Helicoidal flat surfaces in $\mathbb{S}^3$

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Joint work with João Paulo dos Santos

Geometric aspects on capillary problems and related topics

Aim

- Classification of helicoidal flat surfaces in $\mathbb{S}^3$ in terms of their first and second fundamental forms and by linear solutions of the corresponding angle function.

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In $\mathbb{R}^3$, a helicoidal surface can be written as

$$X(u,v) = (u\cos v, u\sin v, \lambda(u) + hv),$$

where $h \in \mathbb{R}$ and $\lambda(u)$ is a smooth function.

G the 1-parameter subgroup of the isometries $\phi_{\alpha,\beta}(t) : \mathbb{S}^3 \to \mathbb{S}^3$:

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When $\beta \neq 0$, $\varphi_\beta(t)$ fixes the set $l = \{(z,0) \in \mathbb{S}^3\}$. So, $\{\varphi_\beta(t)\}$ consists of rotations around l and $\{\psi_\alpha(t)\}$ are translations along l.

Definition

A *helicoidal* surface in $\mathbb{S}^3$ is a surface invariant under the action $\phi_{\alpha,\beta} : \mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{S}^3$ of $\mathbb{S}^1$ on $\mathbb{S}^3$ given by

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- $\alpha = -\beta$ is also, up to a rotation in $\mathbb{S}^3$, a Clifford translation.

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- Baikoussis, Koufogiorgos, 1998:
  Helicoidal surfaces with prescribed mean or Gaussian curvature.

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- Kokubu, Umehara, Yamada, 2004:
  Global properties of flat surfaces with admissible singularities.
- Martinéz, dos Santos, Tenenblat, 2013:
  Complete classification of the helicoidal flat surfaces in terms of meromorphic data as well as by means of linear harmonic functions.

Natural parametrization

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Then we have

$$\begin{aligned} X_t &= \phi_{\alpha,\beta}(t) \cdot (-\alpha x_2, \alpha x_1, 0, \beta x_3), \\ X_s &= \phi_{\alpha,\beta}(t) \cdot \gamma'(s), \end{aligned}$$

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and

$$N = \phi_{\alpha,\beta}(t) \cdot \big(\beta x_3(x_2' x_3 - x_2 x_3'), \beta x_3(x_1 x_3' - x_1' x_3), \beta x_3(x_1' x_2 - x_1 x_2'), -\alpha x_3'\big).$$

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Proposition 1

A helicoidal surface, locally parametrized as before, is a flat surface if and only if the following equation

$$\beta^2 \varphi'' \sin^3 \varphi \cos \varphi - \beta^2 (\varphi')^2 \sin^4 \varphi + \alpha^2 (\varphi')^4 \cos^4 \varphi = 0 \qquad (2)$$

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Proof: Exercise.

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The aim here is to characterize the flat surfaces when ω is linear, i.e.,

$$\omega(u, v) = \omega_1(u) + \omega_2(v) = \lambda_1 u + \lambda_2 v + \lambda_3, \quad \lambda_i \in \mathbb{R}.$$

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Theorem (Bianchi-Spivak)

$c_a, c_b : I \subset \mathbb{R} \to \mathbb{S}^3$ curves parametrized by arclength, with curvatures κ_a and κ_b, and whose torsions are $\tau_a = 1$ and $\tau_b = -1$.

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- parameterize a flat surface in $\mathbb{S}^3$,
- the fundamental forms are as before,
- ω satisfies $\omega_1'(u) = -\kappa_a(u)$ and $\omega_2'(v) = \kappa_b(v)$.

Given $r > 1$, consider the curve $\gamma_r : \mathbb{R} \to \mathbb{S}^3$ (*base curve*) given by

$$\gamma_r(u) = \frac{1}{\sqrt{1+r^2}} \left(r\cos\frac{u}{r}, r\sin\frac{u}{r}, \cos ru, \sin ru \right).$$

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Consider now

$$\begin{aligned} c_a(u) &= \frac{1}{\sqrt{1+a^2}}(a,0,-1,0)\cdot\gamma_a(u), \\ c_b(v) &= \frac{1}{\sqrt{1+b^2}}T(\gamma_b(v))\cdot(b,0,0,-1), \end{aligned}$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Theorem 1

The map $X(u,v) = c_a(u) \cdot c_b(v)$ is a parametrization of a flat surface in $\mathbb{S}^3$, whose the fundamental forms are given by

$$\begin{aligned} I &= du^2 + 2\cos\left(\left(\frac{1-a^2}{a}\right)u + \left(\frac{b^2-1}{b}\right)v + c\right) dudv + dv^2, \\ II &= 2\sin\left(\left(\frac{1-a^2}{a}\right)u + \left(\frac{b^2-1}{b}\right)v + c\right) dudv. \end{aligned}$$

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g_a &= \frac{1}{\sqrt{1+a^2}}(a, 0, -1, 0), \\
g_b &= \frac{1}{\sqrt{1+b^2}}(b, 0, 0, -1), \\
Y(u,v) &= \gamma_a(u) \cdot T(\gamma_b(v)).
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$Y(u,v)$ is invariant by helicoidal motions if

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where $u(t)$ and $v(t)$ are smooth functions. A straightforward computation shows that

$$u(t) = u + z(t) \quad \text{and} \quad v(t) = v + w(t),$$

where

$$z(t) = \frac{a(b^2 - 1)}{a^2 b^2 - 1}\beta t \quad \text{and} \quad w(t) = \frac{b(1 - a^2)}{a^2 b^2 - 1}\beta t,$$

with

$$\alpha = \frac{b^2 - a^2}{a^2 b^2 - 1}\beta.$$

Constant angle surfaces

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$$E_1(z, w) = (iz, iw), \quad E_2(z, w) = (-i\overline{w}, i\overline{z}), \quad E_3(z, w) = (-\overline{w}, \overline{z}).$$

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Let M be a helicoidal surface in $\mathbb{S}^3$, parametrized as before. Then, M is flat if and only if it is a constant angle surface.

Proof: It is an application of the preview characterization of flat surfaces in $\mathbb{S}^3$ (Proposition 1).

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Proof: Consider the unit normal vector field N associated to the local parametrization X of M given in (1).

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Let M be a helicoidal flat surface in $\mathbb{S}^3$. Then M admits a local parametrization such that the fundamental forms are given as before and ω is a linear function.

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Remark:

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  Characterization of constant angle surfaces in the Berger sphere: such surfaces are determined by a 1-parameter family of isometries of the Berger sphere and by a geodesic of a 2-torus in $\mathbb{S}^3$.

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Given a number $\epsilon > 0$, the Berger sphere $\mathbb{S}^3_\epsilon$ is defined as the sphere $\mathbb{S}^3$ endowed with the metric

$$\langle X, Y\rangle_\epsilon = \langle X, Y\rangle + (\epsilon^2 - 1)\langle X, E_1\rangle\langle Y, E_1\rangle.$$

Using the Montaldo-Onnis characterization, there exists a local parametrization $F(u, v)$ of M given by

$$F(u, v) = A(v)b(u),$$

where

$$b(u) = \left(\sqrt{c_1}\cos(\alpha_1 u), \sqrt{c_1}\sin(\alpha_1 u), \sqrt{c_2}\cos(\alpha_2 u), \sqrt{c_2}\sin(\alpha_2 u)\right)$$

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and $A(v) = A(\xi, \xi_1, \xi_2, \xi_3)(v)$ is a 1-parameter family of 4×4 orthogonal matrices commuting with a complex structure of $\mathbb{R}^4$, ξ is a constant and the functions $\xi_i(v)$, $1 \leq i \leq 3$, satisfy

$$\cos^2(\xi_1(v))\xi_2'(v) - \sin^2(\xi_1(v))\xi_3'(v) = 0. \tag{4}$$

Taking $\epsilon = 1$, we can reparametrize the curve b such that the new curve is a base curve γ_a:

Taking $\epsilon = 1$, we can reparametrize the curve b such that the new curve is a base curve γ_a: writing $s = 2\sqrt{c_1 c_2}$, we have

$$b(s) = \frac{1}{\sqrt{1+a^2}} \left(a\cos\frac{s}{a}, a\sin\frac{s}{a}, \cos(as), \sin(as) \right),$$

where $a = \sqrt{c_1/c_2}$.

The matrix

$$A(v) = A(\xi, \xi_1, \xi_2, \xi_3)(v) = \begin{pmatrix} \alpha(v) \\ J_1\alpha(v) \\ \cos\xi J_2\alpha(v) + \sin\xi J_3\alpha(v) \\ -\cos\xi J_3\alpha(v) + \sin\xi J_2\alpha(v) \end{pmatrix},$$

where

$$\alpha(v) = (\cos\xi_1 \cos\xi_2, -\cos\xi_1 \sin\xi_2, \sin\xi_1 \cos\xi_3, -\sin\xi_1 \sin\xi_3)$$

and J_1, J_2 and J_3 are orthogonal matrices given explicitly, can be written as

$$A(v) = A(\xi) \cdot \tilde{A}(v),$$

where

$$A(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin\xi & \cos\xi \\ 0 & 0 & -\cos\xi & \sin\xi \end{pmatrix} \quad \text{and} \quad \tilde{A}(v) = \begin{pmatrix} \alpha(v) \\ J_1\alpha(v) \\ J_3\alpha(v) \\ J_2\alpha(v) \end{pmatrix}.$$

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$$A(v)\cdot b(s) = A(\xi)X(v,s).$$

$X(v,s)$ can be written as

$$X(v,s) = \frac{1}{\sqrt{1+a^2}}(x_1, x_2, x_3, x_4),$$

with

$$\begin{aligned}
x_1 &= a\cos\xi_1 \cos\left(\frac{s}{a} + \xi_2\right) + \sin\xi_1 \cos(as + \xi_3), \\
x_2 &= a\cos\xi_1 \sin\left(\frac{s}{a} + \xi_2\right) + \sin\xi_1 \sin(as + \xi_3), \\
x_3 &= -a\sin\xi_1 \cos\left(\frac{s}{a} - \xi_3\right) + \cos\xi_1 \cos(as - \xi_2), \\
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x_3 &= -a\sin\xi_1\cos\left(\frac{s}{a}-\xi_3\right)+\cos\xi_1\cos(as-\xi_2),\\
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On the other hand,

$$
\phi_{\alpha,\beta}(t)\cdot X(v,s)=\frac{1}{\sqrt{1+a^2}}(z_1,z_2,z_3,z_4),
$$

where

$$
\begin{aligned}
z_1 &= a\cos\xi_1\cos\left(\frac{s}{a}+\xi_2+\alpha t\right)+\sin\xi_1\cos\left(as+\xi_3+\alpha t\right),\\
z_2 &= a\cos\xi_1\sin\left(\frac{s}{a}+\xi_2+\alpha t\right)+\sin\xi_1\sin\left(as+\xi_3+\alpha t\right),\\
z_3 &= -a\sin\xi_1\cos\left(\frac{s}{a}-\xi_3+\beta t\right)+\cos\xi_1\cos\left(as-\xi_2+\beta t\right),\\
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As the surface is helicoidal, we have

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(ii) $\xi_1(v)$ constant. In this case, $v(t)$ and $s(t)$ are given by

$$s(t) = s + \frac{a(b^2-1)}{a^2b^2-1}\beta t \quad \text{and} \quad v(t) = v + \frac{b(1-a^2)}{a^2b^2-1}\beta t,$$

that coincide with the expressions obtained in Theorem 1.

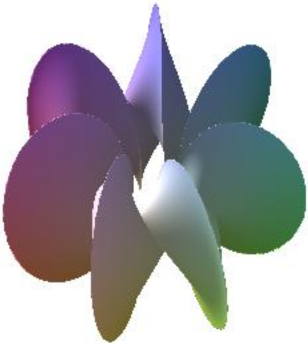
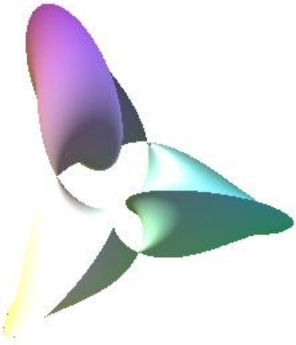


Figura: a=2 and b=3.



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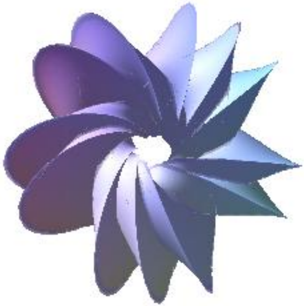

Figura: a=$\sqrt{2}$ and b=3.