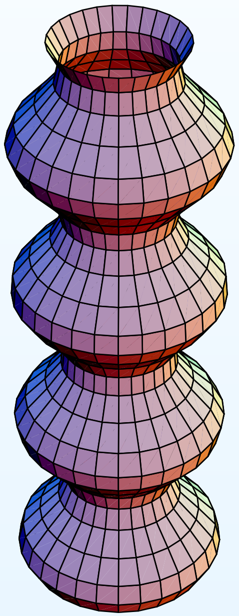
Theorem. In $\mathbb{R}^n$ the only enbedded compact mean curvature hypersurfaces are the spheres.

Onduloinds

In $\mathbb{R}^n$ there are exist noncompact surfaces with constant mean curvature. For example the Delaunay surfaces...



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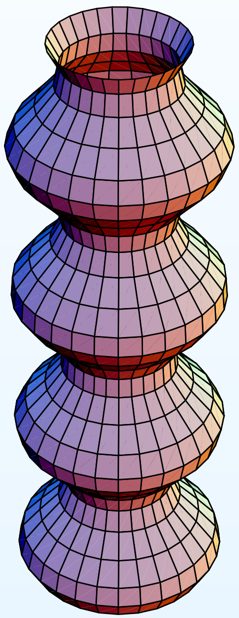

Theorem (S. 2010 $\&$ Schlenk-S. 2011): It is possible to build overdetermined solutions in domains that look like full onduloids in $\mathbb{R}^n$, $n \geq 2$, for the function $f(t) = \lambda\, t$.

A strong parallelism with minimal surfaces

Traizet *(GAFA, 2013)*

$\{$minimal bigraphs in $\mathbb{R}^3\}$ $\cup$ $\{$the half-plane$\}$ $\longleftrightarrow$ domains in $\mathbb{R}^2$ that support a positive solution to the problem

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Corollary. If $\partial\Omega$ is unbounded and connected then, Ω is a half-plane.

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Del Pino, Pacard, Wei *(DUKE, 2015)*

It is possible to find positive solutions to

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3. Perturbations of the De Giorgi-Bombieri-Giusti epigraph in $\mathbb{R}^9$.

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Natural questions for overdetermined problems:
1) "Is the half-space the only overdet. epigraph in $\mathbb{R}^n$ $(n \leq 8)$?"

2) "If Ω is diffeomorphic to a half-space and is overdet. in $\mathbb{R}^n$ $(n \leq 8)$, is it true that Ω is a half-space?"

Main theorem

Theorem [Ros-Ruiz-S.]

Let f be a locally Lipschitz function and $\Omega \subset \mathbb{R}^2$, be a domain that support a positive bounded solution of the overdetermined elliptic system

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = 1 & \text{on } \partial\Omega \end{cases}$$

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From now on, we take f, Ω and u satisfying the hypothesis of the theorem.

Starting point and steps of the proof

Regularity: Overdetermined domains are in fact of class $C^{2,\alpha}$ (Kinderlehrer-Nirenberg, Vogel).

Farina-Valdinoci (*ARMA*, 2009): If Ω is of class C^3, u is increasing in one variable and $|\nabla u|$ is bounded, then Ω is a half-plane and u is one-dimensional.

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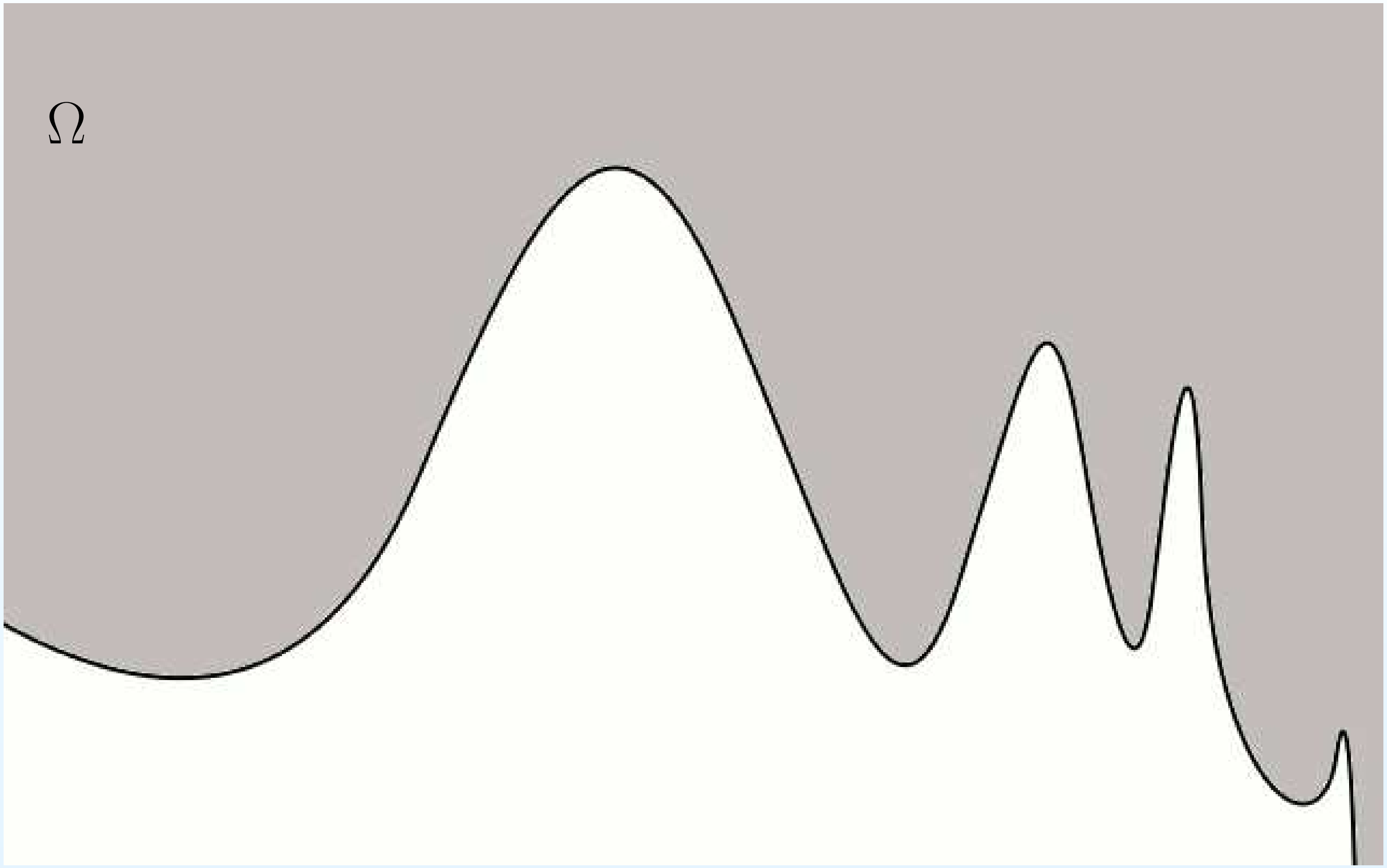
The steps of the proof of the main theorem will be the following:

1. We start by showing that $\|u\|_{C^{2,\alpha}}$ is bounded in $\overline{\Omega}$.
2. Then, we prove that either u is increasing in one variable or Ω contains an internally tangent half-plane.
3. To finish, we show that if $\|u\|_{C^{2,\alpha}}$ is bounded and Ω contains an internally tangent half-plane, then Ω is a half-plane.

Boundedness of the curvature (and then of $|\nabla u|$)



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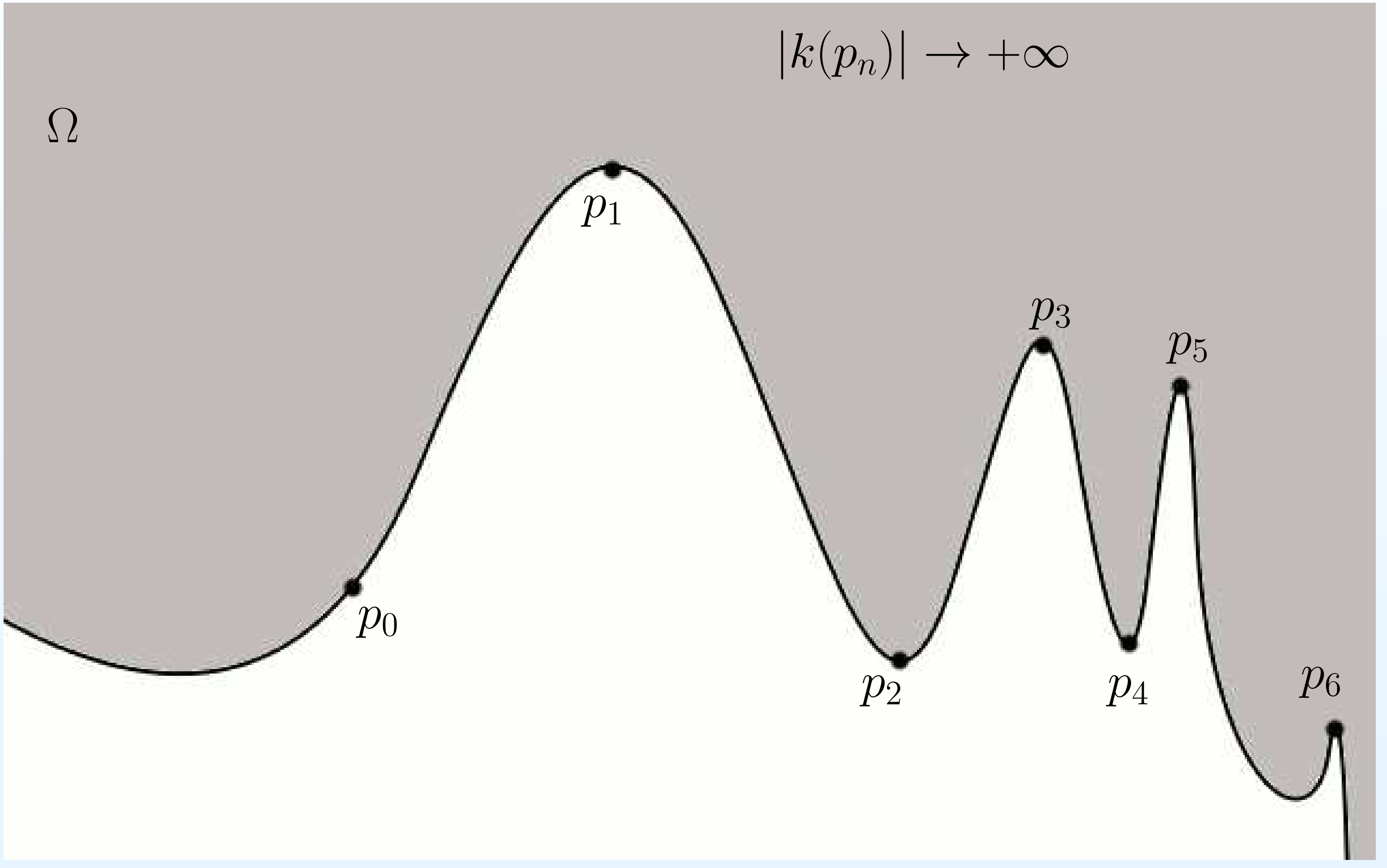

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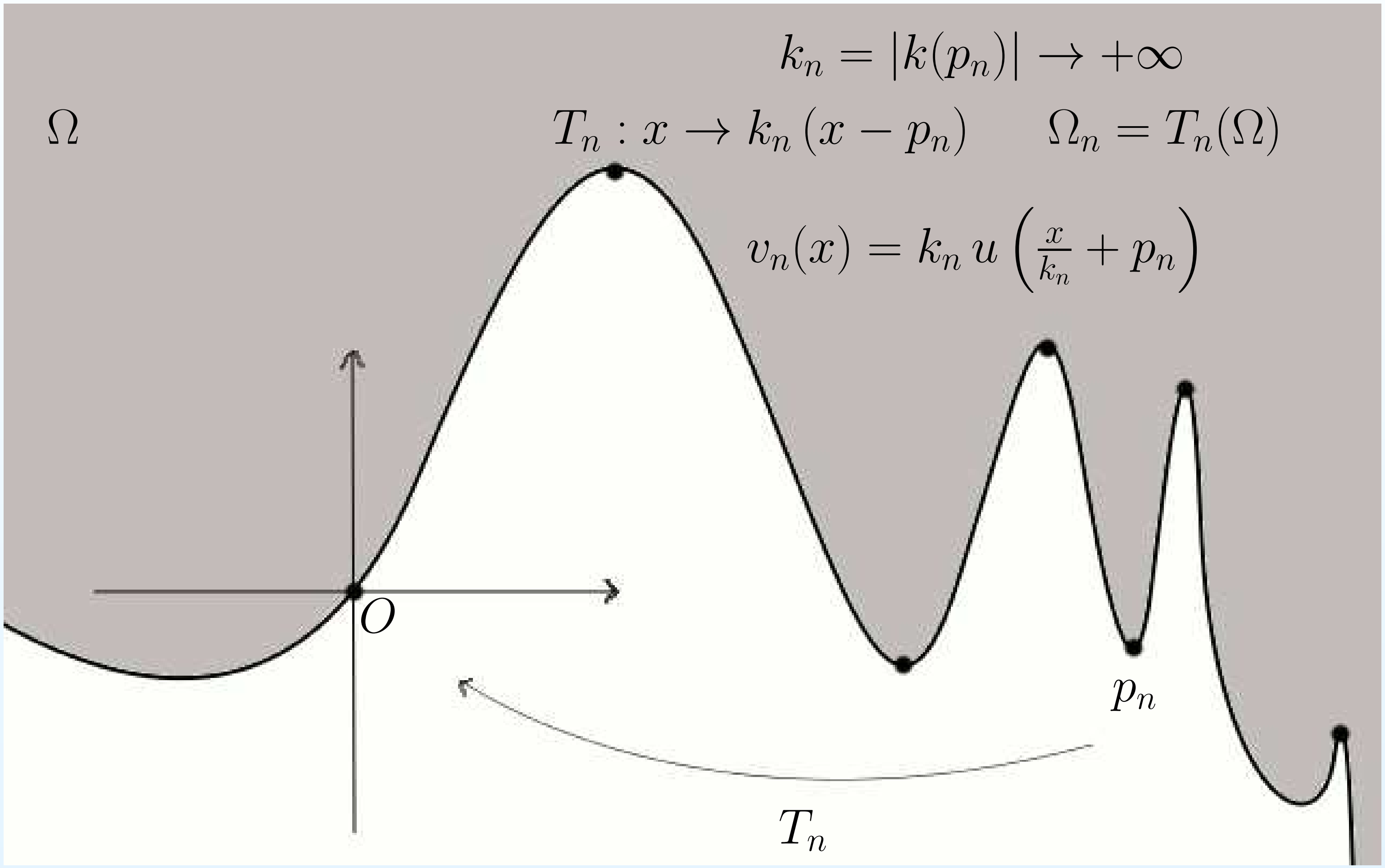

Convergence to an harmonic overdetermined domain Ω_∞ with $\partial\Omega_\infty$ connected and unbounded and $|k(O)| = 1$. $\longrightarrow$ Impossible

Limit directions

Definition. We say that $v \in \mathbb{S}^1$ is a limit direction (LD) for $\partial\Omega$ if there exists $p_n \in \partial\Omega$ such that $|p_n| \to +\infty$ and

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We can fix the coordinates of $\mathbb{R}^2$ in order that $O = (0,0) \in \partial\Omega$, $\partial\Omega$ is tangent to the x-axis in O, and the normal inward half-line at O (contained in Ω by the moving plane) is the positive part of the y-axis.

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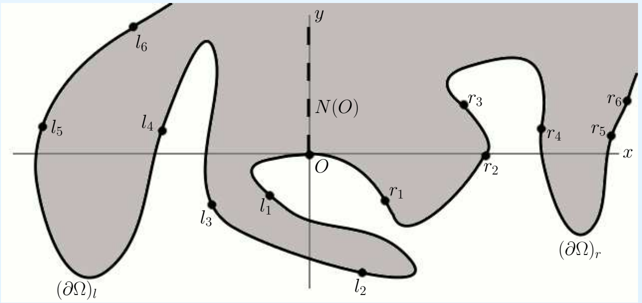

Angle between limit directions

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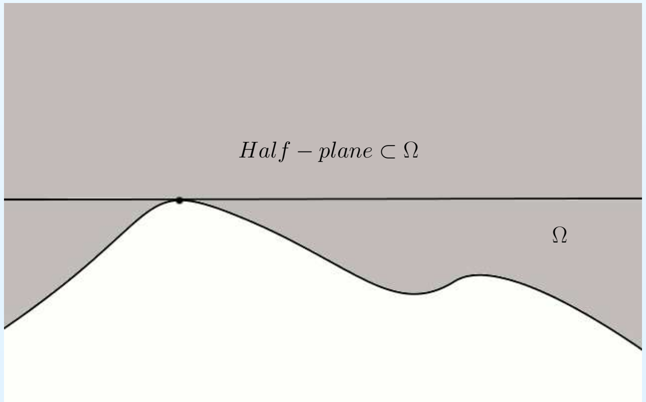
Remark : If the angle is bigger than π for any choice of v_l and v_r, then Ω contains an internally tangent half-plane.



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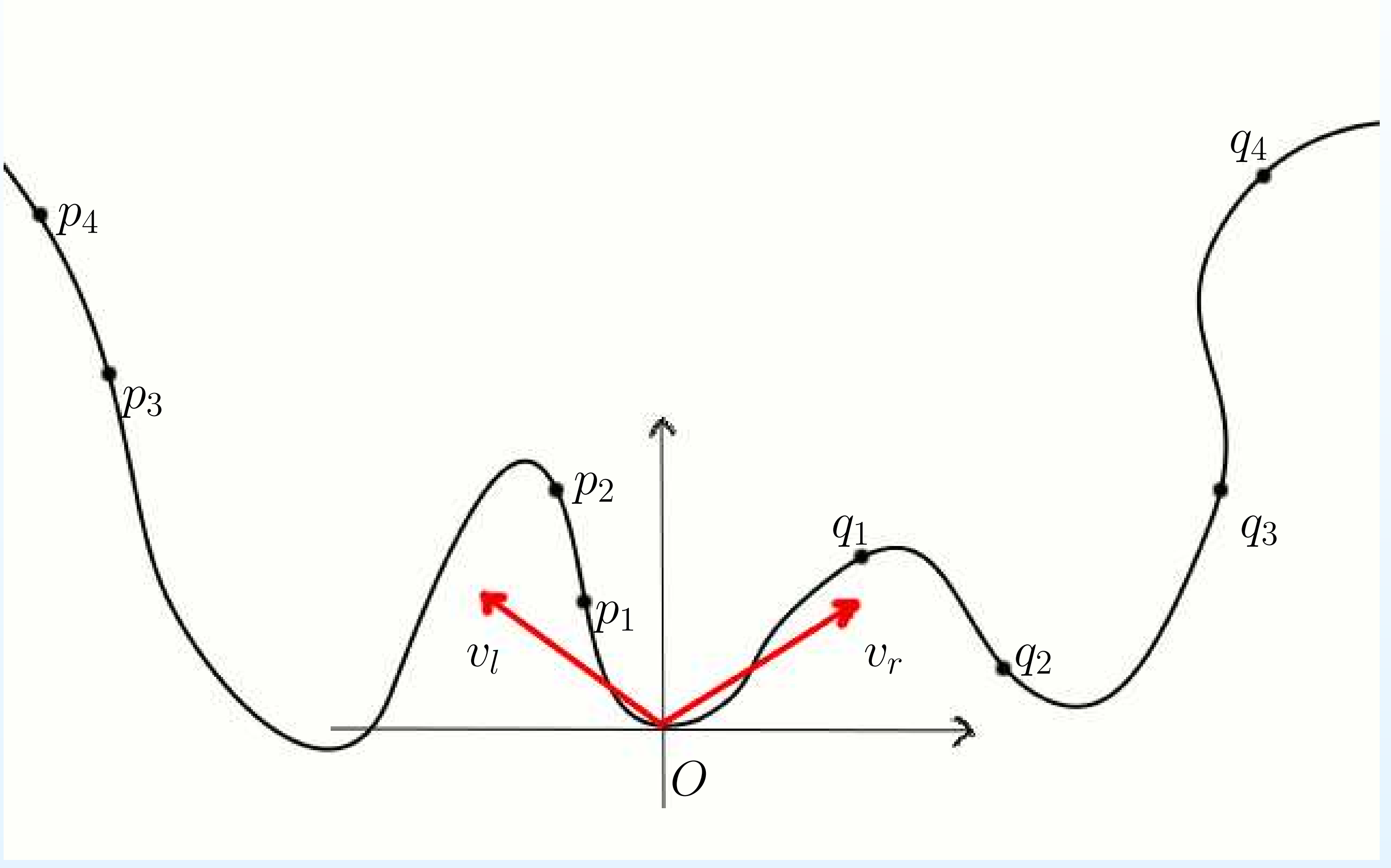

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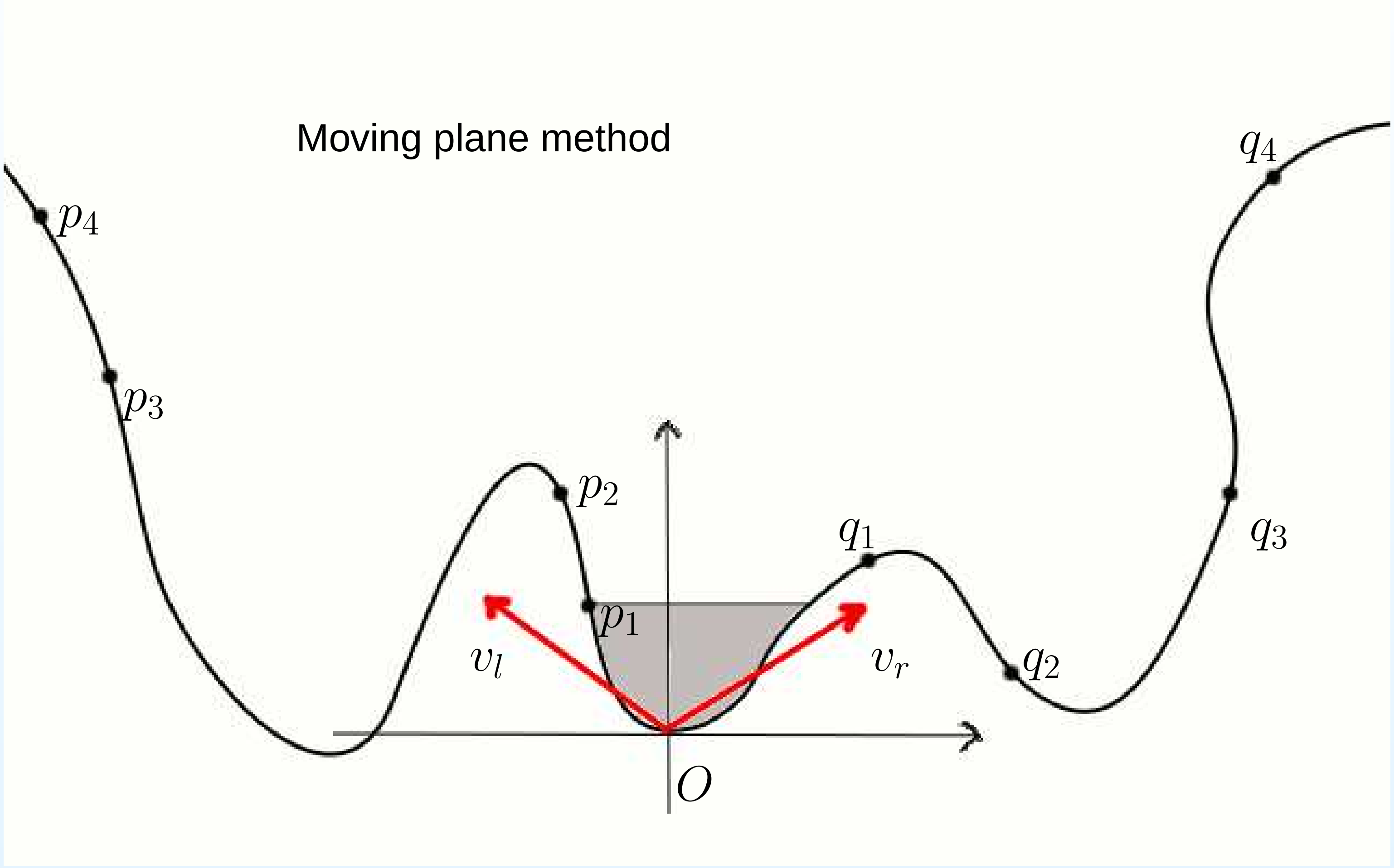

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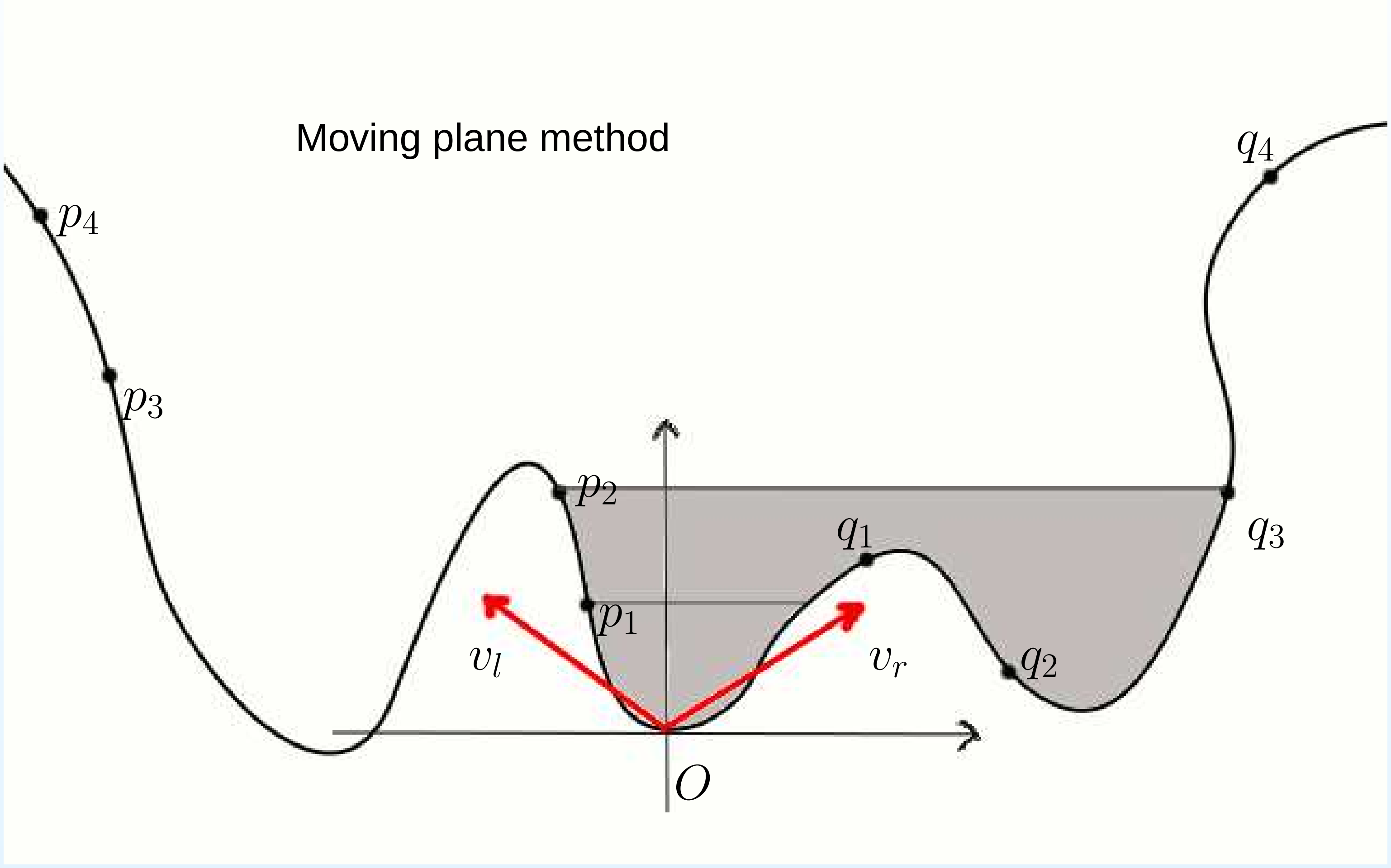

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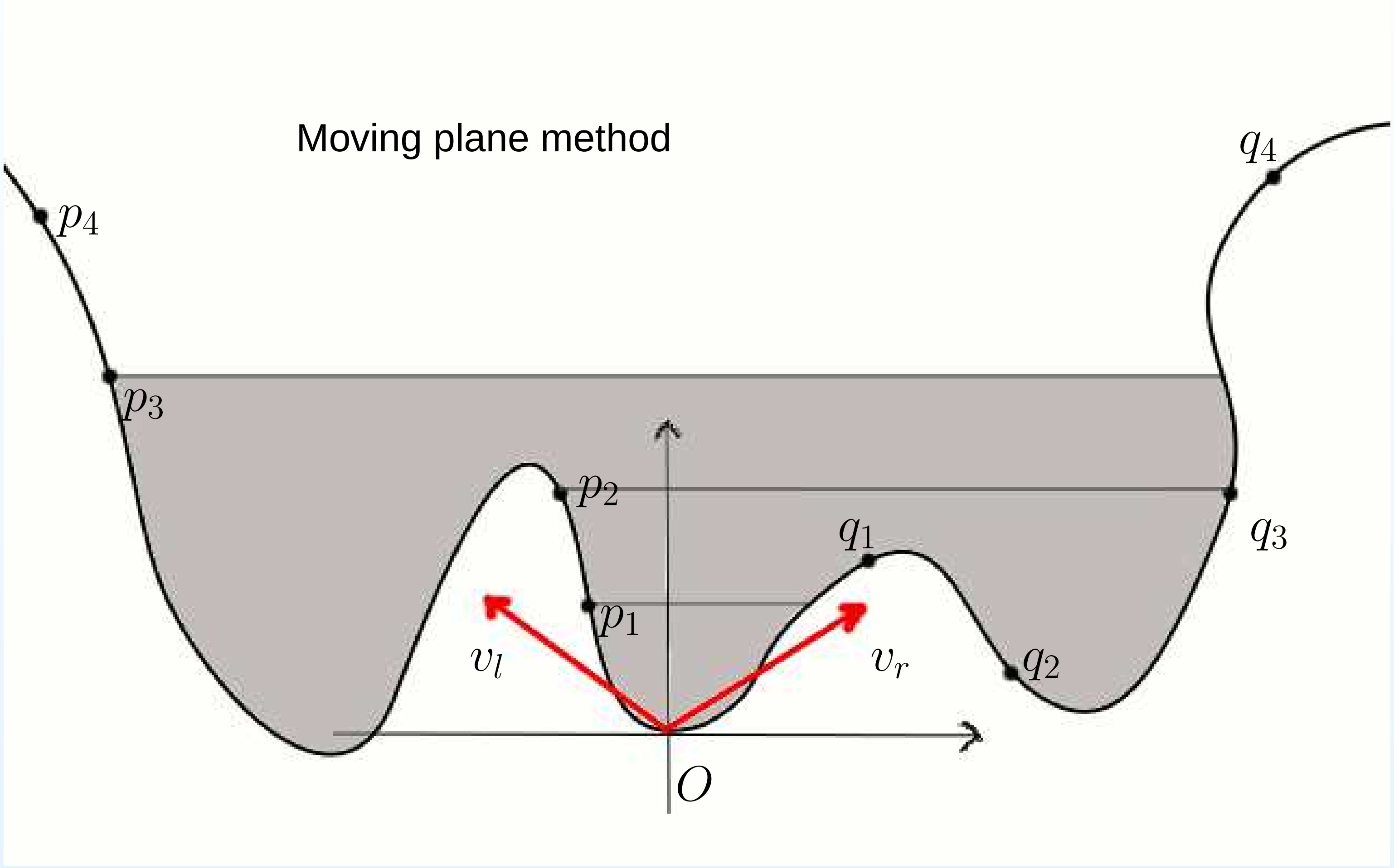
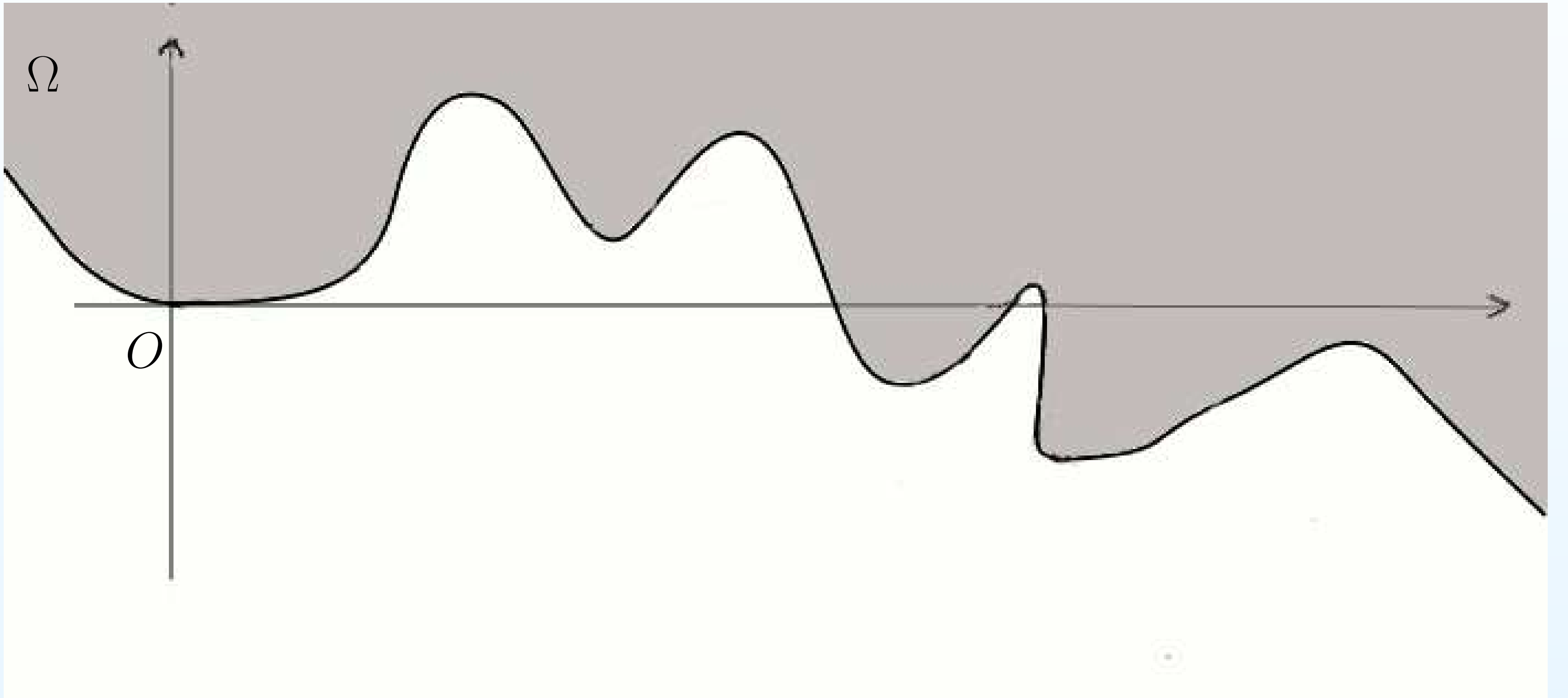


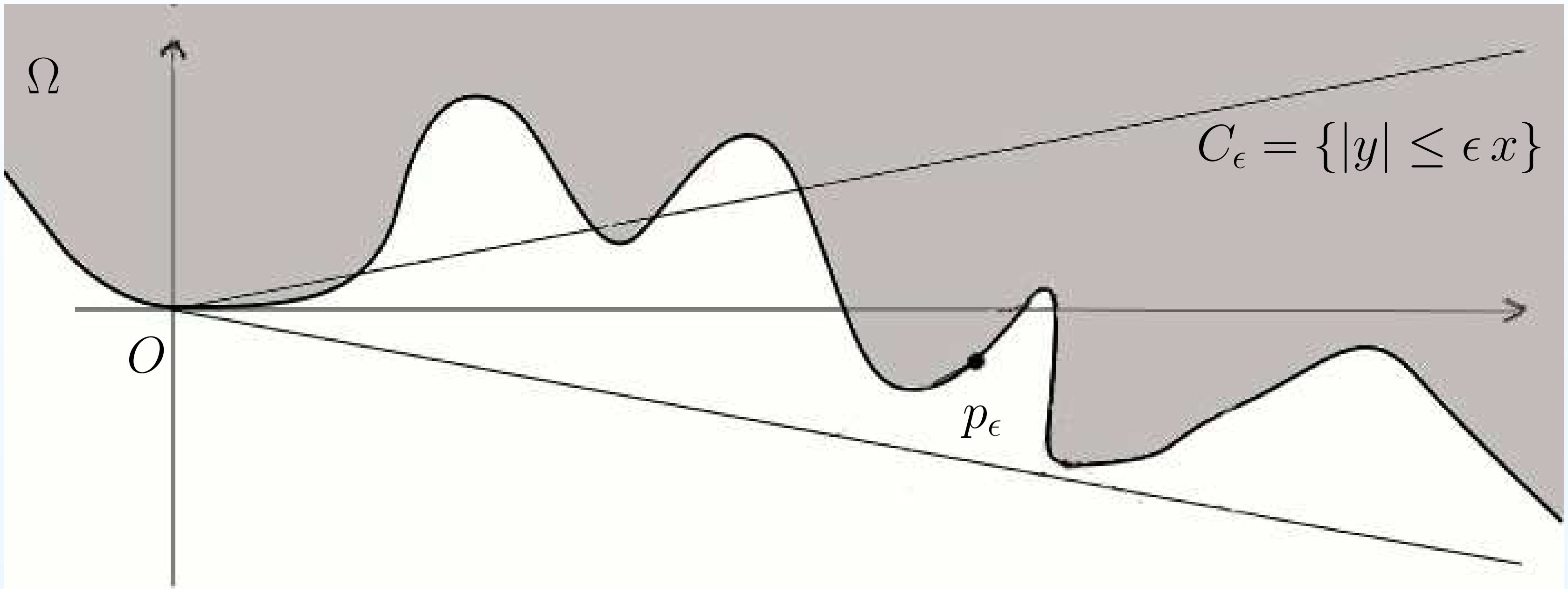
Conclusion : u is increasing in one variable. Since ∇u is bounded, Ω is a half-plane.

Following the boundary of the domain till ∞



$(1, 0)$ is the last limit direction at the right (possible thanks to a rotation).

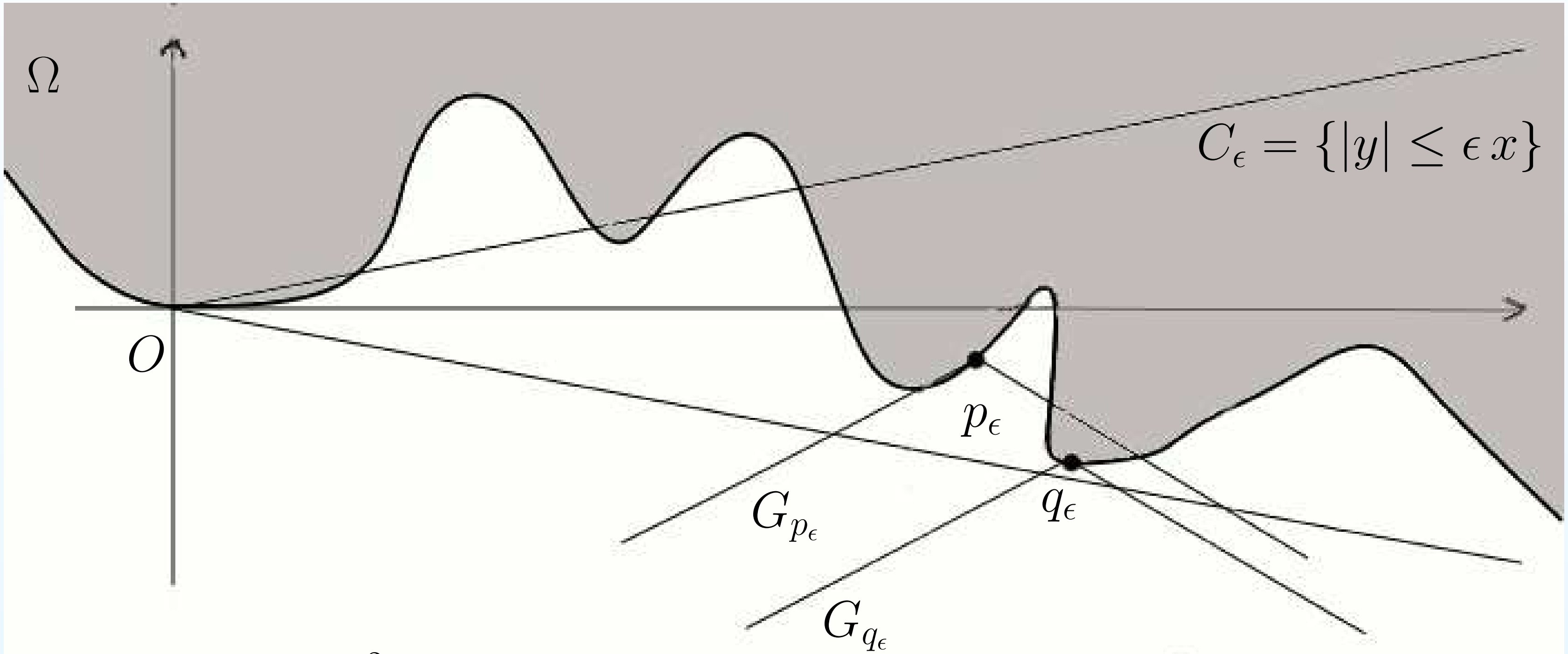
Following the boundary of the domain till ∞



$dist(p_\epsilon, O) \geq \epsilon^{-2}$

$(1, 0)$ is the last limit direction at the right (possible thanks to a rotation).

Following the boundary of the domain till ∞



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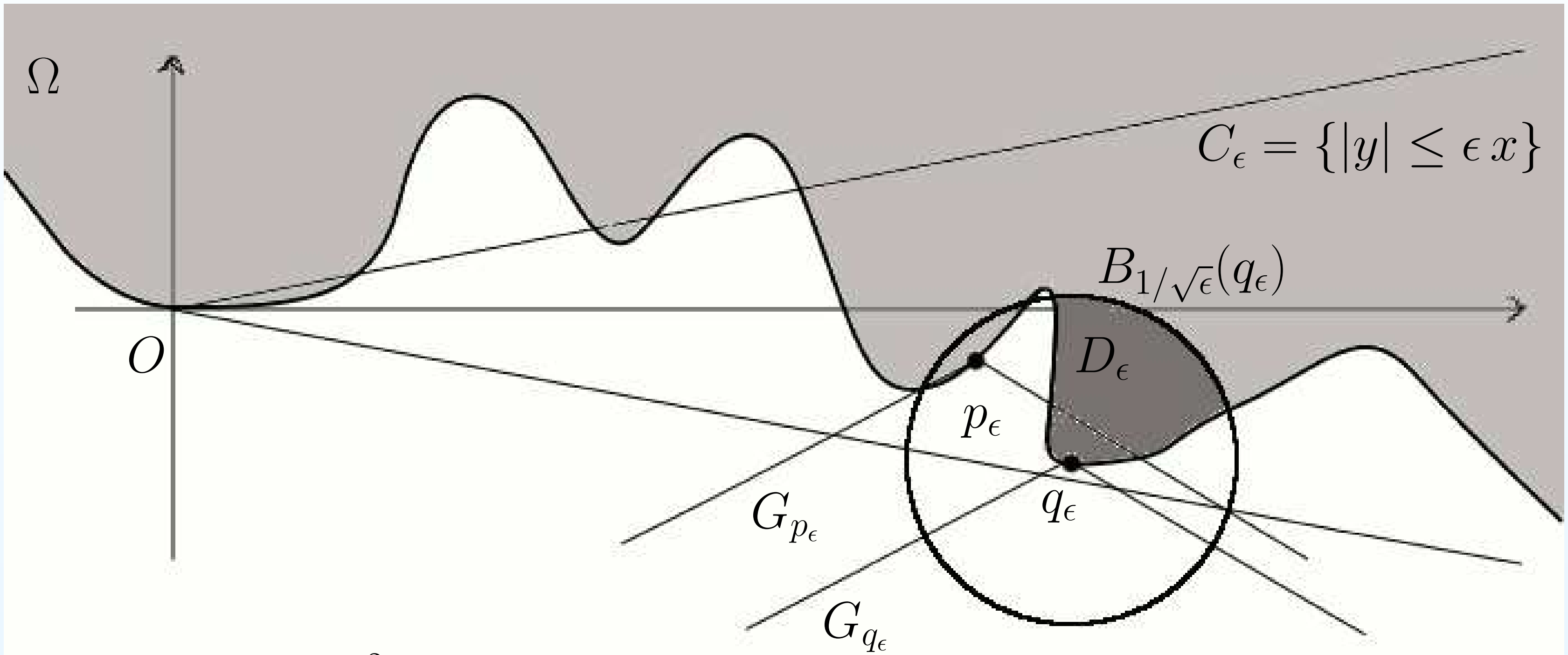
Following the boundary of the domain till ∞



$dist(p_\epsilon, O) \geq \epsilon^{-2}$

$(1, 0)$ is the last limit direction at the right (possible thanks to a rotation).

Move q_ϵ to the origin, and pass to the limit $\epsilon \to 0$. D_n converges to an overdet. domain Ω_∞ contained in a half plane. Then Ω_∞ is a half-plane. We built an overdet. half-plane starting from Ω.

Radial solutions converging to u_∞

Proposition. Assume that for $y \in [0, +\infty[$ we have

$$\begin{cases} \varphi''(y) + f(\varphi(y)) = 0 \\ \varphi(0) = 0,\ \varphi'(0) = 1, \lim_{t\to+\infty} \varphi(y) = L > 0. \end{cases}$$

Then, there exists $R_0 > 0$ such that for any $R > R_0$ the problem:

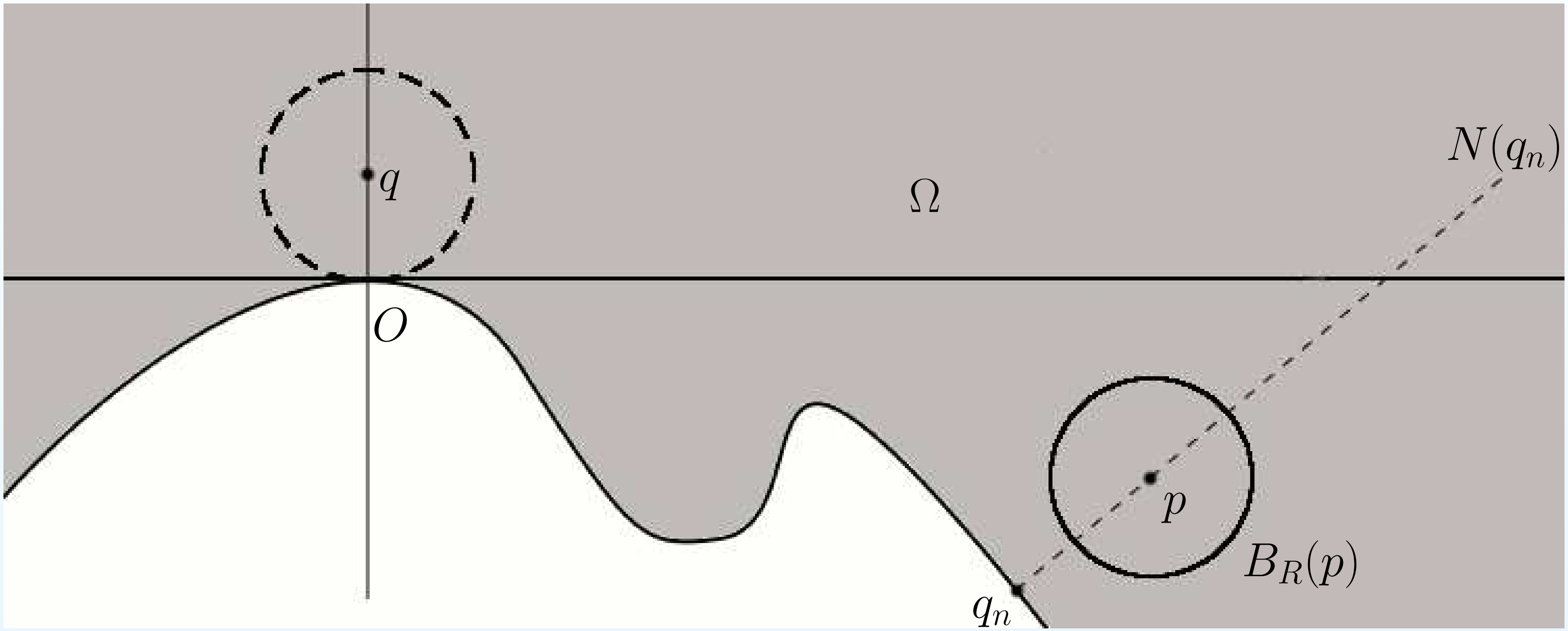
(1)
$$\begin{cases} \Delta u + f(u) = 0 & x \in B_R(O), \\ u = 0, & x \in \partial B_R(O) \end{cases}$$

has a positive radially symmetric solution u_R, and as $R \to +\infty$

i) $u_R < L$ and $\forall\, \rho \in (0,1)$, $u_R|_{B_{\rho R}(O)}$ converges unif. to L.

ii) The functions $v_R(z) = u_R(z - (0,R))$ converges to $u(x,y) = \varphi(y)$ locally in compact sets of $H = \{y > 0\}$.

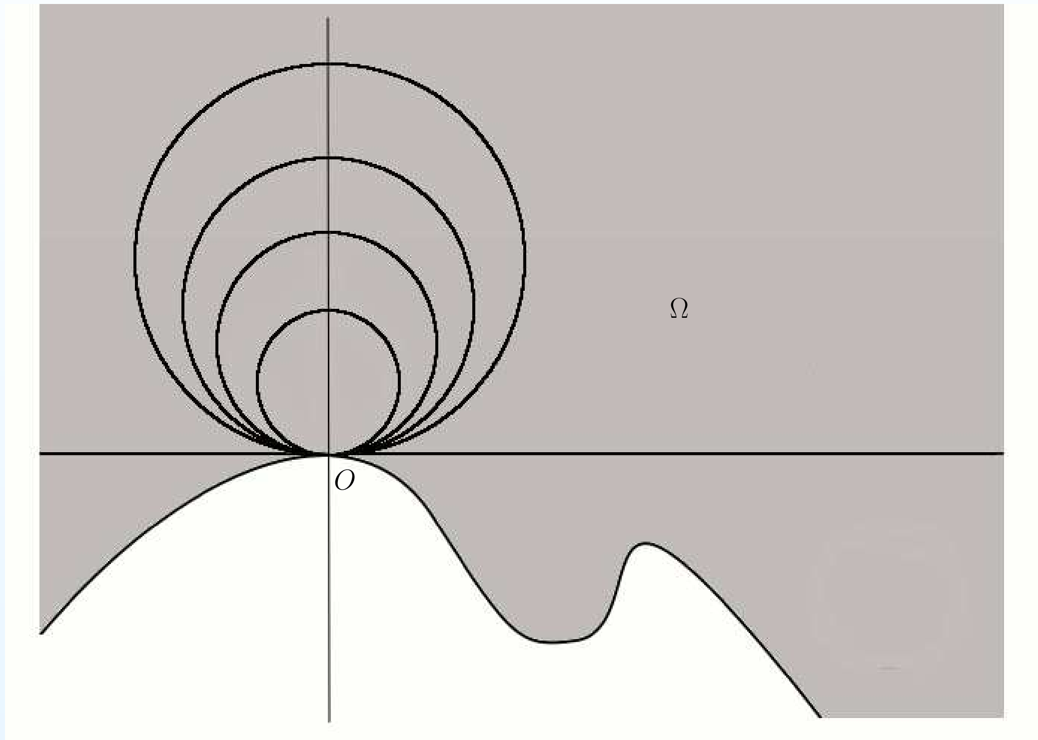
Moving a radial solution under the graph of u



The previous result allows us to put the graph of a radial solution in $B_R(p)$ below the graph of u.

Proposition. It is possible to move this graph without touching the graph of u till we reach the position of the ball $B_R(q)$.

Comparison



We take $R \to +\infty$, and we obtain that the graph of the overdetermined solution in the half-space is below the graph of u.

The maximum principle says us that Ω is a half-plane.