Characterizations of a Clifford hypersurface in a unit sphere

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Theorem (Brendle 2013)

The only embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus.

Theorem (Andrews-Li 2015)

Every embedded CMC torus in $\mathbb{S}^3$ is rotationally symmetric.

In fact, Andrews-Li gave a complete classification of embedded constant mean curvature tori in $\mathbb{S}^3$.

Remark

For higher-dimensional analogues, one possible approach is to characterize a Clifford hypersurface among embedded constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$. Unfortunately, even when $H = 0$, it is well-known that there exist infinitely many mutually noncongruent embedded minimal hypersurfaces in $\mathbb{S}^{n+1}$ which are homeomorphic to the Clifford hypersurface due to Hsiang.

In view of this observation, we restrict ourselves to consider compact embedded constant mean curvature hypersurfaces in a unit sphere with two distinct principal curvatures.

Theorem (Andrews-Huang-Li 2015)

Let Σ be a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ, whose multiplicities are m and $n-m$ respectively. If $\lambda + \alpha\mu = 0$ for some positive constant, Σ is congruent to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{1}{\alpha+1}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{\alpha}{1+\alpha}}\right)$.

m-th order mean curvature

The m-th order mean curvature H_m of an n-dimensional hypersurface $M \subset \mathbb{S}^{n+1}$ is defined by the elementary symmetric polynomial of degree m in the principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$ on M as follows:

$$\binom{n}{m} H_m = \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} \lambda_{i_1} \ldots \lambda_{i_m}.$$

Clifford hypersurface

If an n-dimensional Clifford hypersurface in $\mathbb{S}^{n+1}$ has two distinct principal curvatures λ and μ of multiplicities $n-k$ and k, respectively, then it is given by

$$\mathbb{S}^{n-k}\left(\frac{1}{\sqrt{1+\lambda^2}}\right)\times\mathbb{S}^{k}\left(\frac{1}{\sqrt{1+\mu^2}}\right)$$

with $\lambda\mu+1=0$, that is,

$$\mathbb{S}^{n-k}\left(\frac{1}{\sqrt{1+\lambda^2}}\right)\times\mathbb{S}^{k}\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right),$$

where λ satisfies the following identity:

$$\binom{n}{m}H_m=\binom{n-k}{m}\lambda^m+\binom{n-k}{m-1}\binom{k}{1}\lambda^{m-1}\mu+\cdots+\binom{n-k}{1}\binom{k}{m-1}\lambda\mu^{m-1}+\binom{k}{m}\mu^m$$

with $\lambda\mu+1=0$.

Clifford hypersurface

In particular, if one of the principal curvatures is simple, say $k = 1$, then

$$\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right),$$

where λ satisfies the following identity:

$$(n-m)\lambda^m - m\lambda^{m-2} = nH_m.$$

Moreover, if $k = 1$ and $H_m = 0$, then $\lambda = \pm\sqrt{\frac{m}{n-m}}$ and $\mu = \mp\sqrt{\frac{n-m}{m}}$.
Thus a Clifford hypersurface is given by

$$\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right).$$

Introduction

Theorem (Otsuki 1970)

Let M be a minimal hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ.
(I) If the multiplicities of λ and μ are at least 2, then M is locally congruent to a Clifford minimal hypersurface.
(II) If one of λ and μ is simple, then there are infinitely many minimal hypersurfaces other than Clifford minimal hypersurfaces.

Remark

If the multiplicities of two distinct principal curvatures are at least 2, then a compact hypersurface with constant m-th order mean curvature is congruent to a Clifford hypersurface (B.Y. Wu 2009). Thus it suffices to consider the case where one of the two distinct principal curvatures is simple.

Remark

Let Σ be a hypersurface in a space form with two distinct principal curvatures, one of them being simple. Then Σ is a part of rotationally symmetric hypersurface. [Do Carmo-Dajczer 1983]

Consider constant mean curvature hypersurfaces with two distinct principal curvatures λ and μ, μ being simple.

The existence of compact embedded constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ other than the totally geodesic n-spheres and Clifford hypersurfaces was obtained by [Ripoll 1986, Brito-Leite 1990, Wei-Cheng-Li 2010].

Theorem (Perdomo 2010)

For any integer $m \geqslant 2$ and H between $\cot \frac{\pi}{m}$ and $\frac{(m^2-2)\sqrt{n-1}}{n\sqrt{m^2-1}}$, there exists a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature H other than the totally geodesic n-spheres and Clifford hypersurfaces.

Remark

In his construction, λ and μ satisfy that $\lambda > \mu$.

Theorem (Min-S 2015)

Let Σ be an $n(\geqslant 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature $H \geqslant 0$ and with two distinct principal curvatures λ and μ, μ being simple. If $\mu > \lambda$, then Σ is congruent to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where $\lambda = \frac{nH - \sqrt{n^2H^2+4(n-1)}}{2(n-1)}$.

Remark

Constant mean curvature tori in $\mathbb{S}^3$ automatically satisfy the condition that $\mu > \lambda$.

- $F : \Sigma^n \to \mathbb{S}^{n+1}(\subset \mathbb{R}^{n+2})$, an immersion of a compact embedded constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, one of them being simple
- $\nu(x)$, the unit normal vector at $x \in \Sigma$ in $\mathbb{S}^{n+1}$
- h and A, the second fundamental form and the shape operator of Σ, respectively
- The *normalized mean curvature* H is given by

$$H = \frac{1}{n}\mathrm{tr}(h) = \frac{1}{n}\sum_{i=1}^{n} \lambda_i = \frac{1}{n}[(n-1)\lambda + \mu].$$

Since Σ is a compact embedded hypersurface, Σ divides $\mathbb{S}^{n+1}$ into two connected components. Because the mean curvature of $F(\Sigma)$ in $\mathbb{S}^{n+1}$ is constant, we may assume that $H \geqslant 0$ by choosing the suitable orientation of Σ. Let R be the region satisfying that ν points out of R. The *mean curvature vector* $\vec{H}$ satisfies that $\vec{H} = -nH\nu(x)$.

For a positive function Ψ on Σ, we denote by $B_T(x, \frac{1}{\Psi(x)})$ a geodesic ball with radius $\frac{1}{\Psi(x)}$ which touches Σ at $F(x)$ inside the region R in $\mathbb{S}^{n+1}$. Note that our notation $B_T(x, r)$ is different from a geodesic ball $B_r(x)$ centered at x with radius $r > 0$. Then $B_T(x, \frac{1}{\Psi(x)})$ is given by the intersection of $\mathbb{S}^{n+1}$ and a ball of radius $\frac{1}{\Psi(x)}$ centered at $p(x) = F(x) - \frac{1}{\Psi(x)}\nu(x)$ in $\mathbb{R}^{n+2}$. Define the two-point function $Z : \Sigma \times \Sigma \to \mathbb{R}$ by

$$Z(x, y) := \Psi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle . \qquad (1)$$

Then for any $y \in \Sigma$,

$$\begin{cases} Z(x, y) > 0 & \text{if} \quad F(y) \in \mathrm{int} B_T(x, \frac{1}{\Psi(x)}), \\ Z(x, y) = 0 & \text{if} \quad F(y) \in \partial B_T(x, \frac{1}{\Psi(x)}), \\ Z(x, y) < 0 & \text{if} \quad F(y) \notin B_T(x, \frac{1}{\Psi(x)}), \end{cases}$$

since

$$\frac{2}{\Psi(x)} Z(x, y) = |F(y) - p(x)|^2 - \left(\frac{1}{\Psi(x)}\right)^2 .$$

Definition (Andrews-Langford-McCoy 2013)

The *interior ball curvature* k is a positive function on Σ defined by

$$k(x) := \inf\left\{\frac{1}{r} : B_T(x,r) \cap \Sigma = \{x\},\ \ r > 0\right\}.$$

Remark

Because Σ is compact and embedded in $\mathbb{S}^{n+1}$, the function k is a well-defined positive function on Σ.
From the definition of $k(x)$ for every point $x \in \Sigma$, it follows that

$$k(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geqslant 0$$

for all $y \in \Sigma$.

Let $\Phi(x) := \max\{\lambda(x), \mu(x)\}$ be the maximum value of the principal curvatures of Σ in $\mathbb{S}^{n+1}$ at $F(x)$.
Note that the two distinct principal curvature condition guarantees that Σ has no umbilic point and hence $\Phi(x) - H > 0$.

Definition (Brendle 2013, Andrews-Li 2015)

$$\kappa := \sup_{x\in\Sigma} \frac{k(x) - H}{\Phi(x) - H}.$$

For convenience, we will write $\varphi(x) := \Phi(x) - H$.

Remark

Since $k(x) \geqslant \Phi(x)$ and Σ is compact, there exists a constant $K > 0$ satisfying

$$1 \leqslant \kappa < K.$$

Define a positive function

$$\Psi(x) := \kappa\varphi(x) + H = \kappa(\Phi(x) - H) + H$$

on Σ. Then $\Psi(x) \geqslant k(x)$. It follows that

$$Z(x,y) = \Psi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geqslant 0$$

for all $(x,y) \in \Sigma \times \Sigma$.

Remark

- If there exists a point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ satisfying that $Z(\overline{x}, \overline{y}) = 0$, then

  $$\frac{\partial Z}{\partial x_i}(\overline{x}, \overline{y}) = \frac{\partial Z}{\partial y_i}(\overline{x}, \overline{y}) = 0,$$

  since the function Z attains its global minimum at $(\overline{x}, \overline{y})$.
- The global minimum of the function Z is attained at $(x,x) \in \Sigma \times \Sigma$ for all $x \in \Sigma$.

First and second order derivatives of $Z(x, y)$

Consider a pair of points $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ such that $Z(\overline{x}, \overline{y}) = 0$. Then

$$\frac{\partial Z}{\partial x_i}(\overline{x}, \overline{y}) = \frac{\partial Z}{\partial y_i}(\overline{x}, \overline{y}) = 0.$$

Choose geodesic normal coordinates $(x_1, \ldots, x_n)$ at $\overline{x}$ in Σ satisfying that

$$h_{ij} = \lambda_i \delta_{ij}$$

with $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$ and geodesic normal coordinates $(y_1, \ldots, y_n)$ at $\overline{y}$ in Σ.

First and second order derivatives of $Z(x, y)$

$$0 = Z(\overline{x}, \overline{y}) = \Psi(\overline{x})(1 - \langle F(\overline{x}), F(\overline{y})\rangle) + \langle \nu(\overline{x}), F(\overline{y})\rangle$$

$$0 = \frac{\partial Z}{\partial x_i}(\overline{x}, \overline{y}) = \frac{\partial \Psi(\overline{x})}{\partial x_i}\left(1 - \langle F(\overline{x}), F(\overline{y})\rangle - \Psi(\overline{x})\left\langle \frac{\partial F(\overline{x})}{\partial x_i}, F(\overline{y})\right\rangle\right.$$
$$+ \sum_{k=1}^{n} {h_i}^k(\overline{x})\left\langle \frac{\partial F(\overline{x})}{\partial x_k}, F(\overline{y})\right\rangle,$$

$$0 = \frac{\partial Z}{\partial y_i}(\overline{x}, \overline{y}) = -\Psi(\overline{x})\left\langle F(\overline{x}), \frac{\partial F(\overline{y})}{\partial x_i}\right\rangle + \left\langle \nu(\overline{x}), \frac{\partial F(\overline{y})}{\partial x_i}\right\rangle.$$

First and second order derivatives of $Z(x, y)$

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2} \\
&= (1 - \langle F(\overline{x}), F(\overline{y}) \rangle) \\
&\times \left(\Delta_\Sigma \Psi(\overline{x}) - 2 \sum_{i=1}^{n} \frac{|\frac{\partial \Psi(\overline{x})}{\partial x_i}|^2}{\Psi(\overline{x}) - \lambda_i(\overline{x})} + \left(|A(\overline{x})|^2 - n\right) \Psi(\overline{x}) - nH\Psi(\overline{x})^2 + nH \right) \\
&\leqslant (1 - \langle F(\overline{x}), F(\overline{y}) \rangle) \\
&(1 - \langle F(\overline{x}), F(\overline{y}) \rangle) \\
&\times \left(\Delta_\Sigma \Psi(\overline{x}) - \frac{2}{n} \frac{|\nabla^\Sigma \Psi(\overline{x})|^2}{\Psi(\overline{x}) - H} + \left(|A(\overline{x})|^2 - n\right) \Psi(\overline{x}) - nH\Psi(\overline{x})^2 + nH \right),
\end{aligned}
$$

since $\Phi(x) \leqslant k(x) \leqslant \Psi(x)$, for $1 \leqslant i \leqslant n$

$$
\Psi(x) - \lambda_i = \Psi(x) - (nH - \sum_{j \neq i} \lambda_j) = \Psi(x) + \sum_{j \neq i} \lambda_j - nH \leqslant n(\Psi(x) - H).
$$

Simons-type identity

Proposition

Let Σ be a constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ, μ being simple. Then $|\mathring{A}|$ is strictly positive and

$$|\nabla^\Sigma \mathring{A}|^2 = \frac{n+2}{n}|\nabla^\Sigma |\mathring{A}||^2.$$

Remark

It is well-known that a constant mean curvature hypersurface Σ in space forms satisfies

$$|\nabla^\Sigma \mathring{A}|^2 - |\nabla^\Sigma |\mathring{A}||^2 \geqslant \frac{2}{n}|\nabla^\Sigma |\mathring{A}||^2,$$

which is so-called Kato's inequality. It would be interesting to characterize the equality case. This proposition gives a sufficient condition for Kato's inequality to attain the equality.

Simons-type identity

Applying the above Proposition to the function $\varphi = \Phi - H$, where Φ is the maximum value of the principal curvatures, we get the following:

Lemma

Let Σ be a constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ, μ being simple. Then

$$\Delta_\Sigma \varphi - \frac{2}{n}\frac{|\nabla^\Sigma \varphi|^2}{\varphi} + (|A|^2 - n)\varphi - 2nH^2\varphi + nf(n)H\varphi^2 = 0,$$

where the function $f(n)$ is defined by

$$f(n) := \begin{cases} \frac{n-2}{n-1} & \text{if} \quad \Phi = \mu, \\ n-2 & \text{if} \quad \Phi = \lambda. \end{cases}$$

In general, $\Phi(x) \leqslant k(x)$ for every $x \in \Sigma$.

Proposition

Let Σ be an $n(\geqslant 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature H with two distinct principal curvatures, one of them being simple. If $H > 0$. Then

$$k(x) = \Phi(x)$$

for all $x \in \Sigma$.

Proof

Suppose that $\kappa > 1$. Then there exists a point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ with $\overline{x} \neq \overline{y}$ satisfying that $Z(\overline{x}, \overline{y}) = 0$.

$$\begin{aligned}
&\frac{1}{(1 - \langle F(\overline{x}), F(\overline{y}) \rangle)} \left(\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2} \right) \\
&\leqslant \kappa \Delta_\Sigma \varphi(\overline{x}) - \frac{2\kappa}{n} \frac{|\nabla^\Sigma \varphi(\overline{x})|^2}{\varphi(\overline{x})} \\
&+ \left(|A(\overline{x})|^2 - n \right) (\kappa \varphi(\overline{x}) + H) - nH(\kappa \varphi(\overline{x}) + H)^2 + nH \\
&= H|A(\overline{x})|^2 - \kappa^2 nH \varphi(\overline{x})^2 - nH^3 - \kappa n f(n) H \varphi(\overline{x})^2,
\end{aligned}$$

where $f(n) = \frac{n-2}{n-1}$ if $\Phi = \mu$, and $f(n) = n - 2$ if $\Phi = \lambda$.

Proof

Note that

$$|\mathring{A}|^2 = |A|^2 - nH^2 = ng(n)\varphi^2,$$

$$g(n) = \begin{cases} \frac{1}{n-1} & \text{if} \quad \Phi = \mu, \\ n-1 & \text{if} \quad \Phi = \lambda. \end{cases}$$

Then

$$\begin{aligned} &\frac{1}{(1 - \langle F(\overline{x}), F(\overline{y}) \rangle)} \left(\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2} \right) \\ &\leqslant -nH\varphi(\overline{x})^2(\kappa^2 + f(n)\kappa - g(n)) \\ &< -nH\varphi(\overline{x})^2(1 + f(n) - g(n)) \\ &\leqslant 0 \end{aligned}$$

where

$$1 + f(n) - g(n) = \begin{cases} \frac{2(n-2)}{n-1} & \text{if} \quad \Phi = \mu, \\ 0 & \text{if} \quad \Phi = \lambda. \end{cases}$$

Proof

However, since the point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma \setminus D$ is a global minimum point of the function Z, we see

$$0 \leqslant \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2},$$

which is a contradiction. It follows that

$$k(x) = \Phi(x) = \Psi(x)$$

for all $x \in \Sigma$. □

Theorem (Min-S)

Let Σ be an $n(\geqslant 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature $H \geqslant 0$ and with two distinct principal curvatures λ and μ, μ being simple. If $\mu > \lambda$, then Σ is congruent to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where $\lambda = \frac{nH-\sqrt{n^2H^2+4(n-1)}}{2(n-1)}$.

Proof

- If $H = 0$, then Σ is congruent to a Clifford minimal hypersurfaces by the work due to Otsuki.
- It suffices to consider the case $H > 0$. Since $\mu > \lambda$, we have $\Phi = \mu$. By the previous proposition,

$$\Phi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geqslant 0,$$

for all $x, y \in \Sigma$.

Proof

Fix $x \in \Sigma$ and choose an orthonormal frame $\{e_1, \ldots, e_n\}$ in a neighborhood of x such that $h(e_n, e_n) = \Phi$. Let $\gamma(t)$ be a geodesic on Σ such that $\gamma(0) = F(x)$ and $\gamma'(0) = e_n$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) := Z(F(x), \gamma(t)) = \Phi(x)(1 - \langle F(x), \gamma(t)\rangle) + \langle \nu(x), \gamma(t)\rangle .$$

Then, by definition, $f(t) \geqslant 0$ and $f(0) = 0$. A simple computation shows

$$f'(t) = -\langle \Phi(x)F(x) - \nu(x), \gamma'(t)\rangle ,$$

$$f''(t) = \langle \Phi(x)F(x) - \nu(x), \gamma(t) + h(\gamma'(t), \gamma'(t))\nu(\gamma(t))\rangle ,$$

$$\begin{aligned} f'''(t) = &\left\langle \Phi(x)F(x) - \nu(x), \gamma'(t) + (\nabla^{\Sigma}_{\gamma'(t)} h)(\gamma'(t), \gamma'(t))\nu(\gamma(t))\right\rangle \\ &+ \left\langle \Phi(x)F(x) - \nu(x), h(\gamma'(t), \gamma'(t))\nabla_{\gamma'(t)}\nu(\gamma(t))\right\rangle , \end{aligned}$$

where ∇ is the covariant derivative of $\mathbb{R}^{n+2}$.

Proof

In particular, it follows that

$$f(0) = f'(0) = 0,$$
$$f''(0) = \langle \Phi(x)F(x) - \nu(x), F(x) + \Phi(x)\nu(x) \rangle = 0.$$

Moreover the fact that $f \geqslant 0$ implies that $f'''(0) = 0$. Hence

$$0 = f'''(0) = \langle \Phi(x)F(x) - \nu(x), e_n + h_{nnn}(x)\nu(x) \rangle = -h_{nnn}(x).$$

Therefore we get $e_n\lambda = h_{11n} = -\frac{1}{n-1}h_{nnn} = 0$, which implies that λ and μ are constant on Σ by Ostuki. Thus Σ is an isoparametric hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. From the classification of isoparametric hypersurfaces with two principal curvatures due to Cartan, it follows that Σ is congruent to the Riemannian product $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{1}{\sqrt{1+\mu^2}}\right)$, where λ and μ satisfy $nH = (n-1)\lambda + \mu$. $\square$

Introduction

Theorem (Simons 1968)

Let M be a compact minimal hypersurface in $\mathbb{S}^{n+1}$. Then we have

$$\int_M |A|^2 \left(|A|^2 - n\right) \geqslant 0,$$

where $|A|^2$ denotes the squared norm of the second fundamental form on M.

Corollary

Such M is either totally geodesic, or $|A|^2 \equiv n$, or $|A|^2(x) > n$ at some point $x \in M$.

Theorem (Chern-do Carmo-Kobayashi 1968, Lawson 1969)

For $n \geqslant 3$, if $|A|^2 \equiv n$ on M, then M is isometric to a Clifford minimal hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-1}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{1}{n}}\right)$.

Introduction

Theorem (Wang 2003, Perdomo 2004)

Let M be an $n(\geqslant 3)$-dimensional closed minimal hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, one of them being simple. Then

$$\int_M |A|^2 \leqslant n\mathrm{Vol}(M),$$

where $\mathrm{Vol}(M)$ denotes the volume of M. Moreover, equality holds if and only if M is isometric to a Clifford minimal hypersurface.

Remark

The similar curvature integral inequality holds when the m-th order mean curvature H_m vanishes, which was obtained by G. Wei.

Theorem (Min-S 2015)

Let M be an $n(\geqslant 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple (i.e., multiplicity 1). For the unit principal direction vector e_n corresponding to μ, we have

$$\int_M \mathrm{Ric}(e_n, e_n) \geqslant 0,$$

where Ric denotes the Ricci curvature. Moreover, equality holds if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.

Remark

If the multiplicities of two distinct principal curvatures are at least 2, then a closed hypersurface with constant m-th order mean curvature is congruent to a Clifford hypersurface (B.Y. Wu 2009). Thus it suffices to consider the case where one of the two distinct principal curvatures is simple.

Remark

If $H_m \equiv 0$ for $1 \leqslant m < n$, then

$$\mathrm{Ric}(e_n, e_n) = (n-1)(1+\lambda\mu) = (n-1)\left(1 - \frac{m(n-m)}{n(m^2-2m+n)}|A|^2\right).$$

Notations

- M, $n(\geqslant 3)$-dimensional hypersurface in the unit sphere $\mathbb{S}^{n+1}$.
- ∇, the Riemannian connection of M
- $e_1, \ldots, e_n, e_{n+1}$, orthonormal frame fields of the unit sphere such that $e_1, \ldots, e_n$ are tangent to M
- $\omega^1, \ldots, \omega^n, \omega^{n+1}$, the dual coframe.

  $$\omega^{n+1} = 0$$

  on M.
- h and A, the second fundamental form and the shape operator of M such that

  $$\langle A(X), Y \rangle = h(X, Y)$$

  for all $X, Y \in T_pM$
- $\nabla h = \sum_{i,j,k=1}^{n} h_{ijk} \omega^i \otimes \omega^j \otimes \omega^k$, where h_{ijk} is the coefficient function of ∇h such that

  $$\begin{aligned} h_{ijk} \equiv h_{ij;k} &= (\nabla_{e_k} h)(e_i, e_j) \\ &= \nabla_{e_k} h(e_i, e_j) - h(\nabla_{e_k} e_i, e_j) - h(e_i, \nabla_{e_k} e_j) \end{aligned}$$
- $h_{ijk} = h_{ikj}$ by Codazzi equation

Now assume that M is a closed hypersurface in a unit sphere with constant m-th order mean curvature H_m and with two distinct principal curvatures with multiplicities $n-1, 1$. May assume that $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$. We choose the orthonormal frame tangent to M such that $h_{ij} = \lambda_i \delta_{ij}$, that is,

$$\begin{aligned} Ae_i &= \lambda e_i \quad \text{for } i = 1, \ldots, n-1, \\ Ae_n &= \mu e_n. \end{aligned}$$

Since M has two distinct principal curvatures λ and μ,

$$\binom{n}{m} H_m = \binom{n-1}{m} \lambda^m + \binom{n-1}{m-1} \lambda^{m-1} \mu.$$

Therefore

$$H_m = \frac{m}{n} \lambda^{m-1} \left(\frac{n-m}{m} \lambda + \mu \right).$$

Claim: $\lambda^m - H_m$ never vanishes on M.

Proof of claim.
We consider two cases: $m = 1$ and $m \geqslant 2$.

- Suppose $m = 1$.
  - If $H_1 = 0$, then $\lambda \neq 0$. Thus $\lambda - H_1 \neq 0$.
  - If $H_1 \neq 0$, then $\lambda - H_1 = \frac{\lambda - \mu}{n}$. Since $\lambda \neq \mu$, it never vanishes.
- Suppose $m \geqslant 2$.
  - If $H_m \neq 0$, then $\lambda \neq 0$. Therefore $\lambda^m - H_m = \frac{m}{n}\lambda^{m-1}(\lambda - \mu) \neq 0$.
  - If $H_m = 0$ and $\lambda \neq 0$, then $\lambda^m - H_m$ never vanishes.
  - If $H_m = 0$ and $\lambda = 0$ at some point, then it follows from the equation $\lambda^{m-1}(\frac{n-m}{n}\lambda + \mu) = 0$ that $\lambda \equiv 0$. Thus M has constant sectional curvature 1 by the Gauss equation, which implies that M is totally geodesic. However this is a contradiction because $\lambda \neq \mu$.

□

From our claim, we can define a function $w := |\lambda^m - H_m|^{-\frac{1}{n}}$. B.Y. Wu obtained the following useful second order ordinary differential equation on M:

$$\frac{d^2 w}{dv^2} = -w\left(\frac{nH_m - (n-m)\lambda^m}{m\lambda^{m-2}} + 1\right),$$

where v is the arclength parameter of the integral curve with respect to μ.

Remark

In particular, if $H_1 \equiv 0$, then this equation was originally obtained by Otsuki.

Theorem (Otsuki 1970)

Let Σ be a hypersurface immersed in an $(n+1)$-dimensional Riemannian manifold of constant curvature such that the multiplicities of principal curvatures are all constant. Then we have the following:

- The distribution of the space of principal vectors corresponding to each principal curvature is completely integrable.
- If the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.

From the previous theorem, it follows, for each $1 \leqslant i \leqslant n-1$,

$$\nabla_{e_i} \lambda = e_i \lambda = 0.$$

Since $w = |\lambda^m - H_m|^{-\frac{1}{n}} = (s(\lambda^m - H_m))^{-\frac{1}{n}}$ for a fixed constant $s = \pm 1$,

$$\begin{aligned} \nabla_{e_i} w &= \nabla_{e_i} (|\lambda^m - H_m|^{-\frac{1}{n}}) \\ &= -\frac{m}{n} s w^{n+1} \lambda^{m-1} \nabla_{e_i} \lambda. \end{aligned}$$

Moreover, for $i = 1, \ldots, n-1$,

$$\nabla_{e_i} w = 0.$$

For a function $f = f(w)$ on M, we compute the Laplacian of f on M as follows.

$$\Delta f = -\frac{1}{n-1} f'(w) w \ \mathrm{Ric}(e_n, e_n) + \left[f''(w) + (n-1)\frac{f'(w)}{w} \right] (e_n w)^2,$$

where $\mathrm{Ric}(e_n, e_n)$ denotes the Ricci curvature in the direction of e_n.

Theorem (Min-S 2015)

Let M be an $n(\geqslant 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple (i.e., multiplicity 1). For the unit principal direction vector e_n corresponding to μ, we have

$$\int_M \mathrm{Ric}(e_n, e_n) \geqslant 0,$$

where Ric denotes the Ricci curvature. Moreover, equality holds if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.

Proof

Define a function $f(w) = \log w$, where $w = |\lambda^m - H_m|^{-\frac{1}{n}}$.

$$\Delta f = -\frac{\mathrm{Ric}(e_n, e_n)}{n-1} + \frac{n-2}{w^2}(e_n w)^2.$$

Integrating Δf over M gives

$$0 = \int_M -\frac{\mathrm{Ric}(e_n, e_n)}{n-1} + \frac{n-2}{w^2}(e_n w)^2.$$

Therefore

$$\int_M \mathrm{Ric}(e_n, e_n) = (n-1)(n-2)\int_M \frac{(e_n w)^2}{w^2} \geqslant 0.$$

Proof

Moreover equality holds if and only if $e_n w \equiv 0$ on M in the above inequality, which is equivalent that $e_n \lambda \equiv 0$. Thus both λ and μ are constant, which implies that M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$ by Cartan, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$. □

Applications

In particular, if $H_m \equiv 0$ for $1 \leqslant m < n$, then we have

$$\mathrm{Ric}(e_n, e_n) = (n-1)\left(1 - \frac{m(n-m)}{n(m^2-2m+n)}|A|^2\right).$$

Therefore

Corollary

Let M be an $n(\geqslant 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leqslant m < n)$ and with two distinct principal curvatures, one of them being simple. Then

$$\int_M |A|^2 \leqslant \frac{n(m^2-2m+n)}{m(n-m)}\mathrm{Vol}(M),$$

where equality holds if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

Applications

Corollary

Let M be an $n(\geqslant 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature and with two distinct principal curvatures, one of them being simple. Denote by e_n the unit principal direction vector corresponding to μ. If $\mathrm{Ric}(e_n, e_n) \leqslant 0$ on M, then M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.

By making use of the Laplacian of a function of principal curvatures, we obtain a characterization theorem under a pointwise nonnegative Ricci curvature assumption:

Theorem

Let M be an $n(\geqslant 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple. Denote by e_n the unit principal direction vector corresponding to μ. If $\mathrm{Ric}(e_n, e_n) \geqslant 0$ on M, then M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.

Proof

Define a function $f(w) = w^k$ for a number $k < 2 - n$. Then

$$\begin{aligned}\Delta f &= -\frac{1}{n-1} f'(w) w \,\mathrm{Ric}(e_n, e_n) + \left[f''(w) + (n-1)\frac{f'(w)}{w} \right] (e_n w)^2 \\ &= -\frac{1}{n-1} k w^k \,\mathrm{Ric}(e_n, e_n) + k(k+n-2) w^{k-2} (e_n w)^2.\end{aligned}$$

Integrating Δf over M, we have

$$0 = \int_M \Delta f = \int_M \left[-\frac{1}{n-1} k w^k \,\mathrm{Ric}(e_n, e_n) + k(k+n-2) w^{k-2} (e_n w)^2 \right].$$

Proof

Therefore

$$\frac{1}{n-1}\int_M w^k \ \mathrm{Ric}(e_n, e_n) = (k+n-2)\int_M w^{k-2}(e_n w)^2. \qquad (2)$$

From the equality (2) and the assumption that $\mathrm{Ric}(e_n, e_n) \geqslant 0$, it follows that $e_n w \equiv 0$ on M. Therefore M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where λ is the root of the equation $(n-m)\lambda^m - m\lambda^{m-2} = nH_m$.

Applications

In particular, when $H_m \equiv 0$, we give a simple proof of the previous results in [Cheng 2001, Hasanis-Savas-Halilaj 2000, 2005, Otsuki 1970, Wei 2006, 2007, Wu 2009, ...]

Corollary

Let M be an $n(\geqslant 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leqslant m < n)$ and with two distinct principal curvatures, one of them being simple. Either if $|A|^2 \geqslant \frac{n(m^2-2m+n)}{m(n-m)}$ or $|A|^2 \leqslant \frac{n(m^2-2m+n)}{m(n-m)}$ on M, then M is isometric to the Riemannian product $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

Corollary

Let M be an $n(\geqslant 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature and with two distinct principal curvatures. If the Ricci curvature on M is nonnegative, then M is isometric to a Clifford hypersurface.

Theorem (Min-S 2015)

Let M be an $n(\geqslant 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leqslant m < n)$ and with two distinct principal curvatures, one of them being simple. Then we have

$$\begin{cases} \int_M |A|^p \left(|A|^2 - \dfrac{n(m^2 - 2m + n)}{m(n-m)} \right) \leqslant 0 & \text{if } p < \frac{n-2}{n} m, \\ \int_M |A|^p \left(|A|^2 - \dfrac{n(m^2 - 2m + n)}{m(n-m)} \right) \geqslant 0 & \text{if } p > \frac{n-2}{n} m. \end{cases}$$

Moreover, equalities hold if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

Proof

Since $H_m \equiv 0$,

$$\mu = -\frac{n-m}{m}\lambda.$$

Thus as before, we have

$$\int_M w^k \operatorname{Ric}(e_n, e_n) \geqslant 0 \quad \text{if } k > 2-n,$$
$$\int_M w^k \operatorname{Ric}(e_n, e_n) \leqslant 0 \quad \text{if } k < 2-n,$$

where $w = |\lambda|^{-\frac{m}{n}}$. Let $p = -\frac{km}{n}$. Then

$$\begin{cases} \displaystyle\int_M |\lambda|^p \geqslant \frac{n-m}{m}\int_M |\lambda|^{p+2} & \text{if } p < \frac{n-2}{n}m, \\ \displaystyle\int_M |\lambda|^p \leqslant \frac{n-m}{m}\int_M |\lambda|^{p+2} & \text{if } p > \frac{n-2}{n}m. \end{cases}$$

Proof

Therefore using

$$|A|^2 = (n-1)\lambda^2 + \mu^2 = \frac{n(m^2-2m+n)}{m^2}\lambda^2,$$

we finally obtain

$$\begin{cases} \int_M |A|^p \left(|A|^2 - \dfrac{n(m^2-2m+n)}{m(n-m)}\right) \leqslant 0 & \text{if } p < \frac{n-2}{n}m, \\ \int_M |A|^p \left(|A|^2 - \dfrac{n(m^2-2m+n)}{m(n-m)}\right) \geqslant 0 & \text{if } p > \frac{n-2}{n}m. \end{cases}$$

Furthermore, equalities hold if and only if $e_n w \equiv 0$. Thus equalities hold if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

□

It would be interesting to ask if the above results are still true without assuming that M has two distinct principal curvatures.

Thank you for your attention.