Local structure of the space of all triply periodic minimal surfaces in $\mathbf{R}^3$

(Joint work with T. Shoda and P. Piccione)

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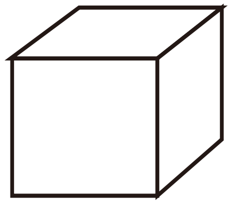
Contents

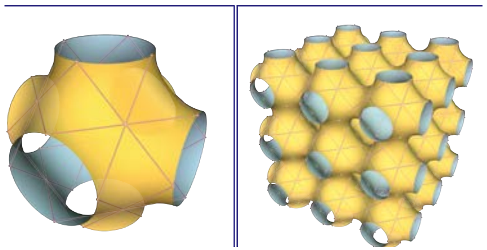
1 Introduction

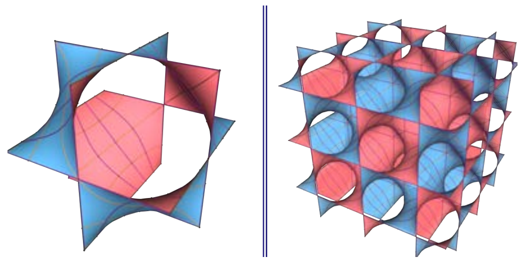
Object: orientable connected embedded triply-periodic minimal surfaces (TPMS's) in $\mathbb{R}^3$. (= cpt. minimal surfaces in flat T^3.)

【The most well-known examples of TPMS's】

Schwarz P surface (19c) Schwarz D surface (19c)

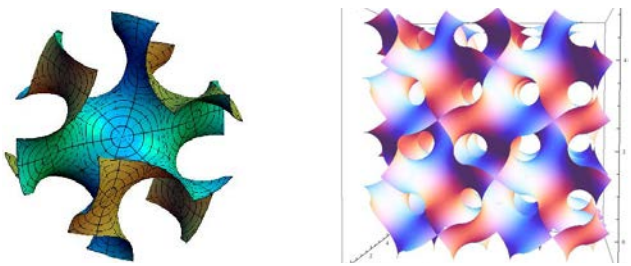


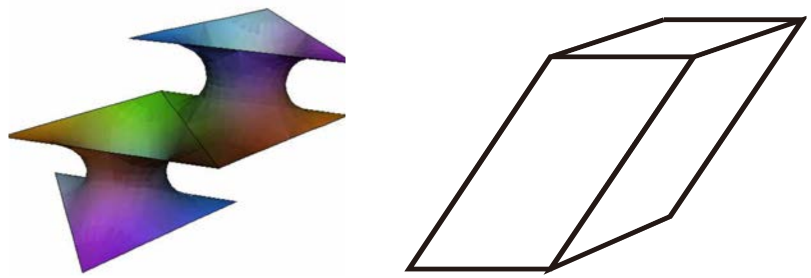




(http://www.indiana.edu/ minimal/archive/Triply/genus3.html)

Alan Schoen's Gyroid(1970) one period of D surface





TPMS($\mathbf{R}^3$) := {orientable connected embedded triply-periodic minimal surfaces (TPMS's) in $\mathbf{R}^3$}

$$\updownarrow$$

CMS($\mathbf{T}^3$) := {orientable connected embedded compact minimal surfaces in flat $\mathbf{T}^3$}. **(g := genus of the considered surface)**

$g = 0$: $\nexists$ **($\longleftarrow$ Gauss-Bonnet Th.)**

$g = 1$: **Totally geodesic subtorus $\mathbf{T}^2 \longleftrightarrow$ planes in $\mathbf{R}^3$**

$g = 2$: $\nexists$ **($\longleftarrow$ Gauss-Bonnet + Gauss map is anti-holo. to S^2)**

$g \geq 3$: **There are many examples.**

- **Classification is difficult.**
- **We study local structures of TPMS($\mathbf{R}^3$).**

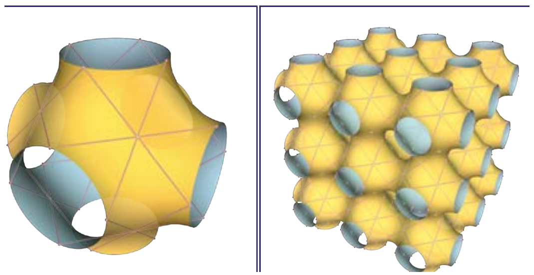
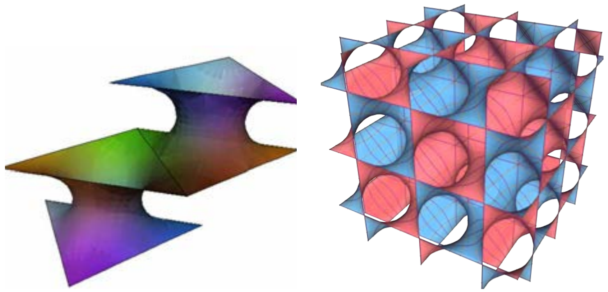
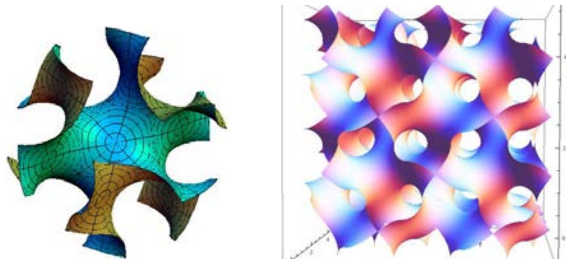
Remark: TPMS's also interest physicists and chemists because they appear in various natural phenomenon: Self-assembly of diblock copolymers in soft matter physics, $\cdots$

Main results (roughly):

(A) For each "generic" $M_0 \in \mathrm{TPMS}(\mathbf{R}^3)$, $\exists \Omega$: neighborhood of M_0 s.t. $\Omega \cap \mathrm{TPMS}(\mathbf{R}^3)$ is 5-dimensional space (up to homothety and congruence in $\mathbf{R}^3$). "5-dimension" corresponds to the space of all lattices in $\mathbf{R}^3$.

Examples of "generic" TPMS's:

Strictly stable TPMS. = The second variation of area is positive for all nontrivial "volume-preserving" variations. Ex: Schwarz P surface, Schwarz D surface, Alan Schoen's Gyroid.

(B)' There are singularities in $\mathrm{TPMS}(\mathbf{R}^3)$.

2 Definitions and main theorems

Σ: 2-dim. oriented compact conn. C^∞ manifold with $g(\Sigma) \geq 3$,

$X : \Sigma \to \mathbf{T}^3_\Lambda := \mathbf{R}^3/\Lambda$, minimal immersion into $\mathbf{T}^3_\Lambda = (\mathbf{T}^3, g_\Lambda)$,

$J[\varphi] := \Delta\varphi - 2K\varphi, \quad K$ is the Gauss curvature of X.

J is the Jacobi operator of X. $\quad H$: mean curvature of surface.

For a variation $X_\epsilon = X + \epsilon(\varphi\vec{n} + \xi) + \mathcal{O}(\epsilon^2)$ of X, $J[\varphi] = 2\,\delta H$.

Consider eigenvalue problem: $(*) \quad J[\varphi] = -\lambda\varphi, \ \ \varphi \in C^{2,\alpha}(\Sigma) - \{0\}$.

Denote by $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ the eigenvalues of $(*)$.

Index of X : $\mathbf{Ind}(X) := \#\{j \mid \lambda_j < 0\}$

$\quad = \mathbf{dim}\{$**variation vector fields which diminishes area**$\}$,

Nullity of X : $\mathbf{Nul}(X) := \#\{j \mid \lambda_j = 0\}$.

Remark. Ind(X) ≥ 1. ($\leftarrow X_\epsilon = X + \epsilon\vec{n}$: parallel surfaces.)

Nul(X) ≥ 3. ($\leftarrow X_\epsilon = X + \epsilon\, \mathbf{e}_i$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis in $\mathbf{R}^3$.)

Notations:

Denote by $\mathcal{T}(\mathbf{T}^3)$ the set of all flat metrics in $\mathbf{T}^3$ (modulo isometry), and by $[\ ]$ the isometry class.

Let Λ_0 be a lattice in $\mathbf{R}^3$. Let $X_0 : \Sigma \to \mathbf{T}^3_{\Lambda_0}$ be a minimal embedding. For any $[\Lambda]$ close to $[\Lambda_0]$, and $\varphi \in C^{2,\alpha}(\Sigma)$ close to 0, we define an embedding $X_{\varphi,\Lambda} : \Sigma \to \mathbf{T}^3$ as

$$X_{\varphi,\Lambda}(p) = \exp^{g_\Lambda}_{X_0(p)}(\varphi(p) \cdot \vec{n}^{g_\Lambda}_{X_0(p)}), \quad p \in \Sigma,$$

where $\exp^{g_\Lambda}$ is the exponential map, and $\vec{n}^{g_\Lambda}_{X_0}$ is the unit normal vector field along X_0 in $(\mathbf{T}^3, g_\Lambda)$. All minimal embeddings near X_0 can be represented in this form.

Recall $X_{\varphi,\Lambda}(p) = \exp^{g_\Lambda}_{X_0(p)}(\varphi(p)\cdot \vec{n}^{g_\Lambda}_{X_0(p)}), \quad p \in \Sigma.$

Theorem A (Rigidity. Meeks(1990)[6] for special cases. Ejiri[1], K-P-S[5]). **Let $X_0 : \Sigma \to \mathbf{T}^3_{\Lambda_0}$ be a compact minimal embedding with $g(\Sigma) \geq 3$ and $\mathrm{Nul}(X_0) = 3$. Then,**

$\exists V$ **: a neighborhood of** $[\Lambda_0]$

in $\mathcal{T}(\mathbf{T}^3) = \{$**flat metrics on** $\mathbf{T}^3\}/\{$**isometries**$\} = \{$**lattices in** $\mathbf{R}^3\}$,

$\exists \Phi : V \to C^{2,\alpha}(\Sigma), \quad \Lambda \mapsto \varphi_\Lambda, \quad C^2$ **mapping, such that**

(i) $\varphi_{\Lambda_0} = 0$,

(ii) $X_\Lambda := X_{\varphi_\Lambda,\Lambda}$ **is a minimal surface in** $(\mathbf{T}^3, g_\Lambda)$,

(iii) $\exists \Omega$**: a neighborhood of** X_0 **s.t.** $\forall \Lambda \in V$, $\forall Y : \Sigma \to (\mathbf{T}^3, g_\Lambda)$**: minimal embedding in** Ω, Y **is congruent to** X_Λ.

That is, in a neighborhood of X_0, there is a 1-1 correspondence between TPMS's and lattices in $\mathbf{R}^3$. Hence the space of TPMS's is (locally) 5-dimensional (up to congruence and homothety).

Remark on Theorem A (Meeks'90 for special cases. Ejiri, KPS): **If X_0 is a compact minimal embedding with genus ≥ 3 and nullity $= 3$, then, in a neighborhood of X_0, there is a 1-1 correspondence between TPMS's and lattices in $\mathbf{R}^3$.**

One of the theorems by Meeks('90) implies: Let X_0 be a TPMS that is represented as two-sheeted covers of S^2 branched over four pairs of antipodal points. Then, X_0 is embedded and the space of TPMS's includes a 5-dimensional family of embedded TPMS's (up to congruence and homothety).

Explicit representations: Set $M := \left\{(w,\zeta) \in \mathbf{C}^2 \;\middle|\; \zeta^2 = \Pi_{j=1}^4 (w - a_j)(w + \overline{a_j}^{-1})\right\}$. **Then,** $X_R(w) := \mathbf{Re}\left[\int_{w_0}^{w} \left(1-w^2,\, i(1+w^2),\, 2w\right)\zeta^{-1} dw\right]$**, and** $X_I(w) := \mathbf{Im}\left[\cdots\right]$, $(w \in M)$ **give embedded TPMS's.**

D and P surfaces admit such representations. Gyroid doesn't.

Theorem B (**Bifurcation. K-P-S[5]**). **Let** $\{X_s \,|\, -\delta < s < \delta\}$, $(\delta > 0)$ **be a continuous family of TPMS's. Assume that each** X_s **is a minimal surface in** $(\mathbf{T}^3, g_{\Lambda(s)})$. **We also assume**

(a) $\forall s \neq 0$, $\mathbf{Nul}(X_s) = 3$. **(i.e. there is no non-trivial nullity.)**

(b) $\forall s > 0$, $\mathbf{Ind}(X_s) - \mathbf{Ind}(X_{-s})$ **is odd. (i.e. at** $s = 0$, **the index jumps with an odd integer.)**

Then, $s = 0$ **is a bifurcation instant for the family** $\{X_s\}$:
i.e. in any neighborhood of X_0, **there exists a sequence** $s_n \in (-\delta, \delta) - \{0\}$ **such that**

$\exists Y_n$: **minimal embedding in** $(\mathbf{T}^3, g_{\Lambda(s_n)})$ **such that**
$s_n \longrightarrow 0$ **and** $\{Y_n\} \longrightarrow X_0$ **in** C^0**-topology, (as** $n \longrightarrow \infty$**).**
Y_n **is not congruent to** X_{s_n} $(\forall n)$.

3 Idea of the proofs of the main theorems

Let $\{e_1, e_2, e_3\}$ be the canonical basis in $\mathbf{R}^3$, and $\pi_\Lambda : \mathbf{R}^3 \to \mathbf{R}^3/\Lambda$ be the projection. For Λ and $i = 1, 2, 3$, set $K_i^\Lambda = (\pi_\Lambda)_*(e_i)$. Then, $\{K_i^\Lambda\}_i$ forms a basis of all killing vector fields in $(\mathbf{T}^3, g_\Lambda)$. For $\varphi \in C^{2,\alpha}(\Sigma)$ close to 0, define a map $f_i^{\varphi,\Lambda} : \Sigma \to \mathbf{R}$ as

$$f_i^{\varphi,\Lambda} = g_\Lambda(K_i^\Lambda, \vec{n}^{g_\Lambda}_{X_{\varphi,\Lambda}}).$$

For an embedding $X : \Sigma \to \mathbf{T}^3$, denote by $\mathcal{H}^\Lambda(X)$ the mean curvature of X in g_Λ. For U_0 : a nbd of 0 in $C^{2,\alpha}(\Sigma)$, V_0: a nbd of $[\Lambda_0]$ in $\mathcal{T}(\mathbf{T}^3)$, consider a map $\widetilde{\mathcal{H}} : U_0 \times \mathbf{R}^3 \times V_0 \longrightarrow C^{0,\alpha}(\Sigma)$,

$$\widetilde{\mathcal{H}}(\varphi, a_1, a_2, a_3, [\Lambda]) := \mathcal{H}^\Lambda(X_{\varphi,\Lambda}) + \sum_{i=1}^{3} a_i f_i^{\varphi,\Lambda}.$$

Then, $\widetilde{\mathcal{H}}^{-1}(\mathbf{0}) = \Big\{(\varphi, 0, 0, 0, [\Lambda]) : \ X_{\varphi,\Lambda}$ **is** g_Λ**−minimal**$\Big\}$**.**

$$\widetilde{\mathcal{H}}^{-1}(\mathbf{0}) = \Big\{(\varphi, 0, 0, 0, [\Lambda]) : \; X_{\varphi,\Lambda} \text{ is } g_\Lambda\text{--minimal}\Big\}.$$

The above representation is based on an idea of N. Kapouleas (1987, 1990), which was then also employed by R. Mazzeo, F. Pacard and D. Pollack (2001), R. Mazzeo and F. Pacard (2001), B. White (1987), J. Pérez and A. Ros (1996).

Recall

$$\widetilde{\mathcal{H}}(\varphi, a_1, a_2, a_3, [\Lambda]) := \mathcal{H}^\Lambda(X_{\varphi,\Lambda}) + \sum_{i=1}^{3} a_i f_i^{\varphi,\Lambda}.$$

For $[\Lambda] \in \mathcal{T}(\mathbf{T}^3)$**, set**

$$\widetilde{\mathcal{H}}_\Lambda : U_0 \times \mathbf{R}^3 \longrightarrow C^{0,\alpha}(\Sigma), \quad \widetilde{\mathcal{H}}_\Lambda(\varphi, a_1, a_2, a_3) := \widetilde{\mathcal{H}}(\varphi, a_1, a_2, a_3, [\Lambda]).$$

Assume $\widetilde{\mathcal{H}}_\Lambda(\varphi, 0, 0, 0) = 0$**. Consider**

$$T_{\varphi,\Lambda} := \mathrm{d}\widetilde{\mathcal{H}}_\Lambda(\varphi, 0, 0, 0) : C^{2,\alpha}(\Sigma) \times \mathbf{R}^3 \longrightarrow C^{0,\alpha}(\Sigma).$$

Then, $\forall(\psi, b_1, b_2, b_3) \in C^{2,\alpha}(\Sigma) \times \mathbf{R}^3$**,**

$$T_{\varphi,\Lambda}(\psi, b_1, b_2, b_3) = J_{x_{\varphi,\Lambda}}(\psi) + \sum_{i=1}^{3} b_i f_i^{\varphi,\Lambda},$$

where $J_{x_{\varphi,\Lambda}}$ **is the Jacobi operator of** $X_{\varphi,\Lambda}$**.** $T_{\varphi,\Lambda}$ **is Fredholm with index 3.**

$T_{\varphi,\Lambda}$ **is surjective.** $\Longleftrightarrow$ $X_{\varphi,\Lambda}$ **is** g_Λ**-minimal with nullity 3.**

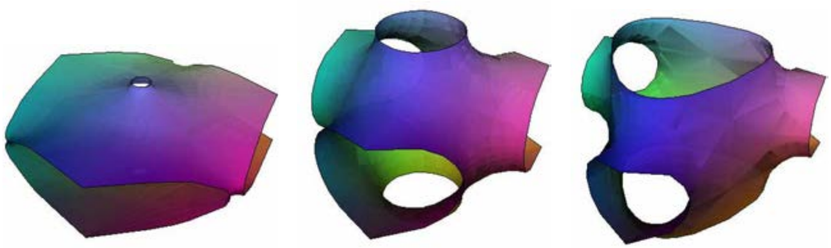
We apply the bifurcation theory (e.g. Kato[3], [4]) to $\widetilde{\mathcal{H}}_\Lambda$**.** $\square$

4 Applications to explicit examples

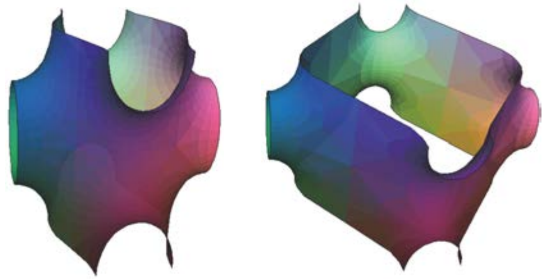
(Most of pictures below were drawn by Prof. Shoichi Fujimori.)

Examples of 1-parameter families of TPMS's:

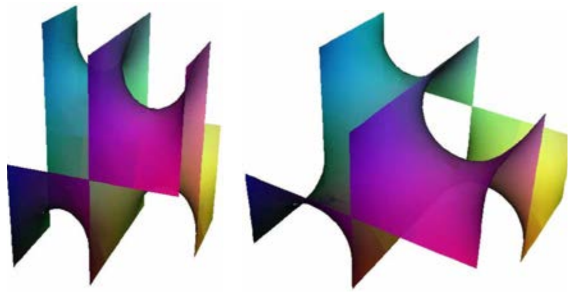
【H-family】



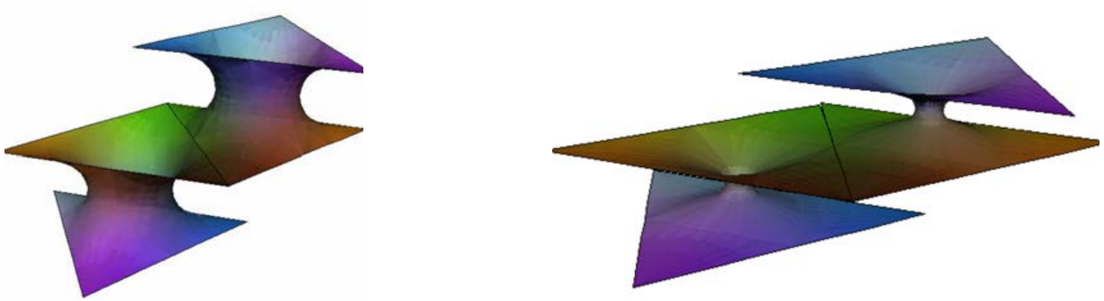
【tCLP-family】



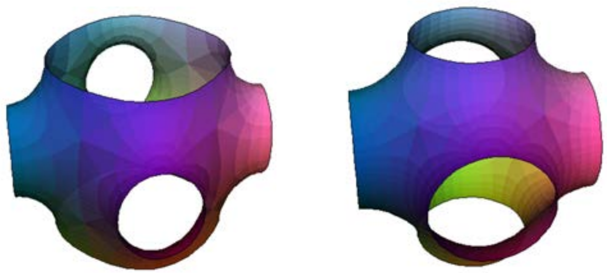
【tD-family】



【rPD-family (Karcher's TT surfaces)】

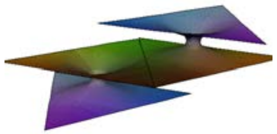


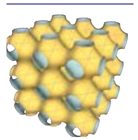
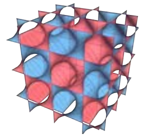
【tP-family】



【Representation of rPD-family】 (Use Weierstrass formula.)

$M_a := \left\{ (w,\zeta) \in \mathbf{C}^2 \;\middle|\; \zeta^2 = w(w^3 - a^3)\left(w^3 + a^{-3}\right) \right\}$, $a > 0$: **Riemann surface.** $\quad X_a(w) := \mathbf{Re}\displaystyle\int_{w_0}^{w} \left(1 - w^2,\, i(1+w^2),\, 2w\right)\zeta^{-1} dw$, $\; w \in M_a$.

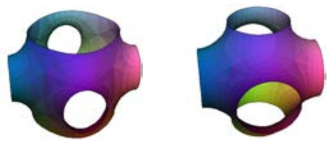


$a = 1/\sqrt{2}$, $b = 14$:**P**  $\quad a = \sqrt{2}$, $b = 14$:**D** 

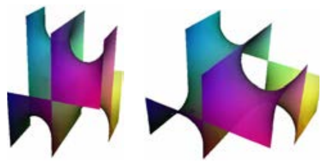
【Representations of tP-family and tD-family】

$N_b := \left\{ (w,\zeta) \in \mathbf{C}^2 \mid \zeta^2 = w^8 + b w^4 + 1 \right\}$, $b \in (2, +\infty)$: **Riemann surface.**

For $w \in N_b$,



tP-family: $\varphi_b(w) = \mathbf{Re}\displaystyle\int_{w_0}^{w} (1 - w^2,\, i(1+w^2),\, 2w)\zeta^{-1} dw$,



tD-family: $\psi_b(w) = \mathbf{Re}\displaystyle\int_{w_0}^{w} i(1 - w^2,\, i(1+w^2),\, 2w)\zeta^{-1} dw$.

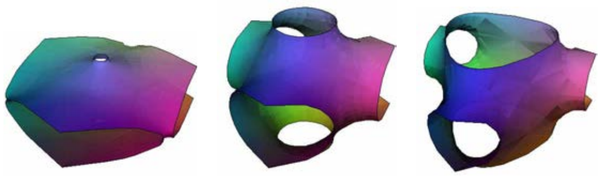
We can apply our main theorems to explicit examples. There is a method to compute the nullities and the indices of TPMS's given by Ejiri-Shoda[2], which includes computation of eigenvalues of 18×18 symmetric matrices whose elements are elliptic integrals! So we need help of numerical computation.

Also, we can find eigenfunctions belonging to zero eigenvalue by using a method given by Montiel-Ros (1991[7]), Ejiri-Kotani (1993).

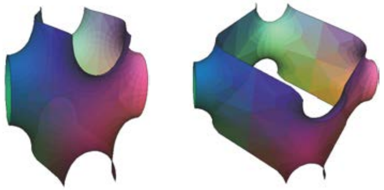
Example 4.1 (Application with numerical computation) It seems there are one bifurcation instant for the H-family, and two bifurcation instants for each of the rPD, tP, and tD families.

This means that there is possibility that we found the existence of new TPMS's which are close to known examples.

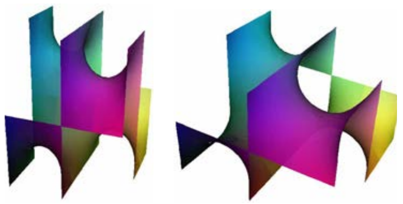
【H】



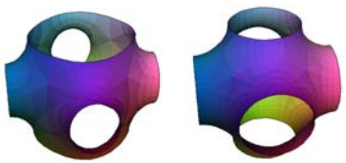
【tCLP】



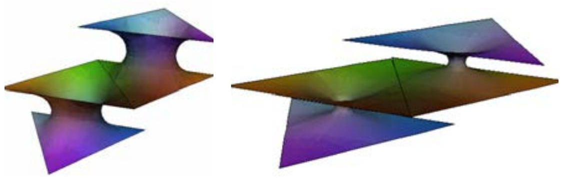
【tD】



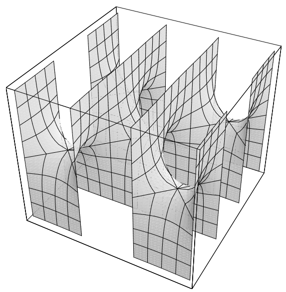
【tP】

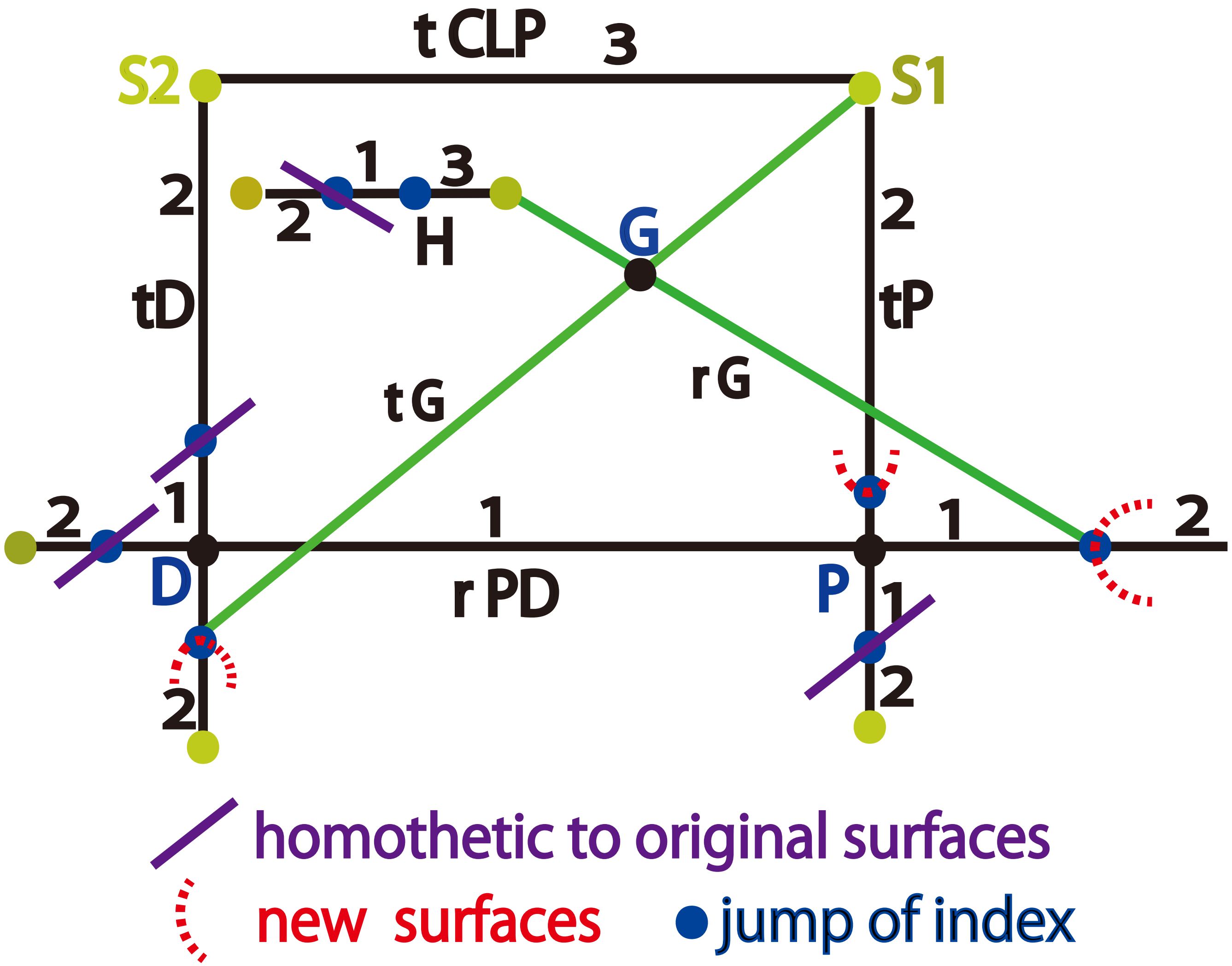


【rPD】



【Scherk surface】





【new surface frpm rPD】

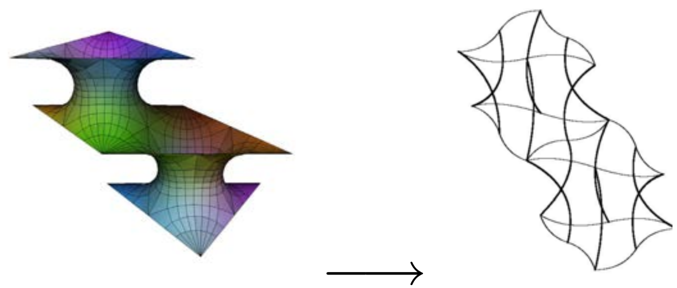
5 Future subjects

(1) Try to verify the “results” about the concrete examples obtained by using numerical computations.

(2) Find explicit representations of the “new” surfaces.

(3) Study the geometry of the surfaces in the bifurcating branches: eg. symmetry-breaking property.

Ex. Bifurcation from the rPD-family: Variation vector field should be the zero eigenfunction.



(4) Study the stability/instability of minimal surfaces in the bifurcating branches.

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[6] W. H. Meeks III, *The theory of triply periodic minimal surfaces*, Indiana Univ. Math. J. 39 (1990), no. 3, 877–936.

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